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A. E. Kipiani

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Some Combinatorial Problems, Connected with Product-isomorphisms of Binary Relations

A. E. KIPIANI*)

Tbilisi, USSR

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Let E be a base set; two subsets A and B of the set E^2 are said to be product-isomorphic (denoted by $\Pi(A, B)$) if there exists a bijection $f: E \to^{1-1} E$ such that $\hat{f}(A) = B$ (where $\hat{f}((x, y)) = (f(x), f(y))$.

In the book of S. Ulam [1] the following two problems are raised:

Problem 1. Let E be an infinite set and $A \subset E^2$; find the cardinality of the set of all subsets in E^2, which are product-isomorphic with the set A.

Problem 2. Assume that E is a continual set, and n is a natural number. Does there exist, for every n, a set having exactly n product-automorphisms?

Introduce the notations:

$$\mathfrak{p}(A) = Card\,\{X \mid X \subset E^2 \,\&\, \Pi(X, A)\}\,;$$
$$\mathfrak{p}_a(A) = Card\,\{f \mid f: E \to^{1-1} E \,\&\, \hat{f}(A) = A\}\,.$$

In connection with Problem 1 we should mention the paper by Kharazishvili [2] where, in particular, for the validity of GCH we have the following relation

$$(\forall A)\,(A \subset E^2 \Rightarrow \mathfrak{p}(A) \in \{1, Card\, E, 2^{Card\, E}\})\,.$$

The paper deals also with geometric characteristics of types of A sets for which the function $\mathfrak{p}(\cdot)$ assumes, respectively, values 1, $Card\, E$ and $2^{Card\, E}$.

Since the general solution of Problem 1 depends on the generalized continuum hypothesis, the consideration of some particular cases of this problem is of a certain interest.

Suppose we have mapping $f: E \to E$; it is well known that the tree whose vertices are the elements of the set E, corresponds to this mapping. It is also evident that the mapping graph f is a uniform set in E^2. Hence, the questions of product-isomorphisms of uniform sets are closely connected with similar questions on tree isomorphisms. The following theorem (Kipiani, Tsakadze) holds.

*) The Georgian Institute of Management of National Economy 6, Chitadze, Tbilisi, USSR.

Theorem 1. Let E be an infinite base set, $Card\, E = \mathfrak{a}$, let Δ_E be diagonal of E^2 and let U be a uniform with respect to the i-th direction subset in E^2 $(i = 1, 2)$. Then

1) if $Card\,(\Delta_E \cap U) = \mathfrak{a}\ \&\ Card\,(\Delta_E \setminus U) = \mathfrak{b}$ then $\mathfrak{p}(U) = \mathfrak{a}^{\mathfrak{b}}$;
2) if $(\exists l)\,(l \subset E^2\ \&\ (l = \{x\} \times E \vee l = E \times \{x\})\ \&\ Card\,(l \cap U) =$ $= \mathfrak{a}\ \&\ Card\,(l \setminus U) = \mathfrak{b})$, then $\mathfrak{p}(U) = \mathfrak{a}^{\mathfrak{b}+1}$;
3) if $Card\ pr_i U = \mathfrak{b} < \mathfrak{a}$, then $\mathfrak{p}(U) = \mathfrak{a}^{\mathfrak{b}}$;
4) in all the remaining cases $\mathfrak{p}(U) = 2^{\mathfrak{b}}$.

Let us further identify the mapping $f: E \to E$ with the graph and denote by $\mathscr{S}(E)$ the group of all permutations of the set E. Then as easily seen, if $f \in \mathscr{S}(E)$, the cardinal number $\mathfrak{p}(f)$ coincides with the cardinality of the set of all elements conjugated to f element in the group $\mathscr{S}(E)$ and $Card\,(\mathscr{S}(E)/\Pi(\ ,\ ))$ is the cardinality of maximal (with respect to inclusion) family of pairwise nonconjugated elements of the group $\mathscr{S}(E)$.

In [3] it is proved that if $Card\, E = \mathfrak{a} \geqq \omega_0$, $\mathfrak{b} = Card\,(f \cap \Delta_E)$ and $\mathfrak{c} =$ $= Card\,(\Delta_E \setminus f)$ then $\mathfrak{p}(f) = \mathfrak{a}^{min\{\mathfrak{b}, \mathfrak{c}\}} . 2^{\mathfrak{c}}$.

The equality $Card\,(\mathscr{S}(E)/\prod(\ ,\ )) = (Card\, \alpha + \omega_0)^{\omega_0}$ (where $Card\, E = \omega_\alpha$) is also proved there.

The paper [2] contains the following result: for any group G, whose cardinality is less or equal to $Card\, E$, one can find a digraph $A \subset E^2$ such that the group of all automorphisms of this digraph is isomorphic with the group G (see the proof in [4], *p.* 54–60). This statement implies the following

Corollary. For any infinite base set E and for any cardinal number $\mathfrak{r} \in\,]\,0, Card\, E\,]$ there exists a digraph $A \subset E^2$ with exactly $\mathfrak{r}$ product-automorphisms.

This result for natural $\mathfrak{r}$ may be easily proved directly. Any such proof however uses the axiom of choice. The question naturally arises: may Problem 2 be solved effectively, i.e. without the axiom of choice?

It turns out that the proof of the following result may be effectively carried out.

Theorem 2. (ZF) Assume that E is a base set, and n is a positive natural number. Then if $Card\, E \in \{\omega_\alpha, 2^{\omega_\alpha}, 2^{2^{\omega_\alpha}}, \ldots\}$, there exists a digraph $A \subset E^2$, with exactly n product-automorphisms.

The following result, in spite of simplicity of the proof, is useful for applications.

Theorem 3. Let E be a base set and $A \subset E^2$. Then

$$\mathfrak{p}(A) \,.\, \mathfrak{p}_a(A) = \begin{cases} 2^{Card\, E} & \text{if} \quad Card\, E \geqq \omega_0 \\ (Card\, E)! & \text{if} \quad Card\, E < \omega_0 \end{cases}$$

Corollary 1. If R is the well ordering relation on the infinite set E, then $\mathfrak{p}(R) =$ $= 2^{Card\, E}$.

Corollary 2. The cardinality of the set of all subsets A, which are the solutions of Problem 2, is equal to the $2^{Card\,E}$.

It should be noted finally that any algebraic system

$$\mathscr{A} = (E; f_1, f_2, \ldots, f_k;\ r_1, r_2, \ldots, r_n)$$

may be represented in E^m for some $m \geqq 1$. Thus, two algebraic systems on the base set E will be isomorphic if and only if the corresponding subsets will be product-isomorphic in E^m (see [1], p. 18–19). Hence, the m-dimensional analogue of Theorem 3 which is also a true statement, asserts that the product of two cardinal numbers, first of which is the number of all isomorphic with $\mathscr{A}$ systems on E, and the second is the number of all automorphisms of the system $\mathscr{A}$, is equal to the $2^{Card\,E}$ (or $(Card\ E)!$ if $Card\ E < \omega_0$).

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