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The Sequentiality and the Fréchet-Urysohn Property with Respect to Ultrafilters

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All spaces are assumed Hausdorff.

Theorem 1. *If $n(\omega^*) > \mathfrak{c}$, then the ultrasequentiality and ultra-Fréchet-Urysohn property coincide, respectively, with the sequentiality and the Fréchet-Urysohn property.*

$n(X)$ denotes the Novak number of X, i.e. the smallest power of a family of nowhere dense sets covering X.

Theorem 2. *Arens space is not p – sequential if p is a P-point in ω^*, on the other hand this space is an ultra-Fréchet-Urysohn space if there are no P-points in ω^* (recall also, that in this space there are no convergent sequences).*

Example [$\Diamond$]. *There exists a non-sequential compact space which is p-Fréchet-Urysohn for some $p \in \omega^*$.*

0. In 1968 M. Katětov [1] introduced the notion of an $\mathscr{F}$-limit point, namely:

Let $\mathscr{F}$ be a filter on ω. A point x in a space X is called an $\mathscr{F}$-limit point of a subset A if there exists a sequence $\{a_n: n \in \omega\} \subset A$ such that $\{n \in \omega: a_n \in Ox\} \in \mathscr{F}$ for every neighborhood Ox of x.

It is evident, that if $\mathscr{F}$ is a Fréchet filter, i.e the filter of cofinite subsets of ω, then an $\mathscr{F}$-limit point is the usual limit of some convergent sequence, lying in the corresponding suset. So, the notion of an $\mathscr{F}$-limit point is the generalization of the notion of a limit of a sequence.

The notions of $\mathscr{F}$-sequentiality and $\mathscr{F}$-Fréchet-Urysohn property are also quite natural, namely:

A space X is called $\mathscr{F}$-Fréchet-Urysohn if every limit point of an arbitrary subset $A \subset X$ is the $\mathscr{F}$-limit for this subset.

A space X is called $\mathscr{F}$-sequential if a subset $A \subset X$ is closed iff there exists no $\mathscr{F}$-limit point for this subset in $X \setminus A$.

In case $\mathscr{F}$ is an ultrafilter p, the notion of p-sequentiality is due to A. P. Kombarov [2] and the notion of p-Fréchet-Urysohn is due to I. Savchenko and V. I. Ponomarev.

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I. When studying these notions it is useful to waive completely the indexation of points of a space by elements of another set.

Let T be any infinite discrete space, let βT be its Stone-Čech compactification or the space of all ultrafilters on T. For $A \subseteqq T$ let $A^* = [A]_{\beta T} \setminus T$. Recall that the sets A^* are the only clopen sets in T^* and that they form the base of this space.

By the type of an ultrafilter ξ on T we understand the set $\mathscr{T}(\xi)$ of all ultrafilters on T received from ξ by means of various bijections T onto itself. It is evident, that the type of an ultrafilter of dispersion character m has the power not greater than $|T|^m$. In particular, for a countable set T the type of any free ultrafilter has the power of continuum.

There exists a canonic one-to-one correspondence between the free filters $\mathscr{F}$ on T and the non-empty subsets F of T^*, namely:

$$\mathscr{F} \leftrightarrow F = \bigcap\{V^*: V \in \mathscr{F}\} \leftrightarrow \mathscr{F} = \{A \subseteqq T: A^* \supseteqq F\} .$$

The corresponding set F will be denoted $F(\mathscr{F})$.

Let X be a topological space, $Y \subset X$, $x \in X$. Let $\mathscr{F}(x)$ denote the filter of neighborhoods of x in X, and let $\mathscr{F}(x)/Y$ denote the family $\{V \cap Y: V \in \mathscr{F}(x)\}$. It is clear, that $x \in [Y]$ iff $\mathscr{F}(x)/Y$ does not contain the empty set, in this case $\mathscr{F}(x)/Y$ is a filter.

Now we can formulate the criteria corresponding to the definitions given above. In the sequel we shall consider only ultrafilters on countable sets. Let T be an infinite countable set, p be any free ultrafilter on it and $\mathscr{T}(p)$ be the type of this ultrafilter.

1. A point x is the p-limit point of $Y \subset X$ iff there exists $T \subset Y$ such that the set $F(\mathscr{F}(x)/Y)$ contains an ultrafilter of the type $\mathscr{T}(p)$.

2. A space X is p-sequential if for every non-closed subset Y there exists $T \subset Y$ such that the set $\bigcup\{F(\mathscr{F}(x)/T: x \in [T] \setminus T\}$ contains an ultrafilter of the type $\mathscr{T}(p)$.

A space is called ultra-Fréchet-Urysohn if it is p-Fréchet-Urysohn for every $p \in \omega^*$.

4. A space is ultra-Fréchet-Urysohn if for every $x \in [Y] \setminus Y$ there exists a subset $T \subset Y$ such that the set $F(\mathscr{F}(x)/T)$ contains ultrafilters of all types.

A space is called ultrasequential if it is p – sequential for every $p \in \omega^*$.

5. A space is ultrasequential if for every non-closed subset Y there exists a subset $T \subset Y$ such that the set $\bigcup\{F(\mathscr{F}(x)/T): x \in [T] \setminus T\}$ contains ultrafilters of all types.

II. The proofs of theorems 1, 2 and the construction of the example.

Let F be a non-emply closed subset of ω^*, $\omega \cup \{F\}$ be the factor-space received from this space by identifying F with the point $\{F\}$.

Let Int be the interior operator on ω^*.

Proposition. *There is a convergent sequence in the space $\omega \cup \{F\}$ iff* $\operatorname{Int} F \neq \emptyset$; *the space $\omega \cup \{F\}$ is Fréchet-Urysohn iff* $F = [\operatorname{Int} F]$.

This is known (see [3]).

It is not hard to prove that it $n(\omega^*) > \mathfrak{c}$ and $\mathscr{E}$ is a family of closed subsets of ω^*,

$|\mathscr{E}| \leqq \mathfrak{c}$ and $\bigcup\mathscr{E}$ contains ultrafilters of all types, then there exists $F \in \mathscr{E}$ such that $\operatorname{Int} F \neq \emptyset$ ([4]).

Next we show how this proposition implies theorem 1.

Let X be a Hausdorff ultrasequential space, Y not closed in X. One can assume, that $|Y| = \aleph_0$. If X is a Hausdorff p-sequential space for some $p \in \omega^*$, then the closure of any countable subset of X has the power not greater than $\mathfrak{c}$. So, $|[Y] \setminus Y| \leqq \mathfrak{c}$. For $x \in [Y] \setminus Y$ let $F_x = F(\mathscr{F}(x)/Y)$, then F_x is a non-empty closed subset of Y^* (when using the asterisk above Y this space is considered with the discrete topology). According to the criterion of ultrasequentiality (see criterion 5 above) the set $\bigcup\{F_x : x \in [Y] \setminus Y\}$ contains ultrafilters of all types, hence (see the previous item) there exists $x_0 \in [Y] \setminus Y$ such that $\operatorname{Int} F_{x_0} \neq \emptyset$. The Proposition implies the existence of a sequence in Y convergent to x_0. But this means that X is sequential.

The proof of the fact that the properties of ultra-Fréchet-Urysohn and Fréchet-Urysohn are equivalent under the assumption $n(\omega^*) > \mathfrak{c}$, is analogous and even simpler.

The proof of theorem 2.

Let $X = \omega \cup \{*\}$ be Arens space (see, for example, [5, chapter 1]). It is easy to see that the set $\tilde{F} = F(\mathscr{F}(*)/\omega)$ (where $\mathscr{F}(*)$ is the filter of all neighborhoods of $*$) can be expressed as $\tilde{F} = [\bigcup\mathscr{A}] \setminus \bigcup\mathscr{A}$, where $\mathscr{A}$ is a infinite countable disjoint family of non-empty clopen subsets of ω^*. As it is easy to see, for any two such subsets $\tilde{F}_1, \tilde{F}_2$ there exists a bijection $\varphi_{12} : \omega \leftrightarrow \omega$ such that the map-extension $\tilde{\varphi}_{12} : \beta\omega \leftrightarrow \beta\omega$ translates $\tilde{F}_1$ onto $\tilde{F}_2$. Furthermore, the family of all such subsets covers ω^*, if there are no P-points in ω^*, and hence every such subset contains ultrafilters of all types. Thus from all this it follows that Arens space is ultra-Fréchet-Urysohn, if there are no P-points in ω^* (see also [4]). Now let us note that there are no convergent sequences in Arens space.

Corollary. *The statement about the property of ultra-Fréchet-Urysohn for Arens space does not depend on* ZFC.

In fact, it is easy to prove that the set $\tilde{F}$ (see above) does not contain any P-point, hence if there exist P-points in ω^*, then Arens space is not ultra-Fréchet-Urysohn. Now it remains to combine our theorem 2 with the statement that there need not be P-points in ω^* (S. Shelah, see, for example [6]).

We start now to construct the example.

Recall that $\diamondsuit$ denotes the set-theoretic assumption which is equivalent to the conjunction of CH and the following assumption:

$\dagger$. There exist a set $\{\lambda_\alpha : \alpha \in \omega_1\}$ of countable limit ordinals and a family $\{S_\alpha : \alpha \in \omega_1\}$ of countable subsets of ω_1 such that $S_\alpha \subset \lambda_\alpha$, $\operatorname{Sup} S_\alpha = \lambda_\alpha$ for every $\alpha \in \omega_1$ and every uncountable subset of ω_1 contains some S_α.

Our construction is completely analogous to Ostaszewski's construction of a non-sequential compact space of countable tightness [7] (see also V. V. Fedorchuk [8]).

Under the assumption of CH all infinite countable subsets of ω_1 can be enumerated by countable ordinals: $\{a_\alpha: \alpha \in \omega_1\}$ and moreover in such a way that $\alpha_\alpha \subseteqq \lambda_\alpha$ for every $\alpha \in \omega_1$.

We shall define a locally compact topology τ on ω_1, in which every initial segment $[0, \beta)$ is open, $[S_\alpha] \supseteq \omega_1 \setminus \lambda_\alpha$ for every $\alpha \in \omega_1$. Remaining properties of τ will be established in the sequel.

Let all points of λ_0 be isolated. Now define the topology in the points of the set $\lambda_1 \setminus \lambda_0$. As this step is completely analogous to the general step we describe the general one.

Thus, suppose that a topology τ_α on λ_α is already defined and satisfies the above conditions. Let $\mathscr{A}_\alpha = \{a_\beta: \beta < \alpha$ and a_β is not contained in any compact subspace of $(\lambda_\alpha, \tau_\alpha)\}$. Hence, the family of compact subspaces of $(\lambda_\alpha, \omega_\alpha)$ generates on every $a \in \mathscr{A}_\alpha$ a proper ideal $\mathscr{I}_\alpha(a)$ with a countable base. Let us suppose that for every $a \in \mathscr{A}_\alpha$ a bijection $\varphi_a: a \leftrightarrow \omega$ is fixed such that the ideal $\mathscr{I}_\alpha$ on ω, generated by the family $\bigcup\{\varphi_a(\mathscr{I}_\alpha(a)): a \in \mathscr{A}_\alpha\}$ is proper. It has, of course, a countable base.

To make the next step of our construction, define the topology in points of $\lambda_{\alpha+1} \setminus \lambda_\alpha$. As $S_\alpha \subseteqq \lambda_\alpha$ and $\operatorname{Sup} S_\alpha = \lambda_\alpha$, hence S_α is not contained in any compact subspace of $(\lambda_\alpha, \tau_\alpha)$. It is easy to see that there exists a discrete family $\mathscr{S}_\alpha$ of compact subspaces in $(\lambda_\alpha, \tau_\alpha)$, every one of which contains some points of S_α. Now divide this family into countably many disjoint subfamilies indexed by the points of $\lambda_{\alpha+1} \setminus \lambda_\alpha$, i.e. $\mathscr{S}_\alpha = \Sigma\{\mathscr{K}_\xi: \xi \in \lambda_{\alpha+1} \setminus \lambda_\alpha\}$. Let the family $\{(\{\xi\} \cup (\bigcup(\mathscr{K}_\xi \setminus \Delta)): \Delta \subset \mathscr{K}_\xi, |\Delta| < \aleph_0\}$ be the base of neighborhoods for the point $\xi \in \lambda_{\alpha+1} \setminus \lambda_\alpha$. It is evident, that a locally compact topology $\tau_{\alpha+1}$ on $\lambda_{\alpha+1}$ satisfying the above inductive conditions is defined.

It is easy to notice that the family $\mathscr{S}_\alpha$ can be divided into subfamilies $\mathscr{K}_\xi$ in many ways. Our idea is to do it in such a way that the main inductive assumption is preserved, i.e. the ideal $\mathscr{I}_{\alpha+1}$ on ω_1, generated by the family $\bigcup\{\varphi_a(\mathscr{I}_{\alpha+1}(a)): a \in \mathscr{A}_{\alpha+1}\}$ must be proper. We omit the bulky proof of the fact that this is really possible.

As a result of the described transfinite process a topology τ on ω_1 is defined. Let $X^* = (\omega_1 \cup \{*\}, \tau^*)$ be Aleksandroff's compactification of the space (ω_1, τ).

1. This compact space X^* can be made nonsequential, namely, there are no sequences convergent to $*$ in it. We have especially omitted this moment when describing the construction in order to make its main idea more explicit. As in the original papers of V. V. Fedorchuk [8] and A. Ostaszewski [7] the space (ω_1, τ) can be made countably compact and hence X^* will be nonsequential.

2. X^* is a p-Fréchet-Urysohn space for some ultrafilter $p \in \omega^*$. In fact, it is easy to see, that the ideal $\mathscr{I} = \bigcup\{\mathscr{I}_\alpha: \alpha \in \omega_1\}$ on ω is proper, hence the dual family $\mathscr{F} = \{\omega_1 \setminus T: T \in \mathscr{I}\}$ is a filter. Let $* \in [M]$, where $M \subset \omega_1$. If M is uncountable, then M contains some S_α and hence $* \in [S_\alpha]$. Therefore, one can assume that M is countable, and hence, $M = a_\alpha$ for some $\alpha \in \omega_1$. It is clear that $a_\alpha \in \mathscr{A}_{\alpha+1}$ and hence on the α-th step of the transfinite process the bijection $\varphi_{a_\alpha}: a_\alpha \leftrightarrow \omega$ was defined such that ... (see above). It follows that the ideal on α_α generated by the family of compact

subspaces of (ω_1, τ), "transferred" on ω by the bijection φ_{a_α}, is contained in $\mathscr{I}$. Hence, the filter of traces on α_α of neighborhoods of the point $*$, is also contained in $\mathscr{F}$. From all this it follows that X^* is p-Fréchet-Urysohn for any ultrafilter p dominating $\mathscr{F}$.

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