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Graph Coloring Problems with Applications in Algebraic Logic

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For symmetric atomic relation algebras the property of being representable, finitely representable, or associative (when associativity is not supposed to hold by definitions) is known to be equivalent to the existence of some edge colorings of complete graphs. Here we give a short survey of the open problems and related results concerning necessary and sufficient conditions, unicity of a representation, and the algorithmic complexity of deciding those properties.

1. Introduction

The aim of this note is to invite the attention of the reader to a topic that offers lots of challenging open problems. Those questions are of definite interest for the reason that they can be interpreted in two eqiuvalent — but entirely different — ways, in two branches of mathematics which, at first sight, have very little connection. Those two subjects are algebraic logic (representations of relation algebras) and combinatorics (edge colorings of graphs). This interesting relationship was discovered by Monk a long time ago (see e.g. [M3]) and was further developed by several authors.

Tne one-to-one correspondence between some types of representations and colorings is explained in detail in [M1]. The link between those two concepts is established by a collection $\mathscr{T}$ of 3-element multisets (triplets) which is uniquely determined by the algebra in question. Such a $\mathscr{T}$ can reflect algebraic properties including representability, finite representability, and associativity. For our discussion we have chosen the language of combinatorics, but in the other interpretation the problems and results have purely algrebraic contents. For a description of this correspondence the reader is referred to [M1] or to the short subsection §0.2 of [T]. A detailed

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discussion of representation theory in algebraic logic can be found in the independently readable Part II of the textbook [HMT].

2. Basic concepts

Throughout, the notation we use is consistent with that in [T]. For a natural number t, $[t]$ denotes the set $\{1, \ldots, t\}$. We denote by $\mathscr{T}(s, t)$ the set of all triplets T on $[t]$ such that precisely s distinct elements of $[t]$ are contained in T ($s = 1, 2, 3$). For a subset S of $\{1, 2, 3\}$, set $\mathscr{T}(S, t) := \bigcup_{s \in S} \mathscr{T}(s, t)$. The set $\mathscr{T}(\{1, 2, 3\}, t)$ will be abbreviated as $[t]^3$.

Complete graphs: The complete graph $K = (V, E)$ has vertex set $V = \{v_i \mid i \in I\}$ (I is finite or infinite) and edge set $E = \{v_i v_j \mid i, j \in I,\ i \neq j\}$. Here the edges $v_i v_j$ are considered to be *unordered* pairs, i.e., $v_i v_j = v_j v_i$. If I is finite, $|I| = n$, then we use the notation $K_n = (V_n, E_n)$, where $V_n = \{v_1, \ldots, v_n\}$ (and then $E_n = \{v_i v_j \mid i, j \in [n],\ i \neq j\}$).

A *triangle* of K is a complete subgraph induced by three vertices. Moreover, we denote by K_4-e the graph obtained from K_4 by deleting an edge e. (This K_4-e has just two triangles.)

Colorings: A *coloring* f of a complete graph $K = (V, E)$ is an *edge coloring* $f: E \to [t]$. Let $\mathscr{T} \subset [t]^3$ be a given family of triplets. In order to avoid unnecessary complications, throughout it will be assumed that *every* $i \in [t]$ *occurs in at least one* $T \in \mathscr{T}$ whenever $\mathscr{T} \neq \emptyset$; for $\mathscr{T} = \emptyset$ we shall have $t = 1$ by definition. A coloring f of K is said to be a

$\mathscr{T}^-$*-coloring* if

(i) for any three distinct vertices $v_i, v_j, v_k \in V$, the 3-element multiset $[f(v_i v_j), f(v_i v_k), f(v_j v_k)]$ belongs to $\mathscr{T}$;

$\mathscr{T}^+$*-coloring* if

(ii) for every $T \in \mathscr{T}$, there are distinct vertices $v_i, v_j, v_k \in V$ such that $[f(v_i v_j), f(v_i v_k), f(v_j v_k)] = T$;

strong $\mathscr{T}$*-coloring* if

(iii) for every $T = [a_0, a_1, a_2] \in \mathscr{T}$, if $f(v_i v_j) = a_q$ for some q, $0 \leqq q \leqq 2$, then there exist v_k and $v_{k'}$, distinct from v_i and v_j, such that $f(v_i v_k) = f(v_i v_{k'}) = a_{q-1}$ and $f(v_i v_{k'}) = f(v_j v_k) = a_{q+1}$ (where subscript addition is taken mod 3);

representation *of* $\mathscr{T}$ it satisfies the properties (i), (ii), and (iii).

Note that in (iii) if $a_0 = a_1$ and $f(v_i v_j) = a_2$ then for $q = 2$ one has to find just one vertex $v_k = v_{k'}$.

Packing of triples: Let $T, T' \in \mathscr{T}$. A *packing* of T and T' is a 'partial representation' of T and T' over K_4, that is a color assignment f' of the five edges of K_4-e in such a way that the multisets of colors occurring on the two triangles of K_4-e are identical to T and T', respectively. A packing is *trivial* if $T = T'$ and f' satisfies the following further requirement: assuming $e = v_3v_4$, $f'(v_iv_1) = f'(v_iv_2)$ for $i = 3$ and 4.

Representable and associative families: A $\mathscr{T} \subset [t]^3$ is called *representable* if it has a representation over some complete graph K. If it has a representation over K_n, for some natural number n, then we say that $\mathscr{T}$ is *finitely representable.*

A $\mathscr{T}$ is *associative* if each non-trivial packing of any two (not necessarily distinct) triplets $T, T' \in \mathscr{T}$ can be completed to a $\mathscr{T}^-$-coloring of K_4. Clearly, for these properties the following hierarchy holds: finitely representable $\Rightarrow$ representable $\Rightarrow$ associative.

Subfamilies of $\mathscr{T}$: Every collection $\mathscr{T} \subset [t]^3$ can be written in the form $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2 \cup \mathscr{T}_3$, setting $\mathscr{T}_i := \mathscr{T} \cap \mathscr{T}(i,t)$ $(i = 1, 2, 3)$.

3. Problems and Results

Throughout we formulate questions and statements in terms of triangle families. In the discussion below we proceed from the most particular structures to the general ones. Let us first consider the representability problem of the $\mathscr{T}(S,t)$, $\emptyset \neq S \subset \{1, 2, 3\}$. For convenience we assume $t \geqq s$ for all $s \in S$.

Theorem (representability of $\mathscr{T}$(S, t))

(1) $\mathscr{T}(1, t)$ is representable if and only if $t = 1$.

(2) $\mathscr{T}(2, t)$ is representable if and only if $t = 2$.

(3) $\mathscr{T}(3, t)$ is representable if and only if $t = 3$.

(1, 2) $\mathscr{T}\{1, 2\}, t)$ is representable for all $t \geqq 2$;
$\mathscr{T}(\{1, 2\}, t)$ is finitely representable if and only if $t = 2$.

(1, 3) $\mathscr{T}(\{1, 3\}, t)$ is representable if and only if there exists a finite projective plane of order $t - 1$.

(2, 3) $\mathscr{T}(\{2, 3\}, t)$ is representable for $3 \leqq t \leqq 5$.

(1, 2, 3) $\mathscr{T}(\{1, 2, 3\}, t)$ is finitely representable for all $t \geqq 3$.

Parts (1), (2), and (3) are easily seen; the others were proved by Tuza [T] (1, 2), Lyndon [L] (1, 3), Corner [C2] (2, 3), and Maddux (unpublished) and Andréka, Jipsen, and Tuza [AJT] (1, 2, 3). The case (1, 3) is hopeless to describe more explicitly since it depends on the existence of finite geometries. For (2, 3), however, one would not expect such difficulties.

Problem 1. Is $\mathscr{T}(\{2,3\},t)$ representable for all t?

As shown in the references given above, in most cases the representations can also be characterized, as follows.

– The representations of $\mathscr{T}(1,t)$ are the monochromatic complete graphs.

– The unique representation of $\mathscr{T}(2,2)$ is K_5 with the edge coloring f such that $f^{-1}(1)=\{v_iv_{i+1}\mid 1\leqq i\leqq 5\}$ and $f^{-1}(2)=\{v_{i+2}v_i\mid 1\leqq i\leqq 5\}$ (subscript addition is taken mod 5).

– The unique representation of $\mathscr{T}(3,3)$ is K_4 with the edge coloring f such that $f^{-1}(i)$ consists of two pairwise disjoint edges for $i=1,2,3$.

– There exist infinitely many non-isomorphic representations of $\mathscr{T}(\{1,2\},t)$ and infinitely many finite representations of $\mathscr{T}(\{1,2\},2)$.

– The representations of $\mathscr{T}(\{1,3\},t)$ are in one-to-one correspondence with the finite affine planes of order $t-1$. (Hence, applying known results on finite geometries, the representation of $\mathscr{T}(\{1,3\},t)$ is not always unique, cf. e.g. [HP].)

– For every sufficiently large n (with respect to t), $\mathscr{T}(\{1,2,3\},t)$ has a representation over K_n.

Problem 2. Determine the smallest integer $n=n(t)$ such that $\mathscr{T}(\{1,2,3\},t)$ has a representation over K_n.

A trivial lower bound, following immediately from the definitions, is $n(t)\geqq t^2+t+1$. Moreover, Andréka, Jipsen, and Tuza [AJT] verified with an explicit construction that $n(t)\leqq(2+o(1))t^2$, and proved with probabilistic methods that almost all colorings of K_n are representations of $\mathscr{T}(\{1,2,3\},t)$ when $n\geqq ct^2\log t$ for some constant c.

A more general form of Problem 2 is this:

Problem 3. Suppose that $\mathscr{T}$ is finitely representable. Estimate the smallest size, $n(\mathscr{T})$, of a representation of $\mathscr{T}$ .Which properties of $\mathscr{T}$ are essential with respect to $n(\mathscr{T})$?

Those few known results may indicate that the 'density' of $\mathscr{T}$ might be relevant in this respect. As we have seen, $[t]^3$ has a very small representation. On the other hand, for 'sparse' triangle families Tuza [T] proved that if $\mathscr{T}_3=\varnothing$ then the size of representations grows exponentially with t.

Another fundamental problem is to draw the line between finite and infinite representability.

Problem 4. Let **RT** denote the class of all representable families $\mathscr{T}$. Describe those $\mathscr{T}\in$ **RT** which are finitely representable.

The following interesting class of examples was found by Comer and Maddux. Call a color $c\in[t]$ *flexible* in a family $\mathscr{T}$ if all triplets $T\in[t]^3$ with $c\in T$ belong to $\mathscr{T}$. It has been shown in [C1] and [M2] that every family containing a flexible color is representable. The following analogous question, however, is still open.

Problem 5. Suppose that $\mathscr{T}$ contains a flexible color. Is $\mathscr{T}$ finitely representable?

A general investigation of triplet-families $\mathscr{T}$ with $\mathscr{T}_i = \emptyset$ for some $i \in \{1,2,3\}$ was done by Tuza [T]. The case of $i = 3$ (i.e., when 3-colored triangles are excluded) is well-understood: All associative, representable, and finitely representable families are characterized. Also, all the cases when the representations are unique (or to the contrary, when there are infinitely many non-isomorphic ones) were determined in [T].

There are two interesting aspects of those representation theorems. First, they provide a method to find explicit constructions of relation algebras which are associative but not representable, and those which are representable but only over an infinite set. For instance, the simplest non-representable but associative one has 4 atoms (including identity) and corresponds to the triangle family $\mathscr{T}(2, 3)$.

Second, by those characterizations, associativity or (finite) representability can be checked by fast altgorihms of at most $O(|\mathscr{T}| + t^2)$ steps whenever $\mathscr{T}_3 = \emptyset$. This running time is surprisingly short, taking into account that associativity itself imposes a requirement for each *pair* of triples, i.e. its check might be quadratic in $|\mathscr{T}|$ (and $|\mathscr{T}|$ can grow as fast as t^3). For $\mathscr{T}_i = \emptyset$, $i = 1$ or 2, the results (again in [T]) are not equally efficient, but representability still remains decidable. Concerning those results, the following problems arise when no restriction is put on $\mathscr{T}$.

Problem 6. How many steps are needed to check associativity?

Of course, $O(t \,.\, |\mathscr{T}|^2) \leqq O(t^7)$ is a trivial upper bound, just by considering all possible packings of triplets of $\mathscr{T}$.

Problem 7. **(a)** Is finite representability decidable?
(b) Is representability decidable?

In particular, it would be of great interest to see a finite algorithm (if there is any) that decides the existence of infinite representations. We note that, by Ramsey's theorem [R], a family with $\mathscr{T}_1 = \emptyset$ is representable if and only if it is finitely representable. For another reason (by its connections with block designs), $\mathscr{T}_2 = \emptyset$ also implies that all representations of $\mathscr{T}$ are finite. A further related result of [T] states that if each pair of colors is supposed to occur in precisely one triplet of $\mathscr{T}$ then associativity, representability, and finite representability are equivalent. (If $\mathscr{T}$ is representable then each pair of colors occurs in at least one of its triplets.)

4. Further Directions

We close this note with two less explicit problems. For the first one, let us recall that associativity can be viewed as 'local' consistency of a family of triplets, while representability means 'global consistency' in a very strong sense. In this context it is quite natural to raise the following question.:

Which algebraic properties correspond to the consistency of a (given) bounded number of triplets

If some results of this kind were available, they would also provide a natural hierarchy among those algebraic properties.

Another class of problems arises when, instead of algebras of binary relations, one considers algebras of relations of higher ranks (called cylindric algebras, see [HMT]). Then one can ask:

Can some of the results concerning representations of relation algebras be extended to cylindric algebras?

Beside similarities between those two types of algebras – cf. e.g. [HMT, Part II, §5.3] and [AJN] – there are some unexpected differences between them that might cause difficulties, see [AN].

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