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Bijections onto Compact Spaces

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In 1935 Banach posed a problem which can be formulated equivalently: When can a metric space have continuous bijection onto a compact metric space? Independently, more general question was raised by Aleksandroff who asked:

When a Hausdorff space X has a continuous bijection onto a compact Hausdorff space Y?

The Aleksandroff question remains still interesting for $X = Z \setminus D$, where Z is a compact Hausdorff space and D is a countable subset. It is known that the class of spaces Z such that for each countable subset $D \subset Z$ there exists a continuous bijection $f: Z \setminus D \to^{onto}_{1-1} Y$ onto a compact Hausdorff space Y, contains compact metric spaces (Parhomenko [4], Rauhvarger [7]), products of compact metric spaces (Proizvolov [6]). In 1972 Belugin [2] extended the results of Parhomenko and the others onto the class of dyadic spaces. On the other hand Ponomarev (see [5]) proved that if we remove from the remainder $\beta\omega \setminus \omega$ of the Čech-Stone compactification of ω, a countable subset D then the resulting space has no continuous bijection onto a compact Hausdorff space. Some more information the reader can find in Arhangelskii's appendix to Kelley's book [3]. Recently Bell [1] introduced a new class of spaces being a generalization of dyadic spaces. The purpose of this note is to show that Belugin's result holds for a class of spaces containing spaces in sense of Bell.

Let T be an infinite set. Denote a Cantor cube by

$$D^T := \{p: p: T \to \{0, 1\}\} .$$

For $s \subset T$, $i: s \to \{0, 1\}$, $p \in D^T$ we shall use the the following notation

$$\begin{aligned} H_s^i &:= \{f \in D^T : f \mid s = i\} \\ G_s(p) &:= \{f \in D^T : f \mid s = p \mid s \text{ and } p^{-1}(0) \subset f^{-1}(0)\} \\ D_s(p) &:= \{f \in G_s(p) : |\{t \in T : f(t) \neq p(t)| \leqq \omega\} . \end{aligned}$$

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One can observe that

(I) $D_s(p)$ is a dense subset of $G_s(p)$ and if $p \in H_s^i$ then

(II) $D_s(p) \subset G_s(p) \subset H_s^i$.

Definition. A subset $X \subset D^T$ is said to be an ω-set iff for each $p \in X$ there exists an $s \subset T$ such that $|s| \leqq \omega$ and $G_s(p) \subset X$.

The following remark will play very important role though it is easy to prove:

(III) The intersection $\bigcap_{n=1}^{\infty} X_n$ of a countable family of

$$\omega\text{-sets } \mathrm{R}_1, X_2, \ldots \text{ is an } \omega\text{-set}.$$

Proof. Let $p \in \bigcap_{n=1}^{\infty} X_n$ and let us choose a sequence $s_1, s_2, \ldots$ of countable subsets of T such that for each n, $G_{s_n}(p) \subset X_n$. Put $s = \bigcup_{n=1}^{\infty} s_n$. The set s is countable and for each n, $G_s(p) \subset G_{s_n}(p) \subset X_n$. Thus, $G_s(p) \subset \bigcap_{n=1}^{\infty} X_n$.

It is obvious that each open subset of D^T is an ω-set. From (III) we get that each $\mathscr{G}_\delta$ subset of ω-set is an ω-set and, in consequence.

(IV) Each $\mathscr{G}_\delta$ subset of ω-set is an ω-set.

Definition. A space Y is said to be a weakly dyadic space if Y is a continuous image of a compact ω-set in D^T.

The remark (IV) yields

(V) Each compact $\mathscr{G}_\delta$ subset of a weakly dyadic space is a weakly dyadic space.

For each ω-set $X \subset D^T$ let us define

$$X_\omega := \{f \in X : |f^{-1}(1)| \leqq \omega\}$$

One can check that the set X_ω has the following properties:

(VI) X_ω is dense in each set $G_s(p) \subset X$ with $|s| \leqq \omega$, and

(VII) X_ω is dense in each $\mathscr{G}_\delta$ subset of X.

Murray G. Bell [1] has defined for any infinite collection T of sets, a space $Cent(T)$, by the following way:

$$Cent(T) := \{S : S \text{ is centered subcollection of } T\} \cup \{\emptyset\}.$$

The family $Cent(T)$ as a topological space can be identified with a subspace $X \subset D^T$;

$$X := \{f : f \text{ is a characteristic function of a centered subcollection of } T\}$$

The space X is a compact subspace of D^T. Notice that for each $p \in X$,

$$G_\phi(p) := \{f \in D^T : p^{-1}(0) \subset f^{-1}(0)\} \subset X.$$

Thus, the set X is an ω-set in the sense of our definition. To prove the main result we need two lemmas.

Lemma 1. Let us assume that $E \subset X$ is a closed subset and $F \subset X$ is an F_σ-subset of an ω-set $X \subset D^T$, $E \cap F = \emptyset$, such that

$$(*) \qquad E \not\subset Int_X(E \cup F)$$

Let us fix a finite set $\sigma = \{\alpha_1, \dots, \alpha_n\} \subset T$ of indexes.

Then there exist points $y \in X \setminus (E \cup F)$ and $z \in E$ such that; 1° $y(\alpha) = z(\alpha)$ for each $\alpha \in \sigma$, and 2° $|\{\alpha \in T: y(\alpha) \neq z(\alpha)\}| \leqq \omega$.

Proof. For each map $i: \sigma \to \{0, 1\}$ let us put

$$(1) \qquad V_i := H^i_\sigma .$$

Notice that

$$(2) \qquad D^T = \bigcup\{V_i: i: \sigma \to \{0, 1\}\} .$$

Define

$$(3) \qquad V := \bigcup\{V_i: X \cap V_i \subset E \cup F\} .$$

Since the set σ is finite, so the sets V_i, and in consequence V, are open in D^T. The condition $(*)$ implies that

$$(4) \qquad E \setminus V \neq \emptyset$$

because $E \subset V$ involves $E \subset V \cap X \subset Int_X(E \cup F)$, a contradiction with $(*)$. Choose a point $p \in E \setminus V$ and a set V_j, $j: \sigma \to \{0, 1\}$ such that

$$(5) \qquad p \in V_j .$$

From (3), (4) and (5) we infer that

$$(6) \qquad V_j \cap [X \setminus (E \cup F)] \neq \emptyset .$$

Let $s \in T$ be a countable set such that

$$(7) \qquad G_s(p) \subset X .$$

Since the set F is of $\mathscr{F}_\sigma$ type and $p \notin F$ we can find a countable subset $w \subset T$ and a map $i: w \to \{0, 1\}$ such that

$$(8) \qquad H^i_w \cap F = \emptyset, \quad p \in H^i_w, \quad s \cup \sigma \subset w, \quad i \,|\, \sigma = j .$$

We have

$$(9) \qquad p \in G_w(p) \subset H^i_w \subset H^j_\sigma = V_j .$$

There are two possibilities

$$(10) \qquad G_w(p) \subset E \quad \text{or} \quad G_w(p) \setminus E \neq \emptyset .$$

a) If $G_w(p) \subset E$ then choose points $y \in X_\omega \cap [V_j \setminus (E \cup F)]$ and $z \in X_\omega \cap G_w(p)$. Such a choice is possible because $X \cap [V_j \setminus (E \cup F)]$ is non-empty $\mathscr{G}_\delta$ subset of X and $X_\omega \cap G_w(p) \neq \emptyset$.

b) If $G_w(p) \setminus E \neq \emptyset$ then $D_w(p) \cap [G_w(p) \setminus E] \neq \emptyset$ because $E \subset X$ is a closed subset and $D_w(p)$ is dense subset of $G_w(p)$. Let $y \in D_w(p) \setminus E$ be an arbitrary point and let us put $z = p$.

The choice of the points $y \in X \setminus (E \cup F)$, $z \in E$ in the both cases completes the proof.

Lemma 2. Let $f: X \to^{onto} Y$, $g: X \to^{onto} Z$ be continuous and closed surjections such that the sets

$$A(x) := f^{-1}f[g^{-1}\,g(x)]\,, \quad x \in X$$

form a partition $W := \{A(x): x \in X\}$ of the space X. Let $q: X \to W$ be the quotient map onto the quotient space W.

Then the map q is closed and there are continuous closed maps $h_1: Y \to W$, $h_2: Z \to W$ such that

$$q = h_1 \circ f \quad \text{and} \quad q = h_2 \circ g\,.$$

Proof. a) We shall show that the quotient map q is closed. Let us fix a set $A(x)$ and an open set $U \subset X$ such that $A(x) \subset U$. It suffices to show that there exists an open set V, $A(x) \subset V \subset U$, such that for each $t \in X$

$$(1) \qquad A(t) \cap V \neq \emptyset \quad \text{implies} \quad A(t) \subset U\,.$$

The condition (1) is equivalent to the fact that q is closed and this means that

$$(2) \qquad \text{the set} \quad \{t \in U: q^{-1}\,q(t) \subset U\} \quad \text{is open}\,.$$

Let us put

$$(3) \qquad U_1 := \{t \in U: f^{-1}\,f(t) \subset U\}\,, \quad U_2 := \{t \in U: g^{-1}\,g(t) \subset U_1\}\,,$$
$$V := \{t \in U: f^{-1}\,f(t) \subset U_2\}\,.$$

Notice that the assumption that the sets $A(x)$, $x \in X$ form a partition implies

$$(4) \qquad A(x) \subset V \subset U_2 \subset U_1 \subset U$$

and

$$(5) \qquad f^{-1}f[A(x)] = A(x) = g^{-1}g[A(x)]\,.$$

Now, we shall verify that the condition (1) is fulfiled. Since

$$A(t) = \bigcup\{f^{-1}\,f(w): w \in g^{-1}\,g(t)\}$$

there exists a point $w \in g^{-1}\,g(t)$ such that $f^{-1}\,f(w) \cap V \neq \emptyset$. In consequence $f^{-1}\,f(w) = f^{-1}\,f(v)$ for some $v \in V$. By the definition of V; $f^{-1}\,f(w) = f^{-1}\,f(v) \subset U_2$. Hence $w \in U_2$ and $g^{-1}\,g(w) = g^{-1}\,g(t) \subset U_1$. Now, according to the definition of U_1, for each $u \in g^{-1}\,g(t)$ we get $f^{-1}\,f(u) \subset U$ i.e., $A(t) \subset U$.

b) Since for each $x \in X$ we have $f^{-1}\,f(x)$ and $g^{-1}\,g(x) \subset A(x)$, so maps h_i are uniquely determined by the conditions

$$q(x) = h_1[f(x)] \quad \text{and} \quad q(x) = h_2[g(x)]\,.$$

Since f and g are quotient maps, so the maps h_i are continuous, and closed because the map q is closed.

Theorem. If Y is a weakly dyadic space then for each countable set $C \subset Y$ there exists a continuous bijection $h: Y \setminus C \to^{onto}_{1-1} W$ onto a compact Hausdorff space W.

Proof. Let $X \subset D^T$ be a compact ω-set and $f: X \to^{onto} Y$ a continuous surjection. Let $C = \{c_1, c_2, \ldots\}$. Without loss of generality we may assume that $Int_Y C = \emptyset$ because

$$Y \setminus Int_Y C = \bigcap \{Y \setminus \{c_k\}: c_k \in Int_Y C\}$$

is a closed $\mathscr{G}_\delta$-subset of a weakly dyadic space and $Y \setminus Int_Y C$ is weakly dyadic, in consequence. Put $C_k = f^{-1}(c_k)$. Notice that

$$C_k \not\subset Int_X \bigcup_{n=1}^{\infty} C_n \quad \text{for each} \quad k = 1, 2, \ldots. \tag{1}$$

Indeed, if $C_k \subset Int_X \bigcup_{n=1}^{\infty} C_n$, then $Y \setminus C \subset f(X \setminus Int_X \bigcup_{n=1}^{\infty} C_n) \subset Y \setminus \{c_k\}$ and on the other hand since $Y \setminus C$ is a dense subset of Y and f is a closed map, we get $f(X \setminus Int_X \bigcup_{n=1}^{\infty} C_n) = Y$, a contradiction. The condition (1) implies that the sets

$$E = C_k \quad \text{and} \quad F = \bigcup_{n \neq k} C_n \cup \bigcup_{n<k} f^{-1} f(y^n), \tag{2}$$

where $f^{-1}(y^1), \ldots, f^{-1}(y^{k-1}) \subset X \setminus \bigcup_{n=1}^{\infty} C_n$, satisfy the condition

$$E \not\subset Int_X(E \cup F). \tag{3}$$

Indeed, suppose that $E \subset Int_X(E \cup F)$. But then, since $C_k \cap \bigcup_{n<k} f^{-1} f(y^n) = \emptyset$ we get that $E = C_k = C_k \setminus \bigcup_{n<k} f^{-1} f(y^n) \subset Int_X \bigcup_{n=1}^{\infty} C_n$ a contradiction with (1).

In view of Lemma 1 we can define by induction points y^k, z^k and sets $A_k = \{\alpha_1^k, \alpha_2^k, \ldots\} \subset T$ such that

(a) $y^k \in X \setminus [\bigcup_{n=1}^{\infty} C_n \cup \bigcup_{n<k} f^{-1} f(y^n)]$, $z^k \in C_k$,
(b) $y^k(\alpha) = z^k(\alpha)$ for each $\alpha \notin A_k$,
(c) $y^k(\alpha_i^j) = z^k(\alpha_i^j)$ for each $i, j < k$.

Consider a decomposition Z of X into sets $\{y^k, z^k\}$ and one-point sets $\{x\}$, $x \notin \{y^1, y^2, \ldots, z^1, z^2, \ldots\}$. Let us check that the quotient map $g: X \to Z$ is closed. To see this, it suffices to verify that for each neighbourhood H_v^i, $v = \{\alpha_1, \ldots, \alpha_n\}$, $i: v \to \{0, 1\}$, of a point $x \in X$ there exists an m such that

$$g^{-1} g[H_v^i \setminus \{y^k, z^k: k < m\}] \subset H_v^i.$$

Choose an m such that $v \cap \bigcup_{n=1}^{\infty} A_n \subset \{\alpha_i^j: i, j \leq m\}$. Let us see that by the condition

(c) for each $\alpha \in v \cap \bigcup_{k=1}^{\infty} A_k$ and $n > m$ we have $y^n(\alpha) = z^n(\alpha)$, and by the condition (b); $y^k(\alpha) = z^k(\alpha)$ for each $\alpha \in v \setminus \bigcup_{n=1}^{\infty} A_n$. Thus $y^n \in H_v^i$ iff $z^n \in H_v^i$ for each $n > m$.

According to lemma 2 the map $h_1: Y \to W$ is continuous, so the space $W = h_1(Y)$ is compact Hausdorff as a closed and continuous image of compact Hasudorff space. The map $h = h_1 \mid Y \setminus C$ is the required bijection.

Final remarks. In 1936 P. S. Aleksandrov introduced the class of dyadic spaces being continuous images of a generalized Cantor discontinuum. The class of the spaces is a natural generalization of the class of compact metric spaces and it is closed with respect to Cartesian products and continuous images. This class has a lot of nice properties and it was the subject of many papers. In 1970 S. Mrówka generalized the class of dyadic spaces defining the class of polyadic spaces (= the continuous images of the products of the one point compactifications of discrete spaces). Centered spaces, more wider class of spaces than the class of polyadic spaces was considered in 1985 by M. G. Bell. The common feature of these generalizations is that many theorems which were originally proved for the class of dyadic spaces holds for their generalizations. The authors have defined the class of weakly dyadic spaces which contains the class of spaces in sense of Bell. We have proved that the Belugin theorem holds for weakly dyadic spaces. It has been observed that one of the most important theorems for dyadic spaces, which says that each closed $\mathscr{G}_\delta$ subset of a dyadic space is dyadic, holds for weakly dyadic spaces (it is still not known if the theorem is valid for centered spaces).

We would like to complete this note with an example of a weakly dyadic space which is not a centered space.

Example. Consider the Cantor cube $D^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\omega}$. Choose a subset $S \subset \mathfrak{c}$ such that $|\mathfrak{c} \setminus S| = \omega$. Define

$$H := \{f \in D^{\mathfrak{c}} : f \mid S = 0\} .$$

It is clear that the set H is homeomorphic to the Cantor cube D^{ω}, so the cardinality of H is equal to $\mathfrak{c}$. Thus we can denumerate points from the set H by indexes from the set S;

$$H = \{x^{\alpha} : \alpha \in S\} .$$

Now, let us define

$$M := \{f \in D^{\mathfrak{c}} : \exists \alpha \in S \text{ such that } f(\alpha) = 1 \text{ and } f \mid \mathfrak{c} \setminus \{\alpha\} = x^{\alpha} \mid \mathfrak{c} \setminus \{\alpha\}\} .$$

The subspace $X := H \cup M$ of the sube $D^{\mathfrak{c}}$ is a closed subspace and satisfies the first axiom of countability i.e., $\chi(X) = \omega$. The weight of the space X is equal to continuum, $w(X) = \mathfrak{c}$, because $M \subset X$ is a discrete subspace and $|M| = \mathfrak{c}$. One can verify that the space X is weakly dyadic (more precisely, X is a compact ω-set in $D^{\mathfrak{c}}$). On the other hand the space X cannot be centered because how it was proved in [1] if X is centered then $w(X) = \chi(X)$.

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