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# Bijections of Scattered Spaces onto Compact Hausdorff

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We shall prove that each scattered and hereditarily paracompact space has a continuous bijection onto a compact Hausdorff space.

In 1935 Stefan Banach posed the following question [see, Colloquium Mathematicum 1, Problem 26 or R. D. Mauldin, The Scottish Book, Problem 1].

When can a metric space have a continuous and one-to-one map onto a compact metric space?

Katětov [3] was one of the first who attacked the Banach problem. He proved that:

A countable regular space has a continuous bijection onto a compact metric space iff it is scattered.

In 1976 Kulpa [4] observed that each completely metrizable space $X$, with $\dim X = 0$, has a continuous bijection onto a compact Hausdorff space. On the other hand, it is known that there exists a $G_\delta$ subset of the plane $\mathbb{R}^2$ which has no continuous bijection onto any compact Hausdorff space. This suggests that the Banach problem can become interesting even for zerodimensional spaces.

The purpose of our paper is to improve the Katětov result. Let us explain the terminology used in the paper. A map $f: X \to Y$ is said to be bijective iff it is one-to-one and onto. A topological space $X$ is said to be scattered iff every non-empty subset $A \subset X$ has an isolated point. For each $A \subset X$ let us denote

$$A^d := \{x \in A: x \text{ is an accumulation point of the set } A\}$$

i.e.

$$A^d := \{x \in A:\ x \in cl_X(A \setminus \{x\})\}$$

and let us put for each ordinal

$$X^{(1)} := X^d\,, \quad X^{(\alpha+1)} := [X^{(\alpha)}]^d$$

and when $\alpha$ is a limit ordinal

$$X^{(\alpha)} := \bigcap\{X^{(\beta)}: \beta < \alpha\}\,.$$

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Let us observe that a topological space $X$ is scattered iff there exists an ordinal $\alpha$ such that $X^{(\alpha)} = \emptyset$.

A proof of the main result will be preceded by three lemmas.

**Lemma 1.** Each locally compact Hausdorff space $X$ has a continuous bijection $f: X \to Y$ onto a compact Hausdorff space $Y$.

**Proof.** Let $Y$ be a topological space obtained from $X$ by inducing a new topology on the set $X$ in the following way: Choose a point $x_0 \in X$. Define a set $U \subset X$ to be open in the space $Y$ iff $U$ is open in the space $X$ and $X \setminus U$ is compact whenever $x_0 \in U$. Such obtained space $Y$ is compact, Hausdorff and the identity map $f: X \to Y$, $f(x) = x$, is continuous.

**Lemma 2.** Let $a \in X$ be a point of a regular space $X$ such that: 1. each open subset $U \subseteq X \setminus \{a\}$ has a continuous bijection onto a locally compact Hausdorff space, 2. the space $X \setminus \{a\}$ is paracompact and $\dim (X \setminus \{a\}) = 0$.

Then the space $X$ has a continuous bijection $f^*: X \to Y^*$ onto a compact Hausdorff space $Y^*$..

**Proof.** For each point $x \neq a$ let us choose an open neighbourhood $U_x$ if $x$ such that $a \notin cl_X U_x$. Now, according to the assumption 2 there exists a covering $\{U_s: s \in S\}$ of the set $X \setminus \{a\}$, consisting of pairwise disjoint clopen sets in $X \setminus \{a\}$, being a refinement of the covering $\{U_x: x \in X \setminus \{a\}\}$. For each $s \in S$ choose a continuous bijection $f_s: U_s \to Y_s$ onto a locally compact Hasudorff space $Y_s$. The family $\{f_s: s \in S\}$ of maps induces a continuous bijection $f: X \setminus \{a\} \to Y$ onto a locally compact Hasudorff space $Y := \oplus \{Y_s: s \in S\}$. Choose a point $a^* \notin Y$. On the set $Y^* := Y \cup \{a^*\}$ let us define a compact Hausdorff topology such that the space $Y^*$ will become the Aleksandroff compactification of the space $Y$, i.e. $U \subset Y^*$ is open in $Y^*$ iff $U \subset Y$ is open in $Y$ or $U \not\subset Y$ and $Y \setminus U$ is a compact subset in $Y$. Next, let us extend the map $f: X \setminus \{a\} \to Y \subset Y^*$ to a map $f^*: X \to Y^*$, putting $f^*(a) := a^*$. It is obvious that $f^*$ is a bijection. Let us check that $f^*$ is a continuous map. Assume that $V \subset Y^*$ is an open set. If $a^* \notin V$ then $V \subset Y$ is an open set in $Y$ and by continuity of $f$, $(f^*)^{-1}(V) = f^{-1}(V)$ is an open set in $X \setminus \{a\}$ and in consequence $f^{-1}(V)$ is an open set in $X$. If $a^* \in V$ then $V = Y^* \setminus Z$ where $Z \subset Y$ is a compact subset. We have

$$(f^*)^{-1}(V) = (f^*)^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z).$$

From the above it follows that it suffices to show that $f^{-1}(Z)$ is closed in $X$. Since $f: X \setminus \{a\} \to Y$ is continuous, so $f^{-1}(Z)$ is closed in $X \setminus \{a\}$. The proof will be completed if we check that $a \notin cl_X f^{-1}(Z)$. The sets $Y_s$, $s \in S$, are open in $Y$. By compactness of $Z$ there exists a finite set of indexes $s_1, \ldots, s_n \in S$ such that $Z \subset Y_{s_1} \cup \ldots \cup Y_{s_n}$ and hence $f^{-1}(Z) \subset U_{s_1} \cup \ldots \cup U_{s_n}$. But since $a \notin cl_X U_s$ for each $s \in S$ we get

$$a \notin cl_X f^{-1}(Z) \subset cl_X[U_{s_1} \cup \ldots \cup U_{s_n}] = cl_X U_{s_1} \cup \ldots \cup cl_X U_{s_n}.$$

**Lemma 3.** For each scattered and hereditarily paracompact space $X$ the covering dimension is equal to zero, $\dim X = 0$.

**Proof.** Since $X$ is a scattered space so $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. The lemma will be proved by induction on a number $\alpha$ such that $X^{(\alpha)} = \emptyset$.

1) $\alpha = 1$. Let $X^{(1)} = \emptyset$. This means that $X$ is a discrete space and it is clear that $\dim X = 0$.

2) $\alpha > 1$. Assume that for each ordinal $\beta < \alpha$ and for each hereditarily paracompact space $X$, $X^{(\beta)} = \emptyset$ implies $\dim X = 0$. We shall show that $X^{(\alpha)} = \emptyset$ implies $\dim X = 0$, too. There are two possibilities:

(a) $\alpha$ is a limit ordinal or (b) $\alpha = \beta + 1$ is a successor of an ordinal $\beta$.

(a) Assume that $\alpha$ is a limit ordinal and $X^{(\alpha)} = \emptyset$. Since for each $\beta < \alpha$ the sets $X^{(\beta)}$ are closed and $X^{(\alpha)} = \cap \{X^{(\beta)}: \beta < \alpha\} = \emptyset$ we get that $X = \bigcup\{U_\beta: \beta < \alpha\}$, where $U_\beta = X \setminus X^{(\beta)}$ is an open set for each $\beta < \alpha$. But $(U_\beta)^{(\beta)} = U_\beta \cap X^{(\beta)} = \emptyset$. In view of the inductive assumption we infer that for each $\beta < \alpha$, $\dim U_\beta = \emptyset$. Now let the family $\{U_s: s \in S\}$ be an open covering of $X$ such that for each $s \in S$ there exists a $\beta < \alpha$ with $cl_X U_s \subset U_\beta$. It is obvious that for each $s \in S$, $\dim cl_X U_s = 0$.

Dowker and Nagami [see [2], p. 214] independently have proved that:

If a weakly paracompact normal space $X$ can be represented as the union of a family $\{U_s: s \in S\}$ of open subspaces such that $\dim \mathrm{cl}_X U_s \leqq n$ for $s \in S$, then $\dim X \leqq n$. From the above we infer that for our space $X$, $\dim X = 0$.

(b) Assume now that $\alpha = \beta + 1$ and $X^{(\alpha)} = \emptyset$. This implies that $X^{(\beta)}$ is discrete and closed subspace of $X$. In order to show that $\dim X = 0$ it suffices to verify that each point $x \in X$ has an open neighbourhood $U$ such that $\dim U = 0$. If $x \notin X^{(\beta)}$ then we may put $U := X \setminus X^{(\beta)}$ because the set $U$ is open, $U^{(\beta)} = U \cap X^{(\beta)} = \emptyset$ and by the inductive assumption $\dim U = 0$. Now, let $a \in X^{(\beta)}$ and $U$ be an open neighbourhood of $a$. Since $X^{(\beta)}$ is a discrete subspace of $X$ we may assume that $U \cap X^{(\beta)} = \{a\}$. Let us notice first that if the point $a$ has a base consisting of clopen sets then $\dim U = 0$. Indeed, let $\{V_t: t \in T\}$ be an arbitrary open covering of the set $U$. Choose a clopen set $U_0 \subset U$, $a \in U_0$ such that $U_0 \subset V_{t_0}$ for some $t_0 \in T$. The family $\{V_t \setminus U_0: t \in T\}$ is an open covering of the set $U \setminus U_0$. But according to the inductive assumption $\dim (U \setminus U_0) = 0$ because

$$[U \setminus U_0]^{(\beta)} \subset [U \setminus \{a\}]^{(\beta)} = (U \setminus \{a\}) \cap X^{(\beta)} = \emptyset .$$

Hence, there exists an open covering $\{U_s: s \in S\}$, $0 \notin S$, of $U \setminus U_0$ consisting of pairwise disjoint sets and being the refinement of the covering $\{V_t \setminus U_0: t \in T\}$. It is clear that the family $\{U_s: s \in S\} \cup \{U_0\}$ is an open covering of $U$ consisting of pairwise disjoint sets and being of refinement of $\{V_t: t \in T\}$.

Thus the proof will be completed if we can show that the point a has a neighbourhood base consisting fo clopen sets.

Let $U$ be an open neighbourhood of $a$. By regularity of $X$ there exist open sets $W$, $V$

such that

$$a \in W \subset cl_X W \subset V \subset cl_X V \subset U .$$

Next, choose for each $x \in U \setminus \{a\}$ an open set $O_x$ such that

(1) $x \in O_x \subset cl_X O_x \subset U \setminus \{a\}$,

(2) $O_x \cap W \neq \emptyset \Rightarrow O_x \subset V$.

Since $[U \setminus \{a\}]^{(\beta)} = \emptyset$ hence in view of the inductive assumption we have $\dim [U \setminus \{a\}] = 0$. And this means that there exists a covering $\{W_s: s \in S\}$ of $U \setminus \{a\}$ consisting of pairwise disjoint and clopen in $U \setminus \{a\}$ sets and being a refinement of the covering $\{O_x: x \in U \setminus \{a\}\}$. The condition (1) implies that the sets $W_s$, $s \in S$ are clopen in $X$, too. Let us define

$$A := \{a\} \cup \bigcup \{W_s: W_s \cap W \neq \emptyset\} .$$

Since the sets $W_s$, $s \in S$, satisfy the condition (2), so the set $A$ has the following property

$$A = W \cup \bigcup \{W_s: W_s \cap W \neq \emptyset\} \subset V .$$

From the above we infer that $A$ is an open neighbourhood of the point $a$. On the other hand, the set $A$ is closed in $U$ because the set $U \setminus A = \bigcap \{W_s: W_s \cap W = \emptyset\}$ is an open subset of $U$. The set $A$ is closed in $X$ because $A \subset V \subset cl_X X \subset U$.

Thus the proof that $\dim X = 0$ is completed.

**Theorem.** Each scattered and hereditarily paracompact space $X$ has a continuous bijection onto a compact Hausdorff space $Y$.

**Proof.** In virtue of the previous remarks the assumptions of the space $X$ imply that $\dim X = 0$ and there exists an $\alpha$ such that $X^{(\alpha)} = \emptyset$.

We shall prove by induction that if $X^{(\alpha)} = \emptyset$ then $X$ has a continuous bijection onto a compact Hausdorff space $Y$.

1) $\alpha = 1$. Let $X^{(1)} = \emptyset$. This means that $X$ is a discrete space and in view of the lemma 1, $X$ has a continuous bijection onto a compact Hausdorff space.

2) $\alpha > 1$. Assume that for each $\beta < \alpha$, $X^{(\beta)} = \emptyset$ implies that $X$ has a continuous bijection onto a compact Hausdorff space. We shall show that if $X^{(\alpha)} = \emptyset$ then $X$ has such a bijection, too. There are two possibilities:

(a) $\alpha$ is a limit ordinal or (b) $\alpha = \beta + 1$ is a successor of an ordinal $\beta$.

2a) Assume that $\alpha$ is a limit ordinal and $X^{(\alpha)} = \emptyset$. Since $\emptyset = X^{(\alpha)} = \bigcap \{X^{(\beta)}: \beta < \alpha\}$ we get $X = \bigcup \{U_\beta: \beta < \alpha\}$ where the sets $U_\beta = X \setminus X^{(\beta)}$, $\beta < \alpha$, are open in $X$. But $\dim X = 0$ implies that there exists an open covering $\{U_s: s \in S\}$ of $X$ being a refinement of $\{U_\beta: \beta < \alpha\}$ consisting of pairwise disjoint sets $U_s$. The sets $U_s$, $s \in S$ have the following property:

(*) for each $s \in S$ there exists a $\beta < \alpha$ such that $U_s^{(\beta)} = \emptyset$. Now, from the inductive assumption for each $s \in S$ there exists a continuous bijection $f_s: U_s \to Y_s$ onto a compact Hausdorff space $Y_s$. The maps $f_s$, $s \in S$, induce a continuous bijection $f: X \to \oplus\{Y_s: s \in S\}$ onto a locally compact Hausdorff space $Y = \oplus\{Y_s: s \in S\}$ and by the lemma 1 $X$ has a continuous bijection onto a compact Hausdorff space.

2b) Assume that $\alpha = \beta + 1$ and $X^{(\alpha)} = \emptyset$. This implies that $X^{(\beta)}$ is a closed and discrete subspace of $X$. For each $x \in X$ let us choose an open set $U_x \subset X$, $x \in U_x$, such that

(1) if $x \notin X^{(\beta)}$ then $U_x \cap X^{(\beta)} = \emptyset$ (for example let $U_x = X \setminus X^{(\beta)}$),

(2) if $x \in X^{(\beta)}$ then $U_x \cap X^{(\beta)} = \{x\}$ (such a choice is possible because $X^{(\beta)}$ is a discrete subspace of $X$).

Let $\{U_s: s \in S\}$ be an open covering of $X$ consisting of pairwise disjoint sets and being a refinement of the covering $\{U_x: x \in X\}$. As in the above, it suffices to show that for each $s \in S$ there exists a continuous bijection $f_s: U_s \to Y_s$ onto a compact Hausdorff space $Y_s$. There are two possibilities:

(a) $U_s \cap X^{(\beta)} = \emptyset$. But then $(U_s)^{(\beta)} = \emptyset$ and according to the inductive assumption there exists a continuous bijection $f_s: U_s \to Y_s$ onto a compact Hausdorff space $Y_s$.

(b) $U_s \cap X^{(\beta)} = \{a\}$. But then the space $U_s$ and the point $a$ fulfil the assumption of the lemma 2, so in this case $U_s$ has such a bijection, too.

The class of hereditarily paracompact spaces contains countable regular spaces and metric spaces. Thus we get the following

**Corollary** (Katětov): Each countable regular scattered space has a continuous bijection onto a compact metric space.

**Corollary.** Each scattered metric space has a continuous bijection onto a compact Hausdorff space.

## References


[1] Problem 26, Colloquium Mathematicum 1 (1948), p. 150.

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[5] Mauldin R. D.: The Scottish Book, Birkhäuser, Boston, 1981.