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The Dimension of Analytic Sets

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Let (X, d) be a metric space. For a subset $A \subseteqq X$ and a positive real number $q > 0$ we consider

$$(1) \qquad N(A, q) = \inf \{k \in \mathbb{N};\ \exists A_1, \dots, A_k,\ \operatorname{diam}(A_i) \leqq q \text{ for } i = 1, \dots, k \ \bigcup_{i=1}^{i=k} A_i \supseteqq A\}$$

the so called covering number of A. Two notions of dimension based on covering numbers were introduced by Wegmann [9] respectively Kahnert [7]:

$$(2) \qquad \mathbf{dim}_m(X) = \inf \{\alpha > 0;\ \exists (X_n)_{n \in \mathbb{N}},\ X = \bigcup_{n=1}^{\infty} X_n \Rightarrow \limsup_{q \to 0} N(X_n, q)\, q^\alpha = 0\}$$

respectively

$$(3) \qquad \mathit{dim}_m(X) = \inf \{\alpha > 0;\ \exists (X_n)_{n \in \mathbb{N}},\ X = \bigcup_{n=1}^{\infty} X_n \Rightarrow \liminf_{q \to 0} N(X_n, q)\, q^\alpha = 0\}\,.$$

They called them upper resp. lower metric dimension. We remark that for countable X $\mathbf{dim}_m(X) = \mathit{dim}_m(X) = 0$ always holds. One reason to consider alternative notions of dimension is that the most used Hausdorff dimension has a bad behaviour w.r.t. sets which are large in the sense of category. Remember that the Hausdorff dimension is defined by

$$(4) \qquad \dim_H(X) = \inf \{\alpha > 0;\ m^\alpha(X) = 0\}$$

where

$$(5) \qquad m^\alpha(A) = \sup_{\varepsilon > 0} \inf \{\sum_{n=1}^{\infty} (\operatorname{diam}(A_n))^\alpha;\ A \subseteqq \bigcup_{n=1}^{\infty} A_n,\ \operatorname{diam}(A_n) \leqq \varepsilon\}$$

is the α-dimensional Hausdorff measure for a subset $A \subseteqq X$. One may also consider all these dimensions in a generalized sense [4] [5]. Let us present an example describing the situation mentioned above.

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Example

Let

$$\mathbb{L} = \left\{x;\ x \text{ irrational such that } \forall n \in \mathbb{N}\ \exists \text{ rational } \frac{p}{q} \text{ such that } \left|x - \frac{p}{q}\right| < \frac{1}{q^n}\right\}$$

be the set of so called Louiville numbers. Then $\mathbb{L}$ is a dense G_δ – set of $\mathbb{R}$ equipped with the Euclidean metric. We obtain $\dim_H(\mathbb{L}) = 0$ [8], but $\mathit{dim}_m(\mathbb{L}) = \mathbf{dim}_m(\mathbb{L}) = 1$ [5].

It is clear that

$$\dim_H(X) \leqq \mathit{dim}_m(X) \leqq \mathbf{dim}_m(X) \tag{6}$$

and it was easy to give examples of spaces (X, d) such that all three numbers coincide (for instance self similar sets [3]).

The basic difference of $\dim_H$ and dim_m was found out by Cajar and Sandau [1]:

Theorem 1

Let A be a closed subset of a complete separable metric space (X, d) satisfying $0 \leqq \mathit{dim}_m(A) < \infty$. Then there is a closed subset D of A such that $\dim_H(D) = 0$ and $\mathit{dim}_m(D) = \mathit{dim}_m(A)$ hold.

This result is applied to obtain

Theorem 2 [1]

For given real $\gamma, \delta, \varepsilon$ with $0 \leqq \gamma \leqq \delta \leqq \varepsilon \leqq 1$ there is a perfect subset P of the reals satisfying

$$\dim_H(P) = \gamma\,, \quad \mathit{dim}_m(P) = \delta \quad \text{and} \quad \mathbf{dim}_m(P) = \varepsilon\,.$$

One problem in using dimension is to find compact sets K within a given set A with a dimension close to the dimension of the given one. To make this precise let A be an analytic set, i.e. a continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$, does there exist compact sets $K \subseteqq A$ such that $\dim(K)$ is arbitrarily close to $\dim(A)$? dim stands for one of the introduced three notions. Since analytic sets cover the most classes of sets, for instance all Borel sets, this seems to be a sufficient approach. The positive answer is wellknown for the Hausdorff dimension for a longer time and it has been an advantage for the use of Hausdorff dimension.

Theorem 3 [2]

Let A be an analytic set of a complete separable metric space. Then $\dim_H(A) = \sup\{\dim_H(K);\ K \subseteqq A,\ K \text{ compact}\}$.

In [2] it is actually proved that m^α is a tight measure $(\alpha > 0)$. Under the same assumptions as for Theorem 3 we could obtain

Theorem 4 [6]

a) $\mathbf{dim}_m(A) = \sup\{\mathbf{dim}_m(K);\ K \subseteqq A \text{ compact}\}$,

b) $\mathit{dim}_m(A) < \infty \Rightarrow \mathit{dim}_m(A) = \sup\{\mathit{dim}_m(K);\ K \subseteqq A \text{ compact}\}$.

The proof uses a refined method of [1]. The restriction for b) is due to the fact that for $\mathit{dim}_m(A) = \infty$ it may happen that A is non-σ-totally bounded (i.e. it is not the union of countable many totally bounded sets) and then our proof in [6] breaks down. This case is an open problem.

Finally, we would like to discuss the condition under which $\dim(A) = \dim(K)$ holds, where $K \subseteqq A$ is a fixed compact set.

To unify our definitions (1) (2) and (3) we introduce certain outer measures, namely

$$m_N^\alpha(A) = \inf\{\sum_{n=1}^{\infty} \liminf_{q\to 0} N(A_n, q)\, q^\alpha;\ A \subseteqq \bigcup_{n=1}^{\infty} A_n\} \tag{6}$$

for $A \subseteqq X$ and $\mathbf{m}_N^\alpha$, where liminf is replaced by limsup in the last formula. It is then easy to see that

$$\mathbf{dim}_m(A) = \inf\{\alpha;\ m_N^\alpha(A) = 0\} \tag{7}$$

respectively

$$\mathbf{dim}_m(A) = \inf\{\alpha;\ \mathbf{m}_N^\alpha(A) = 0\}\,. \tag{8}$$

If one defines for a point $p \in X$

$$\dim(A; p) = \inf\{\dim(A \cap U);\ U \text{ is a neighbourhood of } p\} \tag{9}$$

as the local dimension of p, then we have

Theorem 5

Let A be an analytic subset of some complete separable metric space. Assume $\dim(A) < \infty$ if $\dim = \mathit{dim}_m$. Then there are equivalent:

a) There is a compact set $K \subseteqq A$ such that $\dim(K) = \dim(A)$.

b) There exists a point $p \in A$ such that $\dim(A; p) = \dim(A)$.

Proof. b) $\Rightarrow$ a)

Since $\dim(A; p) \leqq \dim(A)$ we obtain a sequence of neighbourhoods $(U_n)_{n\in\mathbf{N}}$ of p satisfying $\operatorname{diam}(U_n) \downarrow 0$ and $\dim(U_n \cap A) = \dim(A)$. Apply Theorem 3 and 4 to find compact sets $K_n \subseteqq U_n \cap A$ ($U_n \cap A$ is analytic) such that $\dim(K_n) \to$ $\to \dim(A) = \dim(U_n \cap A)$.

Then $K = \bigcup_{n=1}^{\infty} K_n \cup \{x\}$ is a compact subset such that $\dim(A) = \dim(K)$.

a) $\Rightarrow$ b) For $\dim(A) = 0$ is nothing to prove. If $\dim(A) > 0$ choose a sequence $(\beta_n)_{n\in\mathbf{N}}$ with $\beta_n \uparrow \dim(A)$. Let m_n the outer measure according to β_n, i.e. $m_n = m^\beta n$, $m_n = m_N^{\beta_n}$ and $m_n = \mathbf{m}_N^{\beta_n}$, and consider the support $\operatorname{supp}(m_n)$ of the outer measure m_n in A defined as

$$\operatorname{supp}(m_n) = A - \bigcup\{U;\ U \text{ open},\ m_n(A \cap U) = 0\}\,. \tag{10}$$

We claim that $\bigcap_{n=1}^{\infty} \text{supp}(m_n)$ is an non-empty set. We prove this by showing

$$\bigcap_{n=1}^{\infty} \text{supp}(m_n) \cap K \neq \emptyset .$$

Otherwise there are $m_1, \ldots, m_l$ (outer measures) such that

$$\bigcap_{i=1}^{l} \text{supp}(m_i) \cap K = \emptyset$$

by the compactness of K. Hence, K can be covered by open sets U_i with $m_i(U_i \cap K) = 0$ for $i = 1, \ldots, l$. It is easy to see that

$$m_j(U_i \cap K) = 0$$

for $i = 1, \ldots, l$ and $j > l$. Hence $m_j(K) = 0$ and this implies

$$\dim(K) \leqq \beta_{l+1} < \dim(A)$$

as a contradiction. For $p \in \bigcap_{n=1}^{\infty} \text{supp}(m_n) \cap K$ we obtain for all neighbourhoods U of p

$$\dim(U \cap K) \geqq \dim(K) .$$

Hence

$$\dim(A; p) = \dim(A) . \quad \blacksquare$$

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