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Collapsing of Cardinals in Generalized Cohen's Forcing

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We consider generalized Cohen's forcing $C(J)$ in dependence on an ideal J on ω. We prove that either $C(J)$ is a dense subset of $Col(\omega, \mathfrak{c})$ or $C(J)$ is an iteration of ω_1-closed and c.c.c. notions of forcing.

0. Introduction

Let J be an ideal on ω. By generalized Cohen's set of forcing conditions we understand the set $C(J)$ of all $0-1$ valued functions with domain an element of J. $C(J)$ is ordered by the reverse inclusion.

We shall investigate collapsing of cardinals by $C(J)$.

In this paper an ideal on ω means an ideal containing the ideal *fin* of all finite subsets of ω. Let $\varkappa$ be a cardinal number. An ideal J is a $p^*(\varkappa)$-ideal iff for every set $X \subseteq J$ of cardinality less than $\varkappa$ there is $x \in J$ such that for every $y \in X$, $y - x$ is finite. A $p^*(\omega_1)$-ideal we call a p^*-ideal. An ideal J on ω is regular iff for every partition $\{x_n;\ n \in \omega\}$ of ω into finite sets there is an infinite set $a \subseteq \omega$ such that $\bigcup\{x_n;\ n \in a\} \in J$. Notice that the dual filter F to a regular p^*-ideal coincides with the notion of coherent filter introduced in [6] in case F is an ultrafilter. An ideal J on ω is a q-ideal iff for every partition $\{x_n;\ n \in \omega\}$ of ω into finite sets, there is a selector from $\mathscr{P}(\omega) - J$.

Let x, y be arbitrary sets. We write $x \subseteq^* y$ iff $x - y$ is finite, $x = {}^*y$ iff $x \subseteq^* y$ and $y \subseteq^* x$.

Let P be a partially ordered set. We say that $\Theta = \{H_\alpha;\ \alpha \in \varkappa\}$ is a matrix for P if H_α is a maximal antichain in P for all $\alpha \in \varkappa$. A matrix Θ is said to be shattering for P if for each $p \in P$ there is some $H \in \Theta$ such that p is compatible with two members of H at least. A matrix Θ is refining if H_α refines H_β whenever $\beta < \alpha$. A matrix Θ is called a base matrix if Θ is refining and $\bigcup\Theta$ is a dense subset of P. An antichain H is a refinement of a matrix Θ if H refines all members of Θ. Define:

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$\varkappa(P) = \min\{|\Theta|;\ \Theta$ is a shattering matrix for $P\}$,
$h(P) = \min\{|\Theta|;\ \Theta$ has no refinement$\}$,
$n(P) = \min\{|\Theta|;$ there is no Θ-generic filter on $P\}$.
Of course $h(P) \leqq \varkappa(P) \leqq n(P)$.

The following theorem is well known in case $P = \mathscr{P}(\omega)/fin$ (see [1]).

0.1. Theorem. Let $\varkappa \geqq \omega$ be a cardinal number. Let P be a $\varkappa^+$-closed notion of forcing of cardinality at most $2^\varkappa$. Let $h(P) = \varkappa(P)$. Then

(i) Base Matrix Lemma: For each shattering matrix $\Theta = \{H_\alpha;\ \alpha \in \varkappa(P)\}$ there exists a base matrix $\Theta' = \{H'_\alpha;\ \alpha \in \varkappa(P)\}$ such that H'_α refines H_α for every $\alpha \in \varkappa(P)$.

(ii) $\varkappa(P) \leqq \operatorname{cf}(2^\varkappa)$.

(iii) If $\varkappa(P) < 2^\varkappa$ then $\varkappa(P) \leqq n(P) \leqq \varkappa(P)^+$ and P collapses $2^\varkappa$ onto $\varkappa(P)$.

Proof. (i) Let Θ be a shattering matrix. Since $h(P) = \varkappa(P)$, it is enough to find Θ' such that $\bigcup\Theta'$ is a dense subset of P and H'_α refines H_α for all $\alpha \in \varkappa(P)$. We also may assume that Θ is refining.

Since P is $\varkappa^+$-closed and Θ is shattering, for every $p \in P$ there is $\alpha \in \varkappa(P)$ such that p is compatible with $2^\varkappa$ members of H_γ. Let $\alpha \in \varkappa(P)$, denote A_α the family of all $p \in P$ which are compatible with $2^\varkappa$ elements of H_α. Let $\varphi_\alpha\colon A_\alpha \to H_\alpha$ be a one-to-one mapping such that $p, \varphi_\alpha(p)$ are compatible for all $p \in A_\alpha$. For $p \in H_\alpha$, let H_p be a maximal antichain below p such that H_p contains some element $q \leqq r$ whenever $p = \varphi_\alpha(r)$. Let $H'_\alpha = \bigcup\{H_p;\ p \in H_\alpha\}$. Obviously H'_α refines H_α and $\bigcup\Theta'$ is a dense subset of P.

(ii) If $X \subseteq P$ and $|X| < 2^\varkappa$ then the family $H(X)$ of all conditions $p \in P$ such that there is no $q \in X$, $q \leqq p$, is a dense subset of P, since below every condition there are $2^\varkappa$ incompatible conditions. Choose $X_\xi \subseteq P$ for $\xi \in \operatorname{cf}(2^\varkappa)$ such that $|X_\xi| < 2^\varkappa$ and $\bigcup\{X_\xi;\ \xi \in \operatorname{cf}(2^\varkappa)\} = P$. Let $H_\xi \subseteq H(X_\xi)$ be a maximal antichain. Then the matrix $\Theta = \{H_\xi;\ \xi \in \operatorname{cf}(2^\varkappa)\}$ is shattering.

(iii) We find an r.o.(P)-name $\mathbf{f}$ of a function from $\varkappa(P)$ onto $2^\varkappa$.

Let $\Theta = \{H_\alpha;\ \alpha \in \varkappa(P)\}$ be a base matrix. For every condition $p \in \bigcup\Theta$ choose a maximal antichain $\{p_\xi;\ \xi \in 2^\varkappa\}$ below p. We define $\mathbf{f}$ by

$$[\![\mathbf{f}(\alpha^\vee) = \xi^\vee]\!] = \bigvee\{p_\xi;\ p \in H_\alpha\}\,.$$

For $\xi \in 2^\varkappa$, $D_\xi = \{p;\ p \Vdash \exists\alpha\ \mathbf{f}(\alpha) = \xi^\vee\}$ is a dense subset of P. As we assume $\varkappa(P) < 2^\varkappa$, the existence of a $\{D_\xi;\ \xi \in \varkappa(P)^+\}$ – generic filter would mean collapsing of $\varkappa(P)^+$ onto $\varkappa(P)$.

Let B be a complete Boolean algebra and let $a \in B$ be a positive element of B. Then $B \mid a$ is the partial algebra.

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*) Throughout the whole paper, the bold-face letters $\mathbf{f}$, $\mathbf{c}^{-1}(\mathbf{H})$ stand for names $\dot{f}$, $\dot{c}^{-1}(\dot{H})$.

0.2. Lemma. Let B be a complete Boolean algebra and let $D = \{b \in B;\ B \simeq B \mid b\}$ be a dense subset of B. Then B is homogeneous.

Proof. B is atomless. Let $a \in B$, $a \neq \mathbf{0}, \mathbf{1}$ be arbitrary. Choose $A \subseteq D$ a maximal antichain below a and let $X \subseteq D$ be an infinite maximal antichain in B such that $A \subseteq X$. Then for every $b \in A$ there is $X_b \subseteq D$ a maximal antichain below b such that $|X_b| = |X|$. Then the set $Y = \bigcup\{X_b;\ b \in A\}$ is a maximal antichain below a, $Y \subseteq D$ and $|Y| = |X|$. Let f be arbitrary one-to-one function from Y onto X and let $e_b\colon B \mid b \to B \mid f(b)$ be an isomorphism for every $b \in Y$. Then the function h from $B \mid a$ onto B defined by

$$h(x) = \bigvee\{e_b(x \wedge b);\ b \in Y\} \quad \text{for all} \quad x \leqq a$$

is an isomorphism.

1. Approximative forcing

Now we introduce approximative sets depending on an ideal J on ω. Their properties reflect the behaviour of the generalized Cohen's forcing.

Let J be an ideal on ω. $A \subseteq C(J)$ is said to be a J-approximative set if A satisfies the following conditions:
(1) for arbitrary $p, q \in A$ there is $f \in {}^{\omega}2$ such that $p \subseteq^* f$ and $q \subseteq^* f$,
(2) $A \cap {}^{x}2 \neq \emptyset$ for every set $x \in J$,
(3) if $p \in A$ and $q \in C(J)$ and $p =^* q$ then $q \in A$.
Let A be ordered by the reverse inclusion

Let $f \in {}^{\omega}2$ and let $A_f = \{p \in C(J);\ p \subseteq^* f\}$. A_f is a J-approximative set and it is a c.c.c. notion of forcing.

We give another example of an approximative set in case J is generated by an almost disjoint family of subsets of ω. Instead of ω we take ${}^{<\omega}2$. If $f \in {}^{\omega}2$, let $x_f = \{f \mid n;\ n \in \omega\}$. Then $\{x_f\colon f \in {}^{\omega}2\}$ is an almost disjoint family of subsets of ${}^{<\omega}2$. Let J be an ideal generated by this family. Arbitrary collection of functions $p_f\colon x_f \to 2$, $f \in {}^{\omega}2$, can be completed to a J-approximative set A. But choosing functions p_f carefully we obtain a J-approximative set which is not c-c.c.. It is enough to define $p_f(f \mid n) = f(n)$ for $n \in \omega$. The collection $\{p_f;\ f \in {}^{\omega}2\}$ is an antichain in A.

1.1. Lemma. Let A be a J-approximative set. Then
(a) forcing A does not collapse ω_1 iff A is c.c.c.,
(b) if A is c.c.c. then in the generic extension over A, the cardinality of all reals is the same as the cardinality of reals of ground model.

Proof. (a) If $G \subseteq A$ is a generic filter then $g = \bigcup G$ is a function from ω into 2. In the generic extension, $A \subseteq A_g$ and therefore every antichain in A is countable.

(b) There are at most $|\text{r.o.}(A)|^{\omega}$ reals in the generic extension.

1.2. Lemma. Let J be a $p^*(\omega_2)$-ideal and let A be a J-approximative set. Then A is c.c.c..

Proof. Let $\{p_\alpha;\ \alpha \in \omega_1\} \subseteq A$. There is $x \in J$ such that $\operatorname{dom} p_\alpha \subseteq^* x$ for all $\alpha \in \omega_1$. Let $p \in A \cap {}^x2$ and let $f \in {}^\omega 2$ be such that $p \subseteq f$. Then $\{p_\alpha;\ \alpha \in \omega_1\} \subseteq A_f$ and therefore it is not an antichain as A_f is c.c.c..

It is natural to ask whether previous lemma holds for every p^*-ideal. This question can be equivalently reformulated into:

1.3. Problem. Is there a sequence $\{p_\alpha;\ \alpha \in \omega_1\}$ of $0-1$ valued functions such that (1) $\operatorname{dom} p_\alpha \subseteq \omega$, (2) $p_\alpha \subseteq^* p_\beta$ whenever $\alpha < \beta$ and (3) for every $\alpha \neq \beta$ there is $n \in \operatorname{dom} p_\alpha \cap \operatorname{dom} p_\beta$ such that $p_\alpha(n) \neq p_\beta(n)$?

Notice, that such a sequence cannot have its length $\omega_1 + 1$.

2. Decomposition of the generalized Cohen's forcing

Let M be a transitive model of ZFC, $C(J) \in M$ and let $G \subseteq C(J)$ be an M-generic filter. In the following we are working in M.

Let $P(J) = \{c(p);\ p \in C(J)\}$ where $c(p) = \{q \in C(J);\ p =^* q\}$. On $P(J)$ we put the ordering defined as follows: $c(p) \leqq c(q)$ iff $q \subseteq^* p$. It is not hard to verify that $c\colon C(J) \to P(J)$ is a normal function. Therefore, $H = c(G)$ is an M-generic subset of $P(J)$ and G is an $M[H]$-generic subset of $c^{-1}(H)$. In fact $C(J)$ is a dense subset of $P(J) * \mathbf{c}^{-1}(\mathbf{H})$.

$C(fin)$ is the Cohen's forcing while $|P(fin)| = 1$. If J is the ideal generated by one subset of ω over the ideal *fin* then r.o. $(C(J))$ is locally equal to the Cohen's algebra and r.o. $(P(J))$ is atomary. The next lemma describes the final case.

2.1. Lemma. Let J be an ideal which is not one generated over the ideal of finite sets. Let $P = C(J)$ or $P = P(J)$. Then r.o. (P) is homogeneous.

Proof. By Lemma 0.2 it is enough to prove that for every $p \in P$, r.o. $(P) \simeq$ $\simeq$ r.o. $(P) \mid p$. This follows from the fact that whenever $x \in J$ then (since J is not one generated over *fin*) there is a one-to-one function f from ω onto $\omega - x$ such that $f(J) = \{y \subseteq \omega - x;\ y \in J\}$ where $f(J) = \{y \subseteq \omega - x;\ f^{-1}(y) \in J\}$. Therefore P is isomorphic to the set $\{q \in P;\ q \leqq p\}$ for every $p \in P$.

2.2. Lemma. (a) $\varkappa(P(J)) = h(P(J))$.

(b) If J is a p^*-ideal then $P(J)$ is ω_1-closed and $[\![J^\vee$ is a p^*-ideal and $\mathbf{c}^{-1}(\mathbf{H})$ is a $J^\vee$-approximative set$]\!] = \mathbf{1}$ in r.o. $(P(J))$.

Proof. Trivial.

Lemma 2.2 shows that if J is a p^*-ideal then Theorem 0.1 holds for $P(J)$ in case $\varkappa = \omega$.

In the following we denote $\varkappa(J) = \varkappa(P(J))$ and $\mathrm{Add}^*(J) = \min \{|X|;\ X \subseteq J$ and $\forall y \in J\ \exists x \in X\ y - x$ is infinite$\}$. If J is the ideal *fin* or the ideal generated by one set over the ideal *fin* then we put $\mathrm{Add}^*(J) = \mathfrak{c}$.

2.3. Theorem. (a) Forcing $P(J)$ collapses $\mathfrak{c}$ onto $\mathrm{Add}^*(J)$.

(b) $\varkappa(J) \leqq \mathrm{Add}^*(J)$.

First we prove the following lemma. Let $x \in J$ be infinite. Denote

$$A = \{p \in C(J);\ \mathrm{dom}\, p \subseteq x \text{ and } x - \mathrm{dom}\, p \text{ is infinite}\},$$

$$B = \{c(p);\ p \in A\} \text{ and } C = \{c(p);\ p \in {}^x2\}.$$

2.4. Lemma. There is a one-to-one function F from $\mathfrak{c} \times B$ to C such that $F(\alpha, p) \leqq p$ for all $p \in B$ and $\alpha \in \mathfrak{c}$.

Proof. Let $\{(\alpha_\xi, c(p_\xi));\ \xi \in \mathfrak{c}\}$ be an enumeration of $\mathfrak{c} \times B$. We construct $F(\alpha_\xi, c(p_\xi))$ by induction on ξ. Let $\xi \in \mathfrak{c}$. Every element of C is a countable set and there are $\mathfrak{c}$ extensions of p_ξ into a function with domain x. Therefore, there is a function $r \in {}^x2 - \bigcup\{F(\alpha_\zeta, c(p_\zeta));\ \zeta \in \xi\}$ such that $p_\xi \subseteq r$. Put $F(\alpha_\xi, c(p_\xi)) = c(r)$.

Proof of Theorem 2.3. Denote $\varkappa = \mathrm{Add}^*(J)$.

(a) We find a name f of a function from a subset of $\varkappa$ onto $\mathfrak{c}$.

Let $\{x_\xi;\ \xi \in \varkappa\}$ be a family of infinite members of J such that for every $x \in J$ there is an ξ such that $x_\xi - x$ is infinite. For every set x_ξ find $A_\xi, B_\xi, C_\xi, F_\xi$ such that Lemma 2.4 holds. Denote $a_{\xi,\alpha} = \bigvee\{F_\xi(\alpha, q);\ q \in B_\xi\}$ computed in r.o.$(P(J))$. Since elements of C_ξ are pairwise incompatible, $a_{\xi,\alpha} \wedge a_{\xi,\beta} = \mathbf{0}$ for all $\alpha \neq \beta$. Let $\mathbf{f}$ be defined by $[\![\mathbf{f}(\xi^\vee) = \alpha^\vee]\!] = a_{\xi,\alpha}$. We show that $[\![\mathrm{rng}\, \mathbf{f} = \mathfrak{c}]\!] = \mathbf{1}$.

Let $q \in P(J)$ and let $\alpha \in \mathfrak{c}$. Let $q = c(p)$ for some $p \in C(J)$. There exists $\xi \in \varkappa$ such that $x_\xi - x$ is infinite. The conditions q and $F_\xi(\alpha, c(p \mid x_\xi))$ are compatible. Let $r \in P(J)$ be their common extension. Then $r \leqq a_{\xi,\alpha}$ and therefore $r \Vdash \mathbf{f}(\xi^\vee) = \alpha^\vee$.

(b) The M-generic set $H \subseteq P(J)$ is not in M. Therefore $h(P(J)) \leqq \mathfrak{c}$. Moreover, if $\varkappa < \mathfrak{c}$ then $P(J)$ collapses $\varkappa^+$ onto $\varkappa$ and therefore $h(P(J)) \leqq \varkappa$.

2.5. Theorem. Let J be an ideal on ω. The following are equivalent:

(i) forcing $C(J)$ does not collapse ω_1,

(ii) $C(J)$ is an iteration of ω_1-closed and c.c.c. notions of forcing,

(iii) in the generic extension over $C(J)$, every ideal generated by a p^*-ideal of the ground model is a p^*-ideal,

(iv) $C(J)$ is not a dense subset of $Col(\omega, \mathfrak{c})$.

Proof. (i) → (ii). $C(J)$ is a dense subset of the iteration $P(J) * \mathbf{c}^{-1}(\boldsymbol{H})$. Since ω_1 is not collapsed, $P(J)$ is ω_1-closed and $[\![\mathbf{c}^{-1}(\boldsymbol{H})$ is c.c.c.$]\!] = \mathbf{1}$ (Lemma 2.2(b), Theorem 2.3(a) and Lemma 1.1(a)).

(ii) → (iii). Every countable subset of M which is an element of the generic extension is covered by a countable set of M. Therefore, if $K \in M$ is a p^*-ideal then the ideal $\bar{K}$ generated by K in the generic extension is a p^*-ideal.

(iii) → (iv). An ideal which is countably generated and is not one generated over the ideal *fin* cannot be a p^*-ideal. Therefore $C(J)$ does not collapse $\mathfrak{c}$ onto ω.

(iv) → (i). Assume that $C(J)$ is not a dense subset of $Col(\omega, \mathfrak{c})$. Then $C(J)$ does not collapse $\mathfrak{c}$ onto ω (see [5], Lemma 25.11). By Theorem 2.3 it means that J is a p^*-ideal. If $\varkappa(J) = \omega_1$ then $C(J)$ collapses $\mathfrak{c}$ onto ω_1 (because $P(J)$ does) and therefore it cannot collapse ω_1. If $\varkappa(J) > \omega_1$ then J is a $p^*(\omega_2)$-ideal in $M[H]$ and $c^{-1}(H)$ is c.c.c. in $M[H]$ (Lemma 1.2). Therefore ω_1 is not collapsed.

2.6. Corollary. Let J be a regular p^*-ideal. Then
(a) $C(J)$ is an iteration of ω_1-closed and c.c.c. notions of forcing,
(b) in the generic extension over $C(J)$ every ideal generated by a regular p^*-ideal of the ground model is a regular p^*-ideal.

Proof. If J is a regular p^*-ideal then forcing $C(J)$ does not collapse ω_1 (see [7]). Now (a) follows from the preceding theorem.

(b) Let $K \in M[G]$ be an ideal generated by a regular p^*-ideal of M. By 2.5, K is a p^*-ideal. We prove that K is regular.

Let $\{x_n;\ n \in \omega\}$ be a partition of ω into finite sets (in $M[G]$). As ${}^{\omega}\omega \cap M$ is a dominating family in ${}^{\omega}\omega \cap M[G]$ (see [7]), there is an increasing function $f \in {}^{\omega}\omega \cap M$ such that for every $n \in \omega$ there is a k such that $x_k \subseteq \langle f(n), f(n+1))$. Therefore, there is an infinite set $a \subseteq \omega$ such that $\bigcup\{x_n;\ n \in a\} \in K$.

2.7. Corollary. (a) If J is not a p^*-ideal then $M[G] \Vdash 2^{\omega} = (2^{\mathfrak{c}})^M$.
(b) If $\omega_1^M = \omega_1^{M[G]}$ then $M[G] \Vdash 2^{\omega} = \varkappa(J)$.

In these two cases a cardinal $\varkappa$ is collapsed iff $\varkappa(J) < \varkappa \leqq \mathfrak{c}$. Always $\varkappa(J) \leqq \leqq \min\{\mathrm{cf}(\mathfrak{c}), \mathrm{Add}^*(J)\}$.

Proof follows immediately from 0.1, 1.1(b) and 2.3 since $C(J)$ is $\mathfrak{c}^+$-c.c..

2.8. Remark. If in 1.3 "no" is provable then for every p^*-ideal all conditions in Theorem 2.5 hold because in this case $[\![\mathbf{c}^{-1}(\boldsymbol{H})$ is c.c.c.$]\!] = \mathbf{1}$ in r.o.$(P(J))$. Here we can observe something more:

Claim. If 1.3 does not hold then $C(J)$ collapses ω_1 just in case J is not a p^*-ideal.

Proof. Let J be a p^*-ideal and assume that $C(J)$ collapses ω_1. Then $[\![\mathbf{c}^{-1}(\boldsymbol{H})$ is not c.c.c.$]\!] = \mathbf{1}$ in r.o.$(P(J))$. Therefore, there is an r.o.$(P(J))$-name f of a function from ω_1 into $c^{-1}(H)$ such that all values of f are mutually incompatible elements of $C(J)$. Therefore

(1) $\bigvee\{c(q) \wedge [\![f(\alpha^\vee) = q^\vee]\!];\ q \in C(J)\} = \mathbf{1}$ for all $\alpha \in \omega_1$,

(2) $\bigvee\{[\![f(\alpha^\vee) = p^\vee]\!] \wedge [\![f(\beta^\vee) = q^\vee]\!];\ p, q \in C(J)$ are incompatible$\} = \mathbf{1}$ for all $\alpha < \beta < \omega_1$.

As all values $[\![f(\alpha^\vee) = q^\vee]\!]$, $q \in C(J)$, are disjoint, by (1) we have:

(3) $[\![f(\alpha^\vee) = q^\vee]\!] \leqq c(q)$ for all $\alpha \in \omega_1$.

For every $\alpha \in \omega_1$, the set $D_\alpha = \{r \in P(J);\ r$ decides $f(\alpha^\vee)\}$ is a dense subset of $P(J)$. Since $P(J)$ is ω_1-closed, we can find a sequence r_α, $\alpha \in \omega_1$, of elements of $P(J)$ such that $r_\alpha \in D_\alpha$ and $r_\beta \leqq r_\alpha$ for all $\alpha < \beta < \omega_1$. For every $\alpha \in \omega_1$ there is a $q_\alpha \in C(J)$ such that $r_\alpha \Vdash f(\alpha^\vee) = g_\alpha{}^\vee$. If $\alpha \neq \beta$ then $[\![f(\alpha^\vee) = q_\alpha^\vee]\!] \wedge [\![f(\beta^\vee) = q_\beta^\vee]\!] \neq \mathbf{0}$ and so by (2), q_α, q_β are incompatible. According to (3), $r_\alpha \leqq c(q_\alpha)$. Let $p_\alpha \in C(J)$ be an extension of q_α such that $r_\alpha = c(p_\alpha)$. Then the sequence $\{p_\alpha;\ \alpha \in \omega_1\}$ is just such as 1.3 requires. A contradiction.

3. Collapsing of category

If an ideal J is a p^*-ideal and is not regular we can say nothing about collapsing of ω_1. But sometimes we can say something about collapsing of category.

Let us consider the following two ideals on ω:

$$J_0 = \{x \subseteq \omega;\ \Sigma\{1/(n+1);\ n \in x\} < \infty\}\,,$$
$$J_1 = \{x \subseteq \omega;\ \lim_{n\to\infty} |x \cap n|/n = 0\}\,.$$

Both the ideals are p^*-ideals and they are not regular. Moreover they are not q-ideals.

3.1. Theorem. If an ideal J is not regular and is not a q-ideal then the set of reals of the ground model is meager in the generic extension over $C(J)$.

Proof. Let us denote $R = {}^\omega 2$, $Q = {}^{<\omega}2$. As J is not regular and is not a q-ideal, there is a partition $\{x_n;\ n \in \omega\}$ of ω into finite sets such that

(1) if $a \subseteq \omega$ and $\bigcup\{x_n\colon n \in a\} \in J$ then a is finite,

(2) every selector of the partition is in J.

For proving the theorem it is enough to find a family $a_{n,t}$, $n \in \omega$, $t \in Q$, of elements of r.o.$(C(J))$ such that (see [3]):

(a) $\bigwedge_{n\in\omega} \bigwedge_{r\in Q} \bigvee_{s\supseteq r} \bigwedge_{t\supseteq s} a_{n,t} = \mathbf{1}$ and

(b) $\bigvee_{f\in R} \bigwedge_{n\in\omega} \bigvee_{k\in\omega} \bigwedge_{t\supseteq f|k} a_{n,t} = \mathbf{0}$.

If $s \in Q$ put $b(s) = \bigvee\{s \mid x_n;\ x_n \subseteq \operatorname{dom} s\}$, computed in r.o.$(C(J))$, and let π_s be a function from Q into Q defined by $\pi_s(t) = s \cup (t \mid (\operatorname{dom} t - \operatorname{dom} s))$. Of course, if $s \subseteq t$ then $b(s) \leqq b(t)$. Let $\{\pi_n;\ n \in \omega\}$ be an enumeration of the set $\{\pi_s;\ s \in Q\}$. We put $a_{n,s} = b(\pi_n(s))$. Now we verify (a) and (b).

Let $n \in \omega$ and $r \in Q$. There are $v \in Q$ and $m \in \omega$ such that $\pi_v = \pi_n$ and $x_m \cap \cap (\operatorname{dom} r \cup \operatorname{dom} v) = \emptyset$. Then

$$\bigvee_{s\supseteq r} \bigwedge_{t\supseteq s} a_{n,t} = \bigvee_{s\supseteq r} b(\pi_v(s)) \geqq \bigvee\{b(\pi_v(s));\ s \supseteq r \text{ and } x_m \subseteq \operatorname{dom} s\} \geqq$$

$$\geqq \bigvee\{s \mid x_m;\ s \supseteq r \text{ and } x_m \subseteq \operatorname{dom} s\} = \mathbf{1}\,.$$

Now assume that (b) does not hold. Then there is $f \in R$ and $p \in C(J)$ such that $p \leqq \bigvee\{a_{n,f|k};\ k \in \omega\}$ for all $n \in \omega$. As (1) and (2) hold, there are $n \in \omega$ nad $x \in J$ such that $x \cap \operatorname{dom} p = \emptyset$ and $x \cap x_k \neq \emptyset$ for all $k \geqq m$. Therefore, there is a condition $q \leqq p$ such that $x \subseteq \operatorname{dom} q$ and $x_k \subseteq \operatorname{dom} q$ for all $k < m$ and $q \mid x_k \nsubseteq f \mid x_k$ for all $k \geqq m$. Let $s \in Q$ be such that $\operatorname{dom} s = \bigcup\{x_k;\ k < m\}$ and $s(k) \neq q(k)$ for all $k \in \operatorname{dom} s$. Then $q \wedge \bigvee\{b(\pi_s(f \mid k));\ k \in \omega\} = \mathbf{0}$ and therefore there is $n \in \omega$ such that $p \nleqq \bigvee\{a_{n,f|k};\ k \in \omega\}$. A contradiction.

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