Acta Universitatis Carolinae. Mathematica et Physica

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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 31 (1990), No. 2, 101--104

Persistent URL: `http://dml.cz/dmlcz/701960`

Completely Normal Locales

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Received 11 March 1990

In the theory of locales (or 'pointless topologies'), several authors have tried to find a suitable form of separation axioms. The goal of this paper is to investigate a kind of normality for locales, namely the category of completely normal locales. Conjunctive completely normal locales have the following properties:

1. They are closed with respect to sublocales but are not closed with respect to products.
2. They coincide for topological spaces with usual completely normal spaces.
3. The smallest epireflective subcategory of locales containing all normal topological spaces is the category of all completely regular spaces.

All unexplained facts concerning locales can be found in Johnstone [5]. Recall that a *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$$

holds for all $a \in L$, $S \subseteq L$.

A frame homomorphism is a map preserving finite meets and arbitrary joins. If T is a topological space, then the lattice $O(T)$ of all open sets of T is a frame. Let *Frm* be the category of frames. The known facts (see [5]) indicate the importance of the opposite category $Loc = Frm^{op}$. Objects of *Loc* are called *locales*. Of course, sublocales correspond to factorframes and products of locales correspond to sums of frames. Locales isomorphic to some $O(T)$ are called *spatial*. We will restrict our considerations to sober topological spaces in this paper, because the category of sober spaces is isomorphic to the category of spatial locales.

If X is a subcategory of *Loc* then $S(X)$ will denote the subcategory of all sublocales of objects of X and $P(X)$ the subcategory of all products of objects of X. $\hat{X} = S\,P(X)$ is the smallest epireflective subcategory of *Loc* containing X (see [6], p. 82), i.e., $\hat{X}$ is the epireflective hull of X. For example, the Banaschewski-Mulvey compactification for completely regular locales shows that the category of all completely regular locales is the epireflective hull of all completely regular spaces (see [1]).

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Simmons [7], Dowker and Strauss [4] introduce the notion of normality on locales in the following way:

A locale L is called *normal* if L has the following property: $a, b \in L$, $a \vee b = 1 \Rightarrow$ there exists $l \in L$ such that $a \vee l^* = b \vee l = 1$.

Their conception is an extension of the usual normality of topological spaces on locales. The normality of spaces is not closed with respect to subspaces, factorspaces and products. Any normal locale is a homomorphic image of a suitable normal spatial locale. Namely, a locale L is normal iff its locale $Id(L)$ of all ideals in L is normal spatial locale (see [5], p. 69). The corresponding surjective homomorphism $\sigma: Id(L) \to L$ has the form $\sigma(I) = \bigvee I$ for any $I \in Id(L)$.

Normal locales are closed with respect to codense sublocales. Locales which are normal and closed with respect to all sublocales are introduced in the following definition.

1. Definition. A locale L is *completely normal* if L has the following property (N^*): For any $a, b \in L$ there exists $l \in L$ such that $a \leqq b \vee l$, $b \leqq a \vee l^*$.

Remark. Simmons [7] gives the equivalent definition of completely normal locales: For any $a, b \in L$ there exist $x, y \in L$ such that $x \wedge y = 0$, $x \leqq b \leqq a \vee x$, $y \leqq a \leqq b \vee y$.

2. Proposition. Let L be a locale. Then the following assertions are equivalent:

1. L is a completely normal locale.
2. All sublocales of L are normal.
3. For any $a, b \in L$ there exists $l \in L$ such that $a \vee b = (a \wedge b) \vee (a \wedge l) \vee (b \wedge l^*)$.
4. For any $a, b \in L$ there exist $m, n \in L$ such that $m \leqq b$, $n \leqq a$, $m \wedge n = 0$, $a \vee b = (a \vee m) \wedge (b \vee n)$.

Proof. $1 \Rightarrow 3$: We have $a \vee b = (a \vee l^*) \wedge (b \vee l) \wedge (a \vee b) = (a \wedge b) \vee (a \wedge l) \vee (b \wedge l^*)$.

$3 \Rightarrow 4$: If we choose $m = b \wedge l^*$ and $n = a \wedge l$ then $(a \vee m) \wedge (b \vee n) = (a \vee (b \wedge l^*)) \wedge (b \vee (a \wedge l)) = (a \vee b) \wedge (a \vee l^*) \wedge (b \vee l) = a \vee b$.

$4 \Rightarrow 1$ is evident.

$1 \Rightarrow 2$. If $h: L \to M$ is a surjective homomorphism of frames and $a, b \in H$ then elements $c, d \in L$ exist such that $a = h(c)$, $b = h(d)$. Now, we have a suitable $l \in L$ with the property $c \leqq d \vee l$, $d \leqq c \vee l^*$, i.e., $a \leqq b \vee h(l)$, $b \leqq a \vee h(l^*) \leqq a \vee (h(l))^*$.

$2 \Rightarrow 1$. The map $h: L \to L$ such that $h(x) = (a \vee b) \wedge x$ for given $a, b \in L$ and any $x \in L$ is a homomorphism of frames. The fact that $h(L)$ is a normal subframe of L implies an existence of $l \in L$ such that $a \vee h(l) = 1 = b \vee h(l)^*$ holds in $h(L)$. Thus $a \vee b = a \vee [(a \vee b) \wedge l] = (a \vee b) \wedge (a \vee l) \leqq a \vee l$, i.e. $b \leqq a \vee l$. Further, we have $h(l)^* = (a \vee b) \wedge l^*$ in $h(L)$, because $[(a \vee b) \wedge l^*] \wedge [(a \vee b) \wedge l] = 0$ and $(a \vee b) \wedge x \leqq l^* \wedge (a \vee b)$ holds for any $x \in L$ with the property $0 = [(a \vee b) \wedge l] \wedge [(a \vee b) \wedge x] = l \wedge [(a \vee b) \wedge x]$. The consequence is that $a \leqq a \vee b = b \vee h(l)^*_{h(1)} = b \vee [(a \vee b) \wedge l^*] = (a \vee b) \wedge \wedge (b \vee l^*) \leqq b \vee l^*$. Hence L is completely normal.

Remarks. 1. E. Čech [2] defines a completely normal topological space T such that any space imbedded into T is normal. This conception is equivalent with (N^*) on spatial locales.

2. Recall that completely normal spaces are normal. However, if T is an infinite topological space which is not countable compact then its Stone-Čech compactification is normal but not completely normal (see [2], 12.3.5).

3. A Boolean algebra is a completely normal conjunctive locale. A chain is a completely normal locale which is not conjunctive. (Recall that L is a *conjunctive locale* if L has the following property: If $a, b \in L$, $a \nleqq b$ then $c \in L$ exists such that $c \neq 1$, $a \vee c = 1$ $c \geqq b$).

3. Proposition. A normal conjunctive locale is completely regular.

Proof. If L is a normal conjunctive locale and $a, b \in L$, $a \nleqq b$ then $c \in L$ exists such that $a \vee c = 1$, $1 \neq c \geqq b$ and also $l \in L$ exists such that $a \vee l^* = 1$, $c \vee l = = 1$. Further, $h \in L$ exists such that $a \vee h^* = 1$, $l^* \vee h = 1$. Finally, for any $a, b \in L$ there exist elements $l, h \in L$ with properties $a \vee l^* = 1$ and $a \vee h^* = 1 = l^* \vee h$. It means (see [1]) that L is a completely regular locale.

The Proposition 3 implies that sublocales of a normal conjunctive locale are again conjunctive and therefore we shall investigate the category N^* of all completely normal conjunctive locales.

4. Theorem. Let N^*, CR, $NTop$, $NTop^*$ and Top are the categories of all completely normal conjunctive locales, completely regular locales, normal top. spaces, completely normal top. spaces and top. spaces, successively. Then the following assertions hold:

1. $S(N^*) = N^*$.
2. $P(N^*) \nsubseteq N^*$.
3. $N^* \cap Top = NTop^*$.
4. $\widehat{NTop} = \widehat{NTop^*} = CR$.

Proof. 1. Let $K, L \in Frm$, $f: K \to L$ be a surjective frame homomorphism, K be a completely normal frame and let us prove that L has the same property: If $a, b \in L$ then elements $c, d \in K$ exist such that $f(c) = a$, $f(d) = b$. Further, an element $k \in K$ exists such that $c \leqq d \vee k$, $d \leqq c \vee k^*$. Hence $a \leqq b \vee f(k)$, $b \leqq a \vee \vee f(k^*) \leqq a \vee [f(k)]^*$. L is completely normal and Prop. 2 implies conjunctivity of L.

2. Evidently $O(R)$ is a completely normal frame for the usual topological space R of real numbers. Let us suppose that the sum $\Sigma\{L_\alpha: \alpha \in \omega_1\}$ is a completely normal frame, where $L_\alpha = O(R)$ for all $\alpha \in \omega_1$. Consider the surjective homomorphism

$\Sigma\{L_\alpha: \alpha \in \omega_1\} \to O(R^{\omega_1})$. Then $O(R^{\omega_1})$ is a completely normal frame and R is countable compact space (see [8], 21.C.4), a contradiction.

3. The Proposition follows from the fact that completely normal spaces are exactly topological spaces fulfilling 2, 4.

4. Clearly $CR = \widehat{CR} \supseteq \widehat{NTop} \supseteq \widehat{NTop^*}$. If $L \in CR$ then its compactification $\beta(L)$ is a compact completely regular spatial locale (see [1]). Hence $\beta(L)$ is isomorphic to $O(T)$ for a suitable Tykhonov space T. Further, T is homeomorphic to a subspace of a Tykhonov cube $\langle 0, 1\rangle^I$ for a suitable set I (see [3], 7.1.50). If we put $K_\alpha = O(\langle 0, 1\rangle)$ for $\alpha \in I$ then we obtain $\Sigma\{K_\alpha: \alpha \in I\} \to_f O(\langle 0, 1\rangle^I) \to_h \beta(L) \to_g$ $\to_g L$, where f is the natural surjective homomorphism, h is the homomorphism corresponding to the subspace of the Tykhonov cube which is homeomorphic to T and g is the natural surjective homomorphism corresponding to the compactification.

Finally, $\langle 0,1\rangle$ is a completely normal T_1-space. Hence $L \in \widehat{NTop^*}$ and $CR \subseteq \widehat{NTop^*}$.

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