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A Note on Extremally Disconnected Frames

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Some characterizations of extremally disconnected frames by utilizing preopen and semi-preopen nuclei are given.

§ 0. Introduction

This paper presents the results of an investigation done on the notion of extremal disconnectedness in the context of pointless topologies – frames. The paper is divided into four sections. In Section 1, we shall recall some basic facts concerning frames. In Section 2, we shall show some basic equivalents of extremal disconnectedness. In Section 3, we shall develop the machinery for computing with semi-open and preopen nuclei. In Section 4, we shall close the paper with several characterizations of extremally disconnected frames by utilizing preopen and semi-preopen nuclei. The results obtained here are closely related to the work of Jankovic [1], Noiri [3] and Sivaraj [4]. For basics concerning frames see Johnstone [2].

§ 1. Basic facts

A *frame* is defined to be a complete lattice L(with the top element 1 and the bottom element 0) which satisfies the infinite distributive law

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i)$$

for every $x \in L$ and every subset $\{x_i\}_{i \in I}$ of L. Frames can be viewed as generalized topological spaces. Frames which are isomorphic to the frame $O(X)$ of all open subsets of a suitable topological space X are called *topologies* or *spatial frames*.

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However, there are many frames which are not topologies. For example, if we put $L = RO(\mathbf{R})$ to be the lattice of all regular open subsets of real line then L is a non-atomic complete Boolean algebra which is not isomorphic to any topology. Recall that a complete lattice L is a frame iff it is a Heyting algebra – a lattice L is said to be a *Heyting algebra* if, for each pair of elements $a, b \in L$, there exists an element $a \Rightarrow b$ such that

$$c \leqq a \Rightarrow b \quad \text{iff} \quad c \wedge a \leqq b\,.$$

We put $x^* = x \Rightarrow 0$.

We shall call a map from one frame to another a *frame homomorphism* if it preserves arbitrary joins and finite meets. The category of frames will be denoted by *Frm*. We define a *nucleus* on a frame L to be a map $j: L \to L$ satisfying

(i) $a \leqq j(a)$
(ii) $j(a) = j(j(a))$
(iii) $j(a \wedge b) = j(a) \wedge j(b)$

for all $a, b \in L$.

If j is a nucleus on L, we define

$$L_j = \{a \in L: j(a) = a\}\,.$$

Since $j \circ j = j$, the image of j is precisely L_j. Clearly, L_j is a frame and $j: L \to L_j$ is a frame homomorphism.

Let $S \subseteq L$. Then $S = L_j$ for some nucleus j iff S is

(i) closed under $\bigwedge$
(ii) $a \in L$, $b \in S$ implies $a \Rightarrow b \in S$

It is well known (see [2]) that nuclei for topologies are precisely the subspace inclusions. One can easily check that $j: L \to L$ is a nucleus iff $x \Rightarrow j(y) = j(x) \Rightarrow j(y)$.

We shall denote by $N(L)$ the lattice of all nuclei on a frame L.

$N(L)$ is partially ordered by $j \leqq k$ iff $j(a) \leqq k(a)$ for all $a \in L$. One can easily prove that $N(L)$ is a frame as well.

Let a be an element of a frame L. The maps $c_a, u_a: L \to L$, $c_a(x) = a \vee x$, $u_a(x) = a \Rightarrow x$ are nuclei, which, for topologies, correspond to a closed, open subspace respectively. Nuclei of this form are therefore said to be *closed*, *open* respectively. A nucleus which is both open and closed is said to be *clopen*.

We shall denote by $O(L)$ the lattice of open nuclei, by $C(L)$ the lattice of closed nuclei and by $CO(L)$ the lattice of clopen nuclei. We shall define by Δ, ∇ the bottom and the top element of $N(L)$. It is well known that

$$\bigwedge u_{a_i} = u_{\vee a_i}$$
$$u_a \vee u_b = u_{a \wedge b}$$
$$\bigvee c_{a_i} = c_{\vee a_i}$$
$$c_a \wedge c_b = c_{a \wedge b}$$

for all $a, b, a_i \in L$.

For $j \in N(L)$ we put

$$Int(j) = \bigwedge\{k: k \in O(L),\ j \leqq k\} = u_{j^*(0)}$$

$$Cl(j) = \bigvee\{k: k \in C(L),\ k \leqq j\} = c_{j(0)}\,.$$

Then obviously

$$Int(j) \in O(L)$$

$$Cl(j) \in C(L)$$

$$Int(j)^* = Cl(j^*)$$

$$Int(j) = Int\,(j^{**})$$

$$Cl(j^{**})^* = Int(j^*)\,.$$

We say, that j is *dense* iff $Cl(j) = \Delta$. It is easy to check that j is dense iff $j(a) = 0$ implies $a = 0$ for all $a \in L$.

Another important feature of open and closed nuclei is that

$$c_a \vee j = j \circ c_a$$

$$u_a \vee j = u_a \circ j$$

for all $a \in L$, $j \in N(L)$.

Generally, if g preserves $\Rightarrow$ then $g \vee j = g \circ j$.

§ 2. Extremally disconnected frames

2.1. Definition. A frame L is said to be *extremally disconnected* if the closure of every open nuclei on L is open. Recall that, for topologies, the definition coincides with the usual one.

2.2. Lemma. *Let L be a frame. Then the following conditions are equivalent:*

(i) *L is extremally disconnected.*

(ii) *If $u, v \in O(L)$, $u \vee v = \nabla$ then $Cl(u) \vee Cl(v) = \nabla$.*

(iii) *If $a, b \in L$, $a \wedge b = 0$ then there exist $c, d \in L$ such that $c \vee d = 1$, $c \wedge a = 0$, $d \wedge b = 0$.*

Proof. "(ii) $\Leftrightarrow$ (iii)" It is immediate.

"(i) $\Rightarrow$ (ii)" Let $u, v \in O(L)$, $u \vee v = \nabla$. Then $Cl(u) \vee v = \nabla$ and because $Cl(u)$ is open then $Cl(u) \vee Cl(v) = \nabla$.

"(ii) $\Rightarrow$ (i)" Let $u \in O(L)$. Then

$$u \vee Cl(u)^* = \nabla \text{ i.e.}$$

$$Cl(u) \vee Cl(Cl(u)^*) = \nabla \text{ i.e.}$$

$$Cl(u) \vee Int(Cl(u))^* = \nabla \text{ i.e.}$$
$$Int(Cl(u))^* \geqq Cl(u)^* \text{ i.e.}$$
$$Int(Cl(u)) \leqq Cl(u) \text{ i.e.}$$
$$Cl(u) \in O(L).$$

2.3. Definition. A frame L is said to be *regular* (respectively *0-dimensional*) if $a = \bigvee\{x \in L: x \lhd a\}$ (respectively $a = \bigvee\{x \in L: x \lhd x, x \leqq a\}$) for each $a \in L$, here $x \lhd a$ means that $x^* \vee a = 1$ i.e. $Cl(u_x) \leqq u_a$.

2.4. Theorem. *Let L be a regular extremally disconnected frame. Then L is 0-dimensional.*

Proof. Clearly, $x \lhd y$ implies $x \wedge x^* = 0$. Using 2.2. (iii) we have that there exist $c, d \in L$ such that $c \vee d = 1$, $c \wedge x = 0$, $d \wedge x^* = 0$. Then $c \leqq x^*$, $d \leqq \leqq x^{**} \leqq y$. Now, we have that $x^{**} \lhd x^{**}$. The rest is evident.

§ 3. Semi-open and preopen nuclei

3.1. Definition. Let L be a frame, $j \in N(L)$. Then j is said to be

(i) *semi-open* if there exists an open nucleus u such that $Cl(u) \leqq j \leqq u$,
(ii) *α-open* if $Int(Cl(Int(j))) \leqq j$,
(iii) *preopen* if $Int(Cl(j)) \leqq j$,
(iv) *semi-preopen* if $Cl(Int(Cl(j))) \leqq j$.

The set of all semi-open, α-open, preopen, semi-preopen nuclei will be denoted by $SO(L)$, $\alpha(L)$, $PO(L)$, $SPO(L)$ respectively. Clearly, $O(L) \subseteqq \alpha(L) \subseteqq PO(L) \subseteqq SO(L) \cap \cap PO(L)$, $SO(L) \cup PO(L) \subseteqq SPO(L)$.

3.2. Lemma. *Let L be a frame. Then*

(i) $j \in SO(L)$ *iff* $Cl(Int(j)) \leqq j$ *iff* $Cl(Int(j)) = Cl(j)$ $j^*(0)^* \leqq j(0)$.
(ii) $SO(L)$, $\alpha(L)$, $PO(L)$, $SPO(L)$ *are closed under arbitrary meets.*
(iii) $j \in SO(L) \Rightarrow j^{**} \in SO(L)$.
(iv) $j \in SPO(L)$ *iff there is a preopen nucleus* $k \in L$ *such that* $Cl(k) \leqq j \leqq k$ *iff* $Cl(Int(Cl(j))) = Cl(j)$.

Proof. "(i)" j is semi-open iff there exists $u \in O(L)$ such that $Cl(u) \leqq j \leqq u$ iff $Cl(Int(j)) \leqq Cl(u) \leqq j \leqq Int(j) \leqq u$ for some $u \in O(L)$ iff $Cl(Int(j)) \leqq j$ iff $Cl(u_{j^*(0)}) \leqq j$ iff $c_{j^*(0)^*} \leqq j$ iff $j^*(0)^* \leqq j(0)$.

"(ii)" Let $A \subseteqq SO(L)$. Then $Cl(Int(\bigwedge A)) \leqq Cl(Int(a)) \leqq a$ for each $a \in A$ i.e. $Cl(Int(\bigwedge A)) \leqq \bigwedge A$.

"(iii)" Let $j \in Cl(L)$. Then

$$Cl(Int(j^{**})) = Cl(u_{j^{***}(0)}) = Cl(u_{j^*(0)}) = Cl(Int(j)) \leqq j \leqq j^{**}.$$

"(iv)" Let j be semi-preopen. Then $Cl(Int(Cl(j))) \leqq j$ iff $c_{j(0)^{**}} \leqq j$ iff $j(0)^{**} \leqq \leqq j(0)$ iff $j(0)^{**} = j(0)$. We put $k = u_{j(0)^*} \vee j$. The k is preopen and $Cl(k) \leqq j \leqq k$. The reverse direction is obvious.

3.3. Lemma. *Let L be a frame. Then $\alpha(L) \supseteq O(L)$ and it is closed under arbitrary meets and finite joins.*

Proof. It remains to prove that $\alpha(L)$ is closed under finite joins. The rest is evident. Let $j, k \in \alpha(L)$. Then

$$Int(Cl(Int(j \vee k))) \leqq Int(Cl(Int(j) \vee Int(k))) = Int(Cl(u_{j^*(0)} \vee u_{k^*(0)})) =$$

$$= Int(Cl(u_{j^*(0) \wedge k^*(0)})) = Int(c_{(j^*(0) \wedge k^*(0))^*}) = u_{(j^*(0) \wedge k^*(0))^{**}} =$$

$$= u_{j^*(0)^{**}} \vee u_{k^*(0)^{**}} = Int(Cl(Int(j))) \vee Int(Cl(Int(k))) \leqq j \vee k .$$

3.4. Proposition. *Let L be a frame. Then the following conditions are equivalent:*

(i) *L is extremally disconnected.*
(ii) *$SO(L)$ is closed under finite joins.*
(iii) *$SO(L) = \alpha(L)$.*

Proof. "(i) ⇒ (iii)" Let $j \in SO(L)$. Then $Cl(Int(j)) \leqq j$. Clearly,

$$\nabla = j \vee Cl(Int(j))^* = Int(j) \vee Int(Cl(Int(j))^*) =$$

$$= Cl(Int(j)) \vee Cl(Int(Cl(Int(j))^*)) = j \vee Int(Cl(Int(j)))^* .$$

Now, we have $Int(Cl(Int(j))) \leqq j$.

"(iii) ⇒ (ii)" It is evident.

"(ii) ⇒ (i)" Let $u, v \in O(L)$, $u \vee v = \nabla$. Then

$$Cl(u) \vee v = \nabla \text{ i.e.}$$

$$Int(Cl(u)) \vee v = \nabla \text{ i.e.}$$

$$Int(Cl(u) \vee Int(Cl(v)) = \nabla .$$

Clearly $Cl(u) \vee Cl(v) \in SO(L)$. Then

$$\nabla = Cl(Int(Cl(u) \vee Cl(v))) \leqq Cl(u) \vee Cl(v) .$$

3.5. Definition. We say, that a nucleus j of a frame L is *semi-closed* (*semi-preclosed*) iff $j^* \in SO(L)$ ($j^* \in SPO(L)$).

The set of all semi-closed nuclei we will denote by $SC(L)$ ($SPC(L)$).

Clearly, $j \in SC(L)$ ($SPC(L)$) iff $j^{**} \in SC(L)$ ($SPC(L)$).

3.6. Lemma. *Let L be a frame, $A \subseteq SC(L)$ ($A \subseteq SPC(L)$). Then*

$$\bigvee A \in SC(L) \quad (\bigvee A \in SPC(L)) .$$

Proof. We have from 3.2 (ii) that $Cl(Int((\bigvee A)^*)) \leqq (\bigvee A)^*$.

3.7. Definition. The *semi-(pre)closure* of a nucleus j on a frame L is the greatest semi-(pre)closed nucleus lying below j. We denote it by $s\,Cl(j)$ ($sp\,Cl(j)$).

The *semi-interior* of a nucleus j on a frame L is the smallest semi-open nucleus containing j and is denoted by $s\,Int(j)$.

3.8. Lemma. *Let L be a frame, $j \in L$. Then*

(i) $s\,Int(j) = j \vee Cl(Int(j))$

(ii) $s\,Cl(j) = j \wedge Int(Cl(j^{**}))$

Proof. "(i)" Clearly, $Cl(Int(j \vee Cl(Int(j)))) = Cl(Int(j))$. The rest is evident.

"(ii)" Let k be semi-closed, $k \leqq s\,Cl(j)$. Then $k \leqq j$ i.e.

$$k^{**} \leqq Int(Cl(k^{**})) \leqq Int(Cl(j^{**}))\,.$$

Now we have to check that

$$[j \wedge Int(Cl(j))]^{*} \in SO(L)$$

Clearly,

$$Cl(Int((j \wedge Int(Cl(j^{**}))^{*})) = Cl((Cl(j^{**}) \wedge Cl(Int(Cl(j^{**}))))^{*}) =$$

$$= Int((Cl(j^{**}) \wedge Cl(Int(Cl(j^{**})))))^{*} = Int(Cl(j^{**}))^{*} \leqq (j \wedge Int(Cl(j^{**})))^{*}\,.$$

3.9. Lemma. *Let L be a frame, $u \in O(L)$, $S \subseteqq N(L)$. Then*

$$u \vee s = \nabla \quad \textit{for each} \quad s \in S \quad \textit{implies} \quad u \vee \bigwedge S = \nabla\,.$$

Proof. It is transparent.

3.10. Definition. Let j be a nucleus on a frame L. We shall say that

(i) k is *Θ-adherent with respect to j* if

$$Cl(u) \vee j = \nabla \Rightarrow u \vee k = \nabla$$

for each $u \in L$.

(ii) k is *δ-adherent with respect to j* if

$$Int(Cl(u)) \vee j = \nabla \Rightarrow u \vee k = \nabla$$

for each $u \in L$.

The least Θ-adherent (δ-adherent) nucleus with respect to j is called *Θ-closure* (*δ-closure*). We shall write $Cl_{\Theta}(j)$ ($Cl_{\delta}(j)$).

3.11. Lemma. *Let L be a frame, $j \in L$. Let k be Θ-adherent (δ-adherent) with respect to j. Then $Cl(k)$ is Θ-adherent (δ-adherent) with respect to j.*

Proof. Let $u \in O(L)$, $Cl(u) \vee j = \nabla$. Then $u \vee k = \nabla$ i.e. $u \vee Cl(k) = \nabla$. For the δ-adherent case one can proceed similarly.

3.12. Corollary. *Any Θ (δ)-closure is closed.*

3.13. Lemma. *Let L be a frame, $j \in L$. Then*

$$Cl_{\Theta}(j) \leqq Cl_{\delta}(j) \leqq Cl(j).$$

Proof. The first inequality follows from the fact that any δ-adherent nucleus with respect to j is Θ-adherent.

Let us check the second one. We have $Int(Cl(u)) \vee j = \nabla$. Then

$$Int(Cl(u)) \vee Cl(j) = \nabla \text{ i.e. } \quad u \vee Cl(j) = \nabla.$$

3.14. Lemma. *Let L be a frame, $j \in SO(L)$, $Cl(Int(j)) \in O(L)$. Then*

$$Cl_{\Theta}(j) = Cl_{\delta}(j) = Cl(j).$$

Proof. Clearly,

$$Cl(Int(j)) \vee Int(Int(j)^*) = \nabla$$

$$Cl(Int(j)) \vee Cl(Int(Int(j)^*)) = \nabla$$

$$j \vee Cl(Int(Int(j)^*)) = \nabla$$

$$Cl_{\Theta}(j) \vee Int(Int(j)^*) = \nabla$$

$$Cl_{\Theta}(j) \vee Cl(Int(j))^* = \nabla.$$

Then

$$Cl(j) \leqq Cl(Int(j)) \leqq Cl_{\Theta}(j).$$

3.15. Lemma. *Let L be a frame, $j \in PO(L)$. Then*

$$Cl_{\Theta}(j) = Cl_{\delta}(j) = Cl(j).$$

Proof. Clearly, $Int(Cl(j))^* \vee Int(Cl(j)) = \nabla$. Then

$$Cl(Cl(j)^*) \vee j = \nabla \text{ i.e. } \quad Cl(j)^* \vee Cl_{\Theta}(j) = \nabla.$$

Now, we have $Cl(j) \leqq Cl_{\Theta}(j)$.

3.16. Lemma. *Let L be a frame, $j \in SPO(L)$. Then*

$$Cl_{\delta}(j) = Cl(j).$$

Proof. Clearly,

$$Cl(Int(Cl(j)))^* \vee Cl(Int(Cl(j))) = \nabla.$$

Then

$$Int(Int(Cl(j))^*) \vee j = \nabla \text{ i.e. } \quad Int(Cl(Cl(j)^*)) \vee j = \nabla.$$

Using δ-adherence, we have $Cl(j)^* \vee Cl_{\delta}(j) = \nabla$ i.e. $Cl(j) \leqq Cl_{\delta}(j)$.

§ 4. Characterization of extremally disconnected frames

4.1. Theorem. *The following conditions are equivalent for a frame L:*

(i) *L is extremally disconnected.*
(ii) *The closure of every semi-preopen nucleus on L is open.*
(iii) *The δ-closure of every semi-preopen nucleus on L is open.*
(iv) *The δ-closure of every preopen nucleus on L is open.*
(v) *The closure of every preopen nucleus on L is open.*

Proof. "(i) ⇒ (ii)" Let j be semi-preopen. Then

$$Cl(j) = Cl(Int(Cl(j))) \in O(L) .$$

"(ii) ⇒ (iii)" Using 3.16.
"(iii) ⇒ (iv)" Evident, since $PO(L) \subseteqq SPO(L)$.
"(iv) ⇒ (v)" Using 3.15.
"(v) ⇒ (i)" Evident.

Noiri showed that a space is extremally disconnected if and only if $s\,Cl(A) = Cl_{\Theta}(A)$ for each $A \in PO(X) \cup SO(X)$. We shall show a little bit more.

4.2. Theorem. *The following conditions are equivalent for a frame L:*

(i) *L is extremally disconnected.*
(ii) $s\,Cl(j) = Cl_{\Theta}(j)$ *for each* $j \in SPO(L)$.

Proof.

"(i) ⇒ (ii)" Let j be semi-preopen. Then using 4.1 and 3.14

$$Int(Cl(j))^* \vee Int(Cl(j)) = \nabla$$

$$Cl(Cl(j)^*) \vee Int(Cl(j)) = \nabla$$

$$Cl(Cl(j)^*) \vee Cl(Int(Cl(j))) = \nabla$$

$$Cl(Cl(j)^*) \vee j = \nabla$$

$$Cl(j)^* \vee Cl_{\Theta}(j) = \nabla .$$

Now we have $Cl(j) \leqq Cl_{\Theta}(j)$.
"(ii) ⇒ (i)" Let j be an open nucleus on L. Then

$$Int(Cl(j^{**})) = j \wedge Int(Cl(j^{**})) = s\,Cl(j) = Cl_{\Theta}(j) = Cl(j) .$$

Therefore, $Cl(j)$ is open.

4.3. Theorem. *The following conditions are equivalent for a frame L:*

(i) *L is extremally disconnected.*
(ii) *If* $j \in SPO(L)$, $k \in SO(L)$, $j \vee k = \nabla$, *then* $Cl(j) \vee Cl(k) = \nabla$.

(iii) *If* $j \in SPO(L)$, $k \in SO(L)$, $j \vee k = \nabla$, *then* $Cl_\delta(j) \vee Cl_\delta(k) = \nabla$.
(iv) *If* $j \in PO(L)$, $k \in SO(L)$, $j \vee k = \nabla$, *then* $Cl_\Theta(j) \vee Cl_\Theta(k) = \nabla$.
(v) *If* $j \in PO(L)$, $k \in SO(L)$, $j \vee k = \nabla$, *then* $Cl(j) \vee Cl(k) = \nabla$.

Proof. "(i) $\Rightarrow$ (ii)" Let $j \in SPO(L)$, $k \in SO(L)$, $j \vee k = \nabla$. Then $Cl(j) \vee Int(k) = \nabla$. Using 4.1, $Cl(j)$ is open i.e.

$$\nabla = Cl(j) \vee Cl(Int(k)) \leqq Cl(j) \vee k = Cl(j) \vee Cl(k) .$$

"(ii) $\Rightarrow$ (iii)", "(iii) $\Rightarrow$ (v)", "(iv) $\Rightarrow$ (v)" This follows from 3.13, 3.15 and 3.16.
"(i) $\Rightarrow$ (iv)" It follows form and 3.14.
"(v) $\Rightarrow$ (i)" It is evident since $O(L) \subseteq PO(L) \cap SO(L)$.

4.4. Theorem. *The following conditions are equivalent for a frame* L:

(i) L *is extremally disconnected.*
(ii) $Int(j) = s\,Int(j)$ *for each* $j \in SC(L)$.
(iii) *The semi-interior of any semi-closed nucleus on* L *is open.*

Proof. "(i) $\Rightarrow$ (ii)" Let $j \in SC(L)$. Then $s\,Int(j) = j \vee Cl(Int(j))$. We have that $j \leqq Int(Cl(j))$ i.e. $Int(j) \leqq Int(Cl(Int(j)))$. Since any closure of an open nucleus is open as well $Int(j) \leqq Cl(Int(j))$ i.e. $Int(j) = Cl(Int(j)) = s\,Int(j)$.
"(ii) $\Rightarrow$ (iii)" It is transparent.
"(iii) $\Rightarrow$ (i)" Let u be open. Then

$$s\,Int(Cl(u)) = Cl(u) \vee Cl(Int(Cl(u))) = Cl(u)$$

i.e. $Cl(u)$ is open.

4.5. Lemma. *Let* L *be a frame,* $u \in O(L)$, $j \in N(L)$. *Then*

$$Cl(u \vee j) = u \vee Cl(j) .$$

Proof. Is is evident.

4.6. Theorem. *The following conditions are equivalent for a frame* L:

(i) L *is extremally disconnected.*
(ii) *If* $j \in SO(L)$ *and* $k \in SPO(L)$ *then* $Cl(j) \vee Cl(k) = Cl(j \vee k)$.
(iii) *If* $j \in SO(L)$ *and* $k \in SPO(L)$ *then* $j \vee k \in SPO(L)$.
(iv) *If* $j, k \in SO(L)$ *then* $j \vee k \in SO(L)$.

Proof. "(i) $\Rightarrow$ (ii)" Let $j \in SO(L)$ and $k \in SPO(L)$. Then

$$Cl(j) \vee Cl(k) = Cl(Int(j)) \vee Cl(k) =$$

$$= Cl(Int(j) \vee k) \geqq Cl(j \vee k) \geqq Cl(j) \vee Cl(k) .$$

Then $s\,Int(j) = j \vee Cl(Int(j))$.
"(ii) $\Rightarrow$ (iii)" Let $j \in SO(L)$ and $k \in SPO(L)$. Then

$$Cl(Int(Cl(j \vee k))) = Cl(Int(Cl(j)) \vee Int(Cl(k))) =$$
$$= Cl(Int(Cl(j))) \vee Cl(Int(Cl(k))) \leqq Cl(j) \vee k \leqq j \vee k .$$

Now, we have that $j \vee k \in SPO(L)$.

"(iii) ⇒ (i)" Let u, v be open. Then

$$Cl(u \vee v) = Cl(u \vee Cl(v)) \leqq Cl(u \vee Int(Cl(v))) \leqq Cl(Int(Cl(u)) \vee Int(Cl(v))) =$$
$$= Cl(Int(Cl(u) \vee Cl(v))) \leqq Cl(u) \vee Cl(v) \leqq Cl(u \vee v) .$$

"(ii) ⇒ (iv)" as in "(ii) ⇒ (iii)".
"(iv) ⇒ (i)" as in "(iii) ⇒ (i)".

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