Charles Stegall

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More Facts about Conjugate Banach Spaces with the Radon-Nikodym Property

C. STEGALL*)

Austria

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We generalize some results given in [F] which, in turn, answers a question of K. Musiał. A Banach space that is an Asplund space and the unit ball of its dual is a Corson compact in the weak* topology is (hereditarily) weakly compactly generated and has an equivalent Fréchet differentiable norm. A compact space that is simultaneously a Corson compact and (the countable union of compacta) a compact with the Radon Nikodym property is (a Talagrand) an Eberlein compact.

Introduction

A routine consequence of the results of [F] and the interpolation method as used in [S8] is the following result: if a compact space K is a Gul'ko compact and is such that $C(K)$ is generated by an equimeasurable set (such spaces are sometimes called Radon Nikodym compacta) then K is an Eberlein compact. From this it follows that an example given in [T1] (see also [Pol]) is not a Radon Nikodym compact. This has also been observed for rather different reasons by others[1]. Except for the details of constructing Fréchet differentiable norms, which can be found in [F] and its references, this paper can be read independently of [F]. We say that a Banach space Y is an Asplund space if its dual space Y^* has the Radon-Nikodym property. This is equivalent to Z^* being norm separable for any separable linear subspace Z of Y. For references, start with [DU]. The basic point here is that a Banach space that is an Asplund space and the unit ball of its dual is a Corson compact in the weak* topology is weakly compactly generated. The proof of this is only combinatorics of previously known results. The fundamental result, due to Gul'ko, is that a Gul'ko compact is a Corson compact (see [Gu] and [S4] for a differerent point of view).

*) Institut für Mathematik, Johannes Kepler Universität, Altenbergerstrasse, A-4040 Linz, Austria

[1]) See the note at the end of this paper.

Results

We use the following concept.

Definition. *Let $\Phi: X \to \wp(Y)$ be a multivalued map from the metric space X to the subspaces of the metric space Y. A near selection for Φ is a sequence of continuous mappings $\{\phi_n: X \to Y\}$ such that* $\lim \phi_n(x)$ *exists for all $x \in X$ and*

$$\textit{distance}\,(\lim \phi_n(x), \Phi(x)) = 0\,.$$

See [S3] for a direct and easy proof of the following.

Theorem. [JR] *If X is any complete metric space and Y is an Asplund space then any $\Phi: X \to \wp(Y^*)$ that is upper semicontinuous and compact valued with respect to the weak* topology has a near selection.*

If K is a compact space, then $\Phi_K: C(K) \to \wp(K)$ denotes the support mapping, also known as the subdifferential. We use the theorem above in the following case, which depends on [S2]: suppose that X is an Asplund space then there exists a sequence $\{\phi_n: n \in \mathcal{N}\}$ of norm continuous functions $\phi_n: X \to B_{X^*}(0, 1)$ (the unit ball of X^*) such that $\lim_n \phi_n(x) = \phi(x)$ exists in norm for all $x \in X$ and $\phi(x)(x) = \|x\|$ for all $x \in X$. The converse is true also as has been known for some time. In [R] Rodé observed, as a straightforward consequence of some hard results of R. C. James (an outline of the proof of the part that we need is incorporated in the next Proposition), the following: if X is a separable Banach space and $S \subseteq B_{X^*}(0, 1)$ is norm separable, norm closed, convex and for every $x \in X$ there exists an $x^* \in S$ such that $x^*(x) = \|x\|$ then $S = B_{X^*}(0, 1)$. The only complete proof of this result including the results of James with which we are familiar is in [S1]. Suppose such a sequence $\{\phi_n: n \in \mathcal{N}\}$ exists. Something resembling the next Proposition can be found in [R], [S2] and [FG]. It is somewhat stronger than we require.

Lemma. *Let Y be an Asplund space and let K be a weak* compact and convex subset of Y^*. Let $\{\phi_n: Y \to Y^*\}$ be any near selection for*

$$\Phi_K(y) = \{y^* \in K: y^*(y) = \sup_K y\}$$

and let $\phi(y) = \lim_n \phi_n(y)$. Let S be the smallest norm closed and convex set containing $\{\phi(y): y \in Y\}$. Then, $S = K$.

Proof. Suppose that $x_0^* \in K \setminus S$. Choose $x^{**} \in X^{**}$ and $\delta > 0$ so that

$$\sup_S x^{**} + \delta < \langle x^{**}, x_0^* \rangle\,.$$

Recall the theorem of Goldstine: if $x^{**} \in X^{**}$ and $K \subseteq X^*$ is norm compact then for any $\varepsilon > 0$ there exists $x \in X$ so that $\|x\| \leqq \|x^{**}\|$ and

$$|\langle x^{**} - x, x^* \rangle| < \varepsilon$$

for all $x^* \in K$. Construct a sequence $\{x_n: n \in \mathcal{N}\}$, $\|x_n\| \leqq \|x^{**}\|$, in the following way: let x_1 be such that $\|x_1\| \leqq \|x^{**}\|$ and $|\langle x^{**} - x_1, x_0^*\rangle| < 1$ and having the chosen $\{x_1, \ldots, x_n\}$ choose x_{n+1} so that $|\langle x^{**} - x_{n+1}, x^*\rangle| < 1/(n+1)$ for all x^* in

$$\{x_0^*\} \cup \Big(\bigcup_{j \leqq n} \phi_j(A_n \cap B_X(0, n))\Big)$$

where $A_n = [\{x_1, \ldots, x_n\}]$. Since every element of $Z = [\{x_n: n \in \mathcal{N}\}]$ attains its supremum on S, the results of James (see [S1]) say that there exists an increasing sequence of integers $\{m_p\}$ and non negative scalars $\{\lambda_n\}$ such that

$$\sum_{m_p \leqq n < m_{p+1}} \lambda_n = 1$$

for all p and if

$$y_p = \sum_{m_p \leqq n < m_{p+1}} \lambda_n x_n$$

then $\sup_S y_p < \sup_S \limsup x_n + \delta$. Clearly,

$$\langle x^{**}, x_0^*\rangle = \lim_p \langle y_p, x_0^*\rangle \leqq \limsup_p (\sup_S y_p) \leqq \sup_S \limsup x_n + \delta < \langle x^{**}, x_0^*\rangle$$

which is nonsense.

If Y is an Asplund space, $K = B(Y^*)$, $I: Y \to C(K)$ is the canonical operator and $\{\phi_n: C(K) \to Y^*\}$ any near selection for $I^* \circ \Phi_K$ we denote $E^\#$ for a subspace E of Y the smallest norm closed subspace of Y^* containing

$$\bigcup_{x \in E} \bigcup_n \phi_n(I(x)).$$

Usually, we suppress the I and K in this case. The following are special cases of the Lemma above.

To our knowledge, the clever idea of using the results of [JR] in connection with long strings of projections first appears in [F].

Proposition. *If Y is an Asplund space then $Y^\# = Y^*$.*

Proposition. *Let Y be an Asplund space, $T: Y \to C(K)$ any operator, and let $\{\phi_n: C(K) \to Y^*\}$ be any near selection for $T^* \circ \Phi_K$. Define S to be the smallest norm closed subset of Y^* that contains*

$$\bigcup_n \phi_n(C(K)).$$

Then, $T^*(K) \subseteq S$.

Define the following:

$$\mathbf{Y} = \{Z \subseteq Y: Z \text{ is normed closed, linear, and } Z \text{ norms } Z^\#\}.$$

Observe that if $Y \in \mathbf{Y}$ then restriction carries $Y^\#$ isometrically onto Y^*. For each Z in $\mathbf{Y}$, this canonically defines a contractive projection P of Y^* onto $Z^\#$ by $P(x^*)$

is the unique element of $Z^\#$ such that $(P(x^*) - x^*) \mid Z = 0$. Also, if $\mathbf{F}$ is an increasing filter in $\mathbf{Y}$ then

$$\overline{\bigcup_{Z \in \mathbf{F}} Y} \in \mathbf{Y}\,.$$

If T is a subset of a Banach space, denote by $d(T)$ the norm density of T.

Proposition. *Let Y be an Asplund space, $U \subseteq Y$ and $V \subseteq Y^*$ be infinite sets such that $d(U) = d(V)$. Then there exists $Z \subseteq Y$ such that $d(Z) = d(U) = d(V)$, $U \subseteq Z$, $V \subseteq Z^\#$ and Z norms $Z^\#$, hence, $Z \in \mathbf{Y}$.*

Proof. Choose $Y_0 \subseteq X$ such that $Y_0^\#$ contains V, $d(Y_0) = d(U)$ (see above) and we also assume that $U \subseteq Y_0$ and Y_0 norms $[V]$. Having chosen linear spaces $Y_0 \subseteq \ldots \ldots \subseteq Y_n$ with $d(Y_n) = \ldots = d(Y_0) = d(U)$ choose $Y_n \subseteq Y_{n+1}$ so that $d(U) = d(Y_{n+1})$ and Y_{n+1} norms $Y_n^\#$. Let $Z = \overline{\bigcup_n Y_n}$. It is easy to see that $Z^\# = \overline{\bigcup_n Y_n^\#}$ and it follows that $Z \in \mathbf{Y}$ and $d(Y) = d(U) = d(V)$.

The two Propositions above allow the construction of a projectional resolution of the identity for Y^* (see [FG] for this case and [AL] and [S4] for the canonical case). Suppose that $[\omega, \gamma]$ is an ordinal interval and $\{Y_\alpha \colon \omega \leqq \alpha \leqq \gamma\}$ is an increasing family in $\mathbf{Y}$ such that $Y_\gamma = Y$, $d(Y_\alpha) \leqq \alpha$ and for any limit ordinal $\beta \in [\omega, \gamma]$ we have that

$$Y_\beta = \overline{\bigcup_{\omega \leqq \alpha < \beta} Y_\alpha}\,.$$

We make a pair of absolutely trivial remarks concerning this decomposition. Suppose Q_α is the projection of Y^* onto $Y_\alpha^\#$ as defined above. Because the kernels and images have the proper order, these projections commute. Then for any $y^* \in Y^*$ and for any $\varepsilon > 0$ we have that

$$I = \{\alpha \colon \|(Q_{\alpha+1} - Q_\alpha)\, y^*\| \geqq \varepsilon\}$$

is finite. Suppose that $\{\alpha_n\} \subseteq I$ is strictly increasing. Let β be the supremum. Then, with the closure taken in the norm topology,

$$Q_\beta(Y^*) = Y_\beta^\# = \overline{\bigcup_n Y_{\alpha_n}^\#}\,.$$

Choose $x_n^* \in Y_{\alpha_n}^\#$ so that $\{x_\alpha^*\}$ converges to $Q_\beta(x^*)$. It follows that

$$\|x_n^* - Q_{\alpha_n}(x^*)\| \to 0$$

$$\|Q_\beta(x^*) - Q_{\alpha_n}(x^*)\| \to 0$$

$$\|Q_{\alpha_{n+1}}(Q_\beta(x^*) - Q_{\alpha_n}(x^*))\| \to 0$$

which is a contradiction.

Another banal observation is the following: suppose, and just suppose, that we were able to find the family $\{Y_\alpha \colon \omega \leqq \alpha \leqq \gamma\}$ such that each $Y_\alpha^\#$ is weak* closed (equivalently, the projections $\{Q_\alpha\}$ are all weak* continuous). Then, Y is generated by a weakly compact set and the unit ball of Y^* is an Eberlein compact (see [**DFJP**]). This follows trivially by induction on the norm density of Y. If Y is separable there is nothing to do. Suppose that for each α, P_α is a projection on Y such that $P_\alpha^* = Q_\alpha$. For $\alpha > \omega$ choose a weakly compact subset W_α, a subset of the unit ball of $(P_{\alpha+1} - P_\alpha)(Y)$, that generates $(P_{\alpha+1} - P_\alpha)(Y)$; for $\alpha = \omega$, choose a weakly compact, convex and symmetric subset W_ω of $P_\omega(Y)$ that generates $P_\omega(Y)$. We shall show that $W = \bigcup_\alpha W_\alpha$ is weakly compact. Choose any sequence $\{y_n : n \in \mathcal{N}\}$ in W. Since any separable subspace of Y has a norm separable dual, we may, by passing to a subsequence, assume that $\lim y^*(y_n)$ exists for all $y^* \in Y^*$. If $\{y_n\}$ intersects some fixed W_α infinitely often, then it has a weakly converging subsequence. We may suppose that $\{\alpha_n\} \subseteq I$ is a strictly increasing sequence such that

$$(P_{\alpha_{n+1}} - P_{\alpha_n})\, y_n = y_n\,.$$

Thus,

$$\lim y^*(y_n) = y^*((P_{\alpha_{n+1}} - P_{\alpha_n})\, y_n) = ((Q_{\alpha_{n+1}} - Q_{\alpha_n})\, y^*)\,(y_n)\,.$$

But, $\|(Q_{\alpha_{n+1}} - Q_{\alpha_n})\, y^*\| \to 0$ which proves that $\lim y^*(y_n) = 0$ for all $y^* \in Y^*$. Since $0 \in W$, the Eberlein-Smulian theorem says that W is weakly compact. For a fixed non zero $y^* \in Y^*$ choose the minimal α, which cannot be a limit ordinal other than ω, such that $Q_\alpha(y^*) \neq 0$. Either, $Q_\omega(y^*) \neq 0$ or $(Q_\alpha - Q_{\alpha-1})(y^*) = Q_\alpha(y^*) \neq 0$. This proves that y^* does not vanish on W and that Y is weakly compactly generated. This is the case of [S_5] that is correct. Now, to more serious matters.

To show that $Y_\alpha^\#$ is weak* closed it suffices to show that the unit ball of $Y_\alpha^\#$ is weak* compact. This can be done by playing the usual games with the bounded weak* topology [**D**]. We prefer, however, the more pleasant approach given in the first four pages of [**A1**].

Theorem. [**S8**] *The following are equivalent for a compact space K:*

(i) *there exists an Asplund space Y such that K is homeomorphic to a weak* compact subset of Y^*;*

(ii) *there exists an Asplund space Y and an operator $T : Y \to C(K)$ such that the image $T(Y)$ separates the points of K;*

(iii) *there exists an Asplund space Y and an operator $T : Y \to C(K)$ such that the image $T(Y)$ is dense in $C(K)$.*

Definition. [**S8**]*A bounded subset S of a Banach space X is GSP if there exists an Asplund space Y nad an operator $T : Y \to X$ such that $S \subseteq T(B(Y))$.*

Compact spaces that satisfy the conditions of the Theorem above are sometimes called compacta with the Radon-Nikodym property. The canonical examples of such creatures are dispersed compact spaces and Eberlein compacta (see [**S8**]).

Theorem. (*Grothendieck*) *A bounded subset E of $C(K)$ is GSP if and only if for every nice probability measure μ on K and every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subseteq K$ such that $\{f|K_\varepsilon : f \in E\}$ is relatively norm compact in $C(K_\varepsilon)$ and $\mu(K_\varepsilon) >$*

$> 1 - \varepsilon$. *It follows that if a bounded subset E of C(K) is GSP then its closure in the pointwise topology is also GSP.*

Theorem. [S7] *If Y is an Asplund space and K is any weak* compact subset (or, even a subset that is k-analytic in the weak* topology* [**S1**]) *of Y* then the norm density of K is no greater that the weight of K in the weak* topology.*

A related result, in a context more general than the following, can be found in [**H**] (see [**DU**]).

Theorem. (*Stegall, Huff, Morris*) *A Banach space Y is an Asplund space if and only if every norm closed, convex and bounded subset of Y* is the norm closed convex hull of its extreme points.*

Theorem. [**Pol**] *There exists a class $\mathscr{T}$ of topological spaces such that the following are equivalent for a compact space K:*

(i) *K is a Corson compact;*

(ii) *in the simple topology, C(K) is the continuous image of an element of $\mathscr{T}$;*

(iii) *every subspace of C(K) that is closed in the simple topology is the continuous image of an element of $\mathscr{T}$;*

(iv) *there exist $T \in \mathscr{T}$ and $f\colon T \to C(K)$ continuous in the simple topology such that $f(T)$ separates the points of K.*

The proof of the following is contained in the special case of the main theorem below. See [**Di**] and [**F**] about renormings and differentiation.

Theorem. *If Y is an Asplund space and the unit ball of Y* in the weak* topology is a Corson compact, then Y has a projectional resolution of the identity as above with the $Y^{\#}_{\alpha}$ weak* closed. Moreover,*

$$I = \{\alpha\colon \|(Q_{\alpha_{n+1}} - Q_{\alpha_n})\, y^*\| \geqq \varepsilon\}$$

is finite for all y^* in Y and all $\varepsilon > 0$. It follows that every subspace of Y is weakly compactly generated and Y has an equivalent norm that is Fréchet differentiable.*

Let P_Λ denote the Λ-fold product of $[0, 1]$. For each $\lambda \in \Lambda$ define p_λ to be the projection into the λ coordinate. Suppose $K \subseteq P_\Lambda$ is compact and each point in K is countably supported on Λ (the definition of a Corson compact). If $\varrho \subseteq \Lambda$ then we say that ϱ is good if the canonical retraction on P_Λ defined by ϱ leaves K invariant. Clearly, if one has an increasing filter $\mathscr{F}$ of good subsets of Λ then $\bigcup\mathscr{F}$ is also good. If $\{\varrho_n\}$ is increasing, then $\varrho(k) = \varrho(k')$ if and only if $\varrho_n(k) = \varrho_n(k')$ for all $n \in \mathscr{N}$. We shall use ϱ to denote the subset, the retraction on K and the projection defined on $C(K)$ by $\varrho(f) = f \circ \varrho$. A result of Benyamini says that given any infinite $\varrho \subseteq \Lambda$ there exists $\varrho \subseteq \sigma \subseteq \Lambda$ such that σ is good and σ has the same cardinality as ϱ. The continuous image of a Corson compact is a Corson compact (see [**Gu**], [**Pol**] and [**Ne**]).

Main Theorem. *A compact space that is both a Corson compact and a RNP compact is an Eberlein compact.*

Proof of a special case. Let Y be an Asplund space and let K be the unit ball of Y^* in the weak* topology and suppose that it is sitting in P_Λ as a Corson compact. Since Y separates the points of K we may approximate each p_λ as a function on K by a polynomial in any dense subset of Y together with the constants. We may assume that the cardinality of Λ is no greater than $d(Y)$. Fix a cardinal τ smaller than the cardinality of Λ. Fix a closed and linear subspace Z_0 so that $d(Z_0) = \tau$ and fix a subset τ_0 of Λ that has cardinality no greater than τ. A pair (R, ϱ) is valid if $d(R) \leqq \tau$, $\varrho \subseteq \Lambda$ is good, has cardinality no greater than τ, $Z_0 \subseteq R$ and $\tau_0 \subseteq \tau$. We shall designate a valid pair (R, ϱ) as beautiful if it fulfills the following, perhaps redundant, conditions:

(i) $R^\# \cap B(Y^*) \subseteq \varrho(K)$;
(ii) ϱ leaves R invariant;
(iii) R norms $[\varrho(K)]$;
(iv) R generates the algebra $\varrho(C(K))$.

The point of the proof is to verify that if (R, ϱ) is a valid pair then $R^\#$ is weak* closed. There are two parts to the proof. One part of this is to prove that if $\{(R_\xi, \varrho_\xi)\}$ is an increasing (coordinatewise) family of beautiful valid pairs then

$$\{(\overline{\bigcup_\xi R_\xi}, \bigcup_\xi \varrho_\xi)\}$$

is also very beautiful. The other part is to show that if we begin with a valid pair (R, ϱ) then we may construct a larger valid and beautiful pair. With these two parts, the construction of a projectional resolution of the identity follows in exactly the usual way [AL]. We begin with a valid pair (R, ϱ) and we construct a larger valid pair in the following way. Choose $D \subseteq R$ norm dense and for each $x \in D$ choose a countable subset ϱ_x of Λ such that x is in the algebra generated by $\{p_\lambda: \lambda \in \varrho_x\}$. For each $n \in \mathcal{N}$ and each $x \in D$ choose $\sigma_{n,x}$ countable so that $\sigma_{n,x}$ is the support of $\phi_n(x)$. Choose a good σ containing

$$\varrho \cap \left(\bigcup_{x \in D} \varrho_x\right) \cap \left(\bigcup_n \bigcup_{k \in D} \sigma_{n,x}\right)$$

so that (R, σ) is valid. Now, choose $R \subseteq S$ so that S norms $\sigma(K)$, the algebra generated by S (always understood, and the constants) contains $\sigma(C(K))$ and $S^\#$ contains $\varrho(K)$. This is possible because the weight of $\sigma(K)$ in the weak* topology is no more than τ and, because of the Radon-Nikodym property, the norm density of $[\sigma(K)]$ is also no more than τ. We say that (S, σ) extends (R, ϱ). Start with (Z_0, ϱ_0) and let (Z_1, σ_1) be an extension. Let (Z_{n+1}, σ_{n+1}) extend (Z_n, σ_n) and let $Z = \overline{\bigcup_n Z_n}$ and $\sigma = \bigcup_n \sigma_n$. Observe that

$$Z_n^\# \cap B(Y^*) \subseteq \sigma_{n+1}(K) \subseteq Z_{n+2}^\#$$

and σ_{n+1} is invariant on Z_n (which means that $z \circ \sigma_{n+1} = \sigma_{n+1}(z) = z$ for $z \in Z_n$). By continuity, σ is invariant on Z, and because σ is multiplicative, also invariant on the algebra A generated by Z. By the construction, A also contains $\sigma_n(C(K))$

for each $n \in \mathcal{N}$. By the remarks above, we have that $A = \sigma(C(K))$. This means that if i: $Z \to Y$ denotes injection then i^* is one to one on $\sigma(K)$. But,

$$\sigma(K) = \overline{\bigcup_n \sigma_n(K)^*} \supseteq \bigcup_n Z_n^\# \cap B(Y^*) \supseteq \bigcup_n \sigma_n(K) .$$

This proves that $\sigma(K)$ is convex and symmetric and we already know that i^* is one to one on $\sigma(K)$. Thus, $V = \bigcup_n n\,\sigma(K)$ is a weak* closed subspace of Y^* such that $Z_n^\# \subseteq V$ for all n, thus $Z^\# \subseteq V$, and $V \cap Z^\perp = \{0\}$. This means that $V = Z^\#$ and the projection defined above is weak* continuous. The general case can be proved by a bit more omplicated version of this process, which we leave to the reader. We prefer, however, the following.

Proof of the general case. This is rather delicate. Firstly, suppose that K is a Corson compact and S is a GSP subset of $C(K)$ that separates the points of K. It follows from [**S1**] and [**S8**] that we may also assume that S is a subset of the unit ball, is closed, convex, symmetric and $\bigcup_n nS$ is norm dense in $C(K)$. We may also assume that S is closed in the simple topology. Let Y be an interpolation space for S (see [**BL**], [**DFJP**] and [**S8**] for this case). That is, there exists a Banach space Y and an operator T: $Y \to C(K)$ with the following properties:

(i) T^{**} is one to one;
(ii) $S \subseteq T(BY))$;
(iii) $[T^{-1}(S)] = Y$;
(iv) Y is an Asplund space.

There exists a space H in Pol's class and a function h: $H \to S$ that is onto and continuous in the simple topology. We show that $T^{-1} \circ h$: $H \to Y$ is weakly continuous. Since $T^*(CK))$ is norm dense in Y^* we need only show that each $T^*(\mu) \circ T^{-1} \circ h$ is continuous for a probability measure μ in $C(K)^*$. A fundamental result of the Radon Nikodym property in Banach spaces is that $T^*(\mu)$ may be approximated in norm by a finite combination of the extreme points of $T^*(C(K)^*)$, but the extreme points are continuous. Thus, $T^{-1} \circ h : H \to Y$ is weakly continuous and $[T^{-1} \circ h(H)] = Y$. If L is the unit ball of Y^* then the canonical operator from Y to $C(L)$ is, by definition, weak to simple continuous. From Pol's criterion, we know that L is a Corson compact. From the special case, we know that Y is weakly compactly generated and it follows that $C(K)$ is also weakly compactly generated and K is an Eberlein compact.

A Talagrand compact K is a compact space such that $C(K)$ is k-analytic in the weak topology. A Corson compact that is the countable union of Eberlein compacta is a Talagrand compact [**So**], but not necessarily an Eberlein compact [**T1**]. It follows that a Corson compact that is the countable union of Radon-Nikodym compacta is a Talagrand compact. Another immediate consequence of the above is that the example given in [**T1**] has a fragmenting metric (using another vocabulary, this was essentially proved in [**Gr**], see also [**S9**]) but does not have a lsc fragmenting metric (see [**Na**] for facts about this). A compact space K is dual non l_1 if there exists a Banach space X not containing l_1 and K is homeomorphic to a subset of X^* where, of course, X^* has the weak* topology. Obviously, a Radon-Nikodym compact is dual non l_1. We shall use a great deal of the very large literature about Banach spaces not containing l_1 (beginning with **[Ro]**, **[H]** and **[S1]** for more; also **[S10]**). Let P_Λ be as above. Suppose that K is a Corson compact subset of P_Λ. We say that K is solid if for any t in K and any s is in P_Λ such that $s(\lambda) \leqq t(\lambda)$ for all λ then $s \in K$. Any Corson compact is contained in a minimal solid Corson compact. A Gul'ko compact is contained in a solid Gul'ko compact **[So]**. There exist some very exotic Corson compacta; it would be very interesting to know the examples in **[To]** and the examples of Haydon, Talagrand and Kunen (see **[Ne]**) fit in with the results given here.

Theorem. *A compact space K that is both a solid Corson compact and dual non l_1 is a Gul'ko (Vasak) compact.*

Proof. We may assume that there exists a closed, bounded, convex and symmetric subset A of $C(K)$ that does not contain a l_1 basic sequence and $[A] = C(K)$. Assume that K sits in P_Λ as a solid Corson compact. For each $\lambda \in \Lambda$ define p_λ to be the projection into the λ coordinate. For each n and m define

$$\Lambda_{n,m} = \{p_\lambda\colon \text{distance } (p_\lambda, mA) \leqq 2^{-n}\}\,.$$

Let $M \subseteq \mathcal{N}^{\mathcal{N}}$ be such that $\zeta \in M$ if and only if

$$\bigcap_n \Lambda_{n,\zeta(n)} \neq \emptyset\,.$$

Define $\Phi\colon M \to \wp(C(K))$ by $\Phi(\zeta) = \bigcap_n \Lambda_{n,\zeta(n)}$. Fix $\zeta \in M$ and choose ζ_n in M converging to ζ. We may assume

$$\zeta_n(i) = \zeta(i) \quad \text{for all} \quad i \leqq n\,.$$

Choose arbitrarily $r_n \in \Lambda_{n,\zeta(n)}$. We shall show that $\{r_n\}$ has a subsequence that converges weakly to the origin. Firstly, we show that $\{r_n\}$ has a weakly Cauchy subsequence. If not, then it has a subsequence $\{y_m\}$ that is a l_1 basic sequence **[Ro]**. Well known computations (see, for example **[S1]**) yield a $\delta > 0$ such that if $\{z_m\}$ is any sequence such that $\|z_m - y_m\| < \delta$ then $\{z_m\}$ is also a l_1 basic sequence. Choose n sufficiently large so that $2^{-n} < \delta$. There exists p such that if $m \geqq p$ then

$$\text{distance } (y_m, \zeta(n)\,A) < 2^{-n}\,.$$

This means that $\zeta(n)\,A$ contains a l_1 basic sequence which is impossible. As a matter of notational convenience, assume that $\{r_n\}$ is weakly Cauchy. Assume that there exists $t \in K$ so that $\lim r_n(t) \neq 0$. Since K is solid choose $s \in K$ so that s has the same value as t in the r_n coordinate if n is even and is zero in all other coordinates. Clearly, $\lim r_n(s)$ does not exist. This shows that $\{r_n\}$ has a subsequence that converges weakly to zero. Although we shall not repeat the details here (see **[V]**,**[T1]** and [S9]), this argument with the Eberlein-Smulian theorem shows that we may extend Φ to a mutivalued map $\Psi: M \to \wp(C(K))$ that is upper semicontinuous and compact valued in the simple topology. Also, $\Psi(M)$ contains $\{r_\lambda: \lambda \in \Lambda\}$ and separates the points of K. This extension may be defined in the following way:

$$\Psi(\zeta) = \bigcap_n \overline{cs}\Big(\bigcup_{\text{distance}(\eta,\zeta) < 1/n} \Phi(\eta)\Big)$$

where $\overline{cs}(E)$ denotes the smallest norm closed, convex and symmetric set containing E. This means that K is a Gul'ko (Vasak) compact (see [S4]).

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Note. *Since the submission of this mss. for publication we have obtained (June, 1989) more precise information about the references alluded to. We have been indirectly informed that some time ago Reznicenko constructed a compact space that is a Talagrand compact that is not an Eberlein compact and explicitly demonstrated that his example is not a RN compact. In February, 1989, Orihuela and Schachermayer presented us with computations necessary to show that Talagrand's own example* [T1] *is not a RN compact. In response, we immediately reciprocated their generosity by showing how the general result, any Gul'ko compact that is also a RN compact must indeed be an Eberlein compact, follows from* [F] *and an interpolation technique. Also, in February, 1989, we provide Orihuela and Schachermayer with the notes on which this, our present paper, is based. We received, also in June, 1989, the preprint "Every Radon-Nikodym Corson compact space is Eberlein compact", by Orihuela, Schachermayer and Valdivia, which contains the computations mentioned above as well as their own thoughts about related matters. We acknowledge the expression of gratitude to us "Every Radon-Nikodym Corson compact space is Eberlein compact" for discussing these matters with them and providing them with a copy of our notes in February, 1989. We express our gratitude to several audiences in Spain in January, 1988, for patiently listening to our thoughts on long strings of projections, part of which are incorporated into this paper. The question at the end of "Every Radon-Nikodym Corson compact space is Eberlein compact" is stated rhetorically at the end of* [S8]. *Any closed, bounded and convex subset of $L_1(\mu)$, μ a finite measure (hence $L_1(\mu)$ is weakly compactly generated), that is GSP (indeed, that does not contain a l_1 basis) is weakly compact; this is a classic result of Dunford and Pettis. Thus, the space constructed by Rosenthal, a subspace of some $L_1(\mu)$, is not GSG (indeed, not non l_1 generated). References and abundant detail can be found in* [S8]. *A different order of gratitude is due to Michael Bourvier, Dr. med., the doctors, the ordained and secular nursing sisters of St. Josef Krankehaus of Braunau am Inn, particularly the accident station, for their care and advice from March to June of 1989.*