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Generic Well-Posedness in Some Classes of Optimization Problems

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In this paper we consider a class of minimization problems: find $x_0 \in A$ such that $f(x_0) = \min \{f(x): x \in A\}$. Here A is an arbitrary closed subset of a complete metric space X and $f: X \to R$ is an arbitrary continuous real-valued function which is bounded from below. Obviously every such problem is determined by the pair (A, f). The set of all these pairs is endowed with a natural complete metric.

We show that the "majority" of the problems from the class mentioned above are well-posed (i.e. there are uniqueness and continuous dependence of solutions on A and f). Here the "majority" is understood in the Baire category sense, viz. the set of the well-posed problems contains a dense and G_δ-subset of all the complete metric spaces. I.e. its complement is of first Baire category and is considered to be a small set. A class of convex optimization problems is investigated in the same direction.

Similar results have been obtained in [1, 2, 3, 8, 10]. This article generalizes some of them. For further development see [6]. Most of the results here are announced in [11].

1. Notations, Definitions and Main Result

Let (X, p) be a complete metric space. Without loss of generality we may assume that the metric p is less or equal to 1 (if not replace it by the equivalent and complete metric $p'(x, y) = p(x, y)/(1 + p(x, y)))$. Let $\mathscr{B}(X)$ stand for the set of all lower semicontinuous (l.s.c.) bounded from below real-valued functions on X. $\mathscr{B}(X)$ is a (complete) metric space relating to the distance $d(f_1, f_2) = \sup \{|f_1(x) - f_2(x)| : : (1 + |f_1(x) - f_2(x)|): x \in X\}, f_1, f_2 \in \mathscr{B}(X)$. Denote by $\mathscr{F}(X)$ the class of all non-empty closed subsets of X. $\mathscr{F}(X)$ is a complete metric space relating to the Hausdorff distance p_H (see [9]).

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Let (A, f) be from $\mathscr{F}(X) \times \mathscr{B}(X)$. Instead of "the problem to minimize f over A" we will say "the minimization problem (A, f)". Let us recall the well known Tykhonov (see [12]) and Hadamard well-posedness (the latter is taken with respect to the Hausdorff distance on $\mathscr{F}(X)$ and the metric d on $\mathscr{B}(X)$). With $\operatorname{argmin}_A f$ we denote the solution set (possible empty) of the minimization problem (A, f).

Definition 1.1. A minimization problem (A, f) is called well-posed in the sense of Tykhonov (resp. Hadamard), iff it has unique solution $x_0 \in A$ and, moreover for every minimizing sequence $\{x_n\}_{n=1}^{\infty} \subset A$, i.e. $f(x_n) \to f(x_0)$ (resp. for every $(A_n, f_n) \to (A, f)$ and for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in \operatorname{argmin} f_n$) it follows that $x_n \to x_0$.

In order to give another definition of well-posedness let $L_{A,f}(E) = \{x \in X: f(x) \leqq \inf_A f + \varepsilon \text{ and } p(x, A) \leqq \varepsilon\}$, $\varepsilon > 0$. Here $p(x, A) = \inf\{p(x, y): y \in A\}$ is the distance function generated by A. With $\operatorname{diam}(A)$ we designate the diameter of the set A.

Definition 1.2. A minimization problem (A, f) is called well-posed iff $\inf\{\operatorname{diam}.(L_{A,f}(\varepsilon): \varepsilon > 0\} = 0$.

The sets $L_{A,f}(\varepsilon)$, $\varepsilon > 0$ are non-empty and closed for every $\varepsilon > 0$. Moreover if $\varepsilon_1 \leqq \varepsilon_2$ then $L_{A,f}(\varepsilon_1) \subset L_{A,f}(\varepsilon_2)$ which shows that if (A, f) is well-posed then it has unique solution.

Throughout this paper well-posedness is understood in the sense of Definition 1.2. An additional remark is called for. We shall deal with Cartesian products of complete metric spaces (Z_1, p_1) and (Z_2, p_2). On all of them some complete metric generated by the metrics p_1 and p_2 is considered (e.g. $(p_1^2(\cdot, \cdot) + p_2^2(\cdot, \cdot))^{1/2}$).

Before giving the relations between the introduced notions of well-posedness we will prove two preliminary lemmas.

Let $M(A, f) = \inf\{f(x): \in A\}$ be the marginal function for the problem (A, f). With $B[x; \delta]$ (resp. $B(x; \delta)$) we will denote the closed (resp. open) ball in X with center x and radius δ. The corresponding balls in $\mathscr{F}(X)$ and $\mathscr{B}(X)$ will be denoted by the same way (of course instead of x we will put A and f respectively).

With $\mathscr{C}_b(X)$ we designate the set of those f from $\mathscr{B}(x)$ which are continuous. $(C_b(X), d)$ is a complete metric space too.

Lemma 1.3. $M(\cdot, \cdot)$ is upper semicontinuous (*u.s.c.*) at the points of $\mathscr{F}(X) \times C_b(X)$.

Proof. Let $(A_0, f_0) \in \mathscr{F}(X) \times C_b(X)$ and $\varepsilon > 0$ be given. There exists a point $x_0 \in A_0$ such that

$$f(x_0) \leqq \inf_{A_0} f_0 + \varepsilon/3 . \tag{1.1}$$

Now let $\delta \in (0, 1)$ be such that $\delta < \varepsilon/3$ and

$$|f_0(x) - f_0(x_0)| \leqq \varepsilon/3 \quad \text{for every} \quad x \in B[x_0; \delta] . \tag{1.2}$$

Let $(A, f) \in B(A_0; \delta) \times B(f_0; \delta)$. Since $p_H(A, A_0) < \delta$ then there exists $x_1 \in A$ such that $p(x_1, x_0) \leqq \delta$. Having in mind (1.1) and (1.2) we have

$$M(A, f) = \inf_A f \leqq f(x_1) \leqq f_0(x_1) + \varepsilon/3 \leqq f_0(x_0) + \\ + (2\varepsilon)/3 \leqq \inf_{A_0} f_0 + \varepsilon .$$

The proof is completed.

Lemma 1.4. Let $(A, f) \in \mathscr{F}(X) \times \mathscr{B}(X)$. Then for every $\varepsilon > 0$ and $x_\varepsilon \in X$ such that $f(x_\varepsilon) \leqq \inf_A f + \varepsilon$ there exists a function $f_\varepsilon \in \mathscr{B}(X)$ (and if $f \in C_b(X)$ then $f_\varepsilon \in C_b(X)$) such that $(A \cup \{x_\varepsilon\}, f_\varepsilon)$ has a unique solution x_ε and $d(f_\varepsilon, f) \leqq 2\varepsilon$. If in addition $f(x_\varepsilon) \leqq \inf_{A_\varepsilon} f + \varepsilon$, where $A_\varepsilon = \{x \in X: p(x, A) \leqq \varepsilon\}$ then $(A \cup \{x_\varepsilon\}, f_\varepsilon)$ is well-posed.

Proof. Let $\varepsilon > 0$ and $x_\varepsilon \in X$ be as in the Lemma. Following Čoban and Kenderov [2] let us consider the function f_1 defined by

$$f_1(x) = \begin{cases} f(x) - \varepsilon & \text{if } f(x) \geqq f(x_\varepsilon) + \varepsilon \\ f(x_\varepsilon) & \text{if } f(x) \in (f(x_\varepsilon) - \varepsilon, f(x_\varepsilon) + \varepsilon) \\ f(x) + \varepsilon & \text{if } f(x) \leqq f(x_\varepsilon) - \varepsilon \end{cases}$$

In [2] it is proved that if f is continuous, the same is true for f_1. There is no troubles to prove that if f is l.s.c. then f_1 is also l.s.c. It is true also that $d(f_1, f) \leqq \varepsilon$ and that $f_1(x) \geqq f_1(x_\varepsilon)$ for every $x \in A$, i.e. $x_\varepsilon \in \operatorname{argmin}_{A \cup \{x_\varepsilon\}} f_1$. Moreover if in addition $f(x_\varepsilon) \leqq \inf_{A_\varepsilon} f + \varepsilon$ then $x_\varepsilon \in \operatorname{argmin}_{A_\varepsilon \cup \{x_\varepsilon\}} f_1$.

Now let

$$f_\varepsilon(x) = f_1(x) + \varepsilon p(x, x_\varepsilon), \quad x \in X .$$

Obviously $d(f_\varepsilon, f) \leqq 2\varepsilon$ and $f_\varepsilon \in \mathscr{B}(X)$ (resp. $C_b(X)$) if $f \in \mathscr{B}(X)$ (resp. $C_b(X)$). It is not difficult to check that x_ε is a unique minimum point of the problem $(A \cup \{x_\varepsilon\}, f_\varepsilon)$. Indeed let $x \in A$, $x \neq x_\varepsilon$. Then

$$f_\varepsilon(x) = f_1(x) + \varepsilon p(x, x_\varepsilon) \geqq f_1(x_\varepsilon) + \varepsilon p(x, x_\varepsilon) > f_1(x_\varepsilon) = f_\varepsilon(x_\varepsilon) .$$

Suppose in addition that $f(x_\varepsilon) \leqq \inf_{A_\varepsilon} f + \varepsilon$. Then we will prove that $(A \cup \{x_\varepsilon\}, f_\varepsilon)$ is well-posed.

Let $\delta > 0$ be given. Choose $\mu > 0$ so small that

a) $A_\mu \subset A_\varepsilon$
b) $f_1(x) \geqq f_1(x_\varepsilon) - (\delta\varepsilon)/2$ for every $x \in B(x_\varepsilon; \mu)$
c) $\mu < (\delta\varepsilon)/2$

b) is possible because of lower semicontinuity of f_1. Consider $(A \cup \{x_\varepsilon\})_\mu$. Obviously $(A \cup \{x_\varepsilon\})_\mu \subset A_\mu \cup B[x_\varepsilon; \mu]$. Choose arbitrary $x \in L_{A \cup \{x_\varepsilon\}, f_\varepsilon}(\mu)$. We have

$$f_\varepsilon(x) \leqq f_\varepsilon(x_\varepsilon) + \mu = f(x_\varepsilon) + \mu .$$

From the other hand $x \in (A \cup \{x_\varepsilon\})_\mu \subset A_\mu \cup B[x_\varepsilon; \mu]$. Two cases are possible:

1) $x \in A_\mu$. Then (because of $A_\mu \subset A_\varepsilon$)

$$f_\varepsilon(x) = f_1(x) + \varepsilon p(x, x_\varepsilon) \geqq f_1(x_\varepsilon) + \varepsilon p(x, x_\varepsilon) = f(x_\varepsilon) + \varepsilon p(x, x_\varepsilon),$$

therefore

$$p(x, x_\varepsilon) \leqq \mu/\varepsilon \leqq (\delta\varepsilon)/(2\varepsilon) = \delta/2.$$

2) $x \in B[x_\varepsilon; \mu]$. Then

$$f_\varepsilon(x) = f_1(x) + \varepsilon p(x, x_\varepsilon) \geqq f_1(x_\varepsilon) - (\delta\varepsilon)/2 + \varepsilon p(x, x_\varepsilon).$$

Therefore

$$p(x, x_\varepsilon) \leqq \mu/\varepsilon + (\delta\varepsilon)/(2\varepsilon) \leqq \delta.$$

Hence in both cases $p(x, x_\varepsilon) \leqq \delta$, consequently $L_{A\cup\{x_\varepsilon\},f_\varepsilon}(\mu) \subset B[x_\varepsilon; \delta]$. This means that diameters of the sets $L_{A\cup\{x_\varepsilon\},f_\varepsilon}(\mu)$ tend to zero when $\mu \to 0$ hence the problem $(A \cup \{x_\varepsilon\}, f_\varepsilon)$ is well-posed.

Corollary 1.5. The set of the well-posed problems in $\mathscr{F}(X) \times \mathscr{B}(X)$ (resp. in $\mathscr{F}(X) \times C_b(X)$) is dense in $\mathscr{F}(X) \times \mathscr{B}(X)$ (resp. in $\mathscr{F}(X) \times C_b(X)$).

Proof. Let $(A_0, f_0) \in \mathscr{F}(X) \times \mathscr{B}(X)$ $(\mathscr{F}(X) \times C_b(X))$ and $\varepsilon > 0$ be given. Consider $(A_0)_\varepsilon$ and let $x_\varepsilon \in (A_0)_\varepsilon$ be such that $f_0(x_\varepsilon) \leqq \inf_{(A_0)_\varepsilon} f_0 + \varepsilon$. Applying Lemma 1.4 we obtain a function f_ε (which is in $C_b(X)$ if $f_0 \in C_b(X)$) such that $(A_0 \cup \{x_\varepsilon\}, f_\varepsilon)$ is well-posed. But $d(f_\varepsilon, f_0) \leqq 2\varepsilon$ and $p_H(A_0 \cup \{x_\varepsilon\}, A_0) \leqq \varepsilon$.

Now let us turn to the relations between the three kinds of well-posedness.

Proposition 1.6. Let $(A, f) \in \mathscr{F}(X) \times \mathscr{B}(X)$. Consider the following assertions:

a) the problem (A, f) is well-posed in the sense of Hadamard;

b) the problem (A, f) is well-posed;

c) the problem (A, f) is well-posed in the sense of Tykhonov;

Then a) $\Rightarrow b \Rightarrow c$). If f is continuous b) $\Rightarrow a$) and in the case of uniformly continuous f a) $\Leftrightarrow b) \Leftrightarrow$ c).

Proof: a) $\Rightarrow$ b). Let (A, f) be well-posed in the sense of Hadamard with unique solution $x_0 \in A$. Suppose that

$$\inf\{\operatorname{diam}(L_{A,f}(\varepsilon): \varepsilon > 0\} > 2\delta > 0.$$

Then taking $\varepsilon = 1, 1/2, \ldots, 1/n, \ldots$ we obtain a sequence $\{x_n\}_{n=1}^\infty$:

1) $p(x_n, A) \leqq 1/n$

2) $f(x_n) \leqq \inf_A f + 1/n$

3) $p(x_n, x_0) \geqq \delta$.

By Lemma 1.4 for every $n = 1, 2, \ldots$ there exist functions f_n from $\mathscr{B}(X)$ such that $\operatorname{argmin}_{A_n} f_n = \{x_n\}$, where $A_n = A \cup \{x_n\}$ and $d(f_n, f) \leqq 2/n$. But this means that

$(A_n, f_n) \to (A, f)$ and by the Hadamard well-posedness it would follow that $x_n \to x_0$ – a contradiction with 3). Hence (A, f) is well-posed.

The implication b) $\Rightarrow$ c) is a simple consequence of the metric characterization of the Tykhonov well-posedness obtained in [5] by Furi and Vignoli.

Now let f be continuous and (A, f) be well-posed. Then (A, f) has unique solution $x_0 \in A$. Let $(A_n, f_n) \to (A, f)$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in \operatorname{argmin}_{A_n} f_n$. Choose arbitrary $\delta > 0$. By the well-posedness of (A, f) there is a positive ε_0 such that

$$L_{A,f}(\varepsilon_0) \subset B[x_0; \delta]\,. \tag{1.3}$$

Now by upper semicontinuity of the marginal function one may find n_0 such that for every $n \geqq n_0$

$$f_n(x_n) = \inf_{A_n} f_n \leqq \inf_A f + \varepsilon_0/2\,. \tag{1.4}$$

At the end by the uniform convergence of f_n to f there exists n_1 such that for every $n \geqq n_1$ we have

$$|f(x) - f_n(x)| \leqq \varepsilon_0/2 \quad \text{for every} \quad x \in X\,. \tag{1.5}$$

Take $n \geqq \max(n_0, n_1)$. Having in mind (1.3), (1.4) and (1.5) we obtain that

$$f(x_n) \leqq f_n(x_n) + \varepsilon_0/2 \leqq \inf_A f + \varepsilon_0\,, \quad \text{i.e.} \quad x_n \in L_{A,f}(\varepsilon_0)\,.$$

Hence $x_n \in B[x_0; \delta]$ and since δ was arbitrary we may conclude that $x_n \to x_0$.

The proof of the implication c) $\Rightarrow$ b) in the case of uniformly continuous f is routine and is omitted.

Corollary 1.7. Let $f \in C_b(X)$. Then (X, f) is Hadamard well-posed iff (X, f) is Tykhonov well-posed.

Simple examples on the real line show that if f is not continuous then Hadamard well-posedness and well-posedness in the sense of Definition 1.2 may not coincide.

Theorem 1.8. There exists a dense and G_δ-subset D of $\mathscr{F}(X) \times C_b(X)$ such that for every $(A, f) \in D$ the corresponding minimization problem is well-posed.

Proof. Let us consider the sets

$$D_n = \{(A, f)\colon \inf\{\operatorname{diam}(L_{A,f}(\varepsilon)\colon \varepsilon > 0\} < 1/n\}\,,$$

$n = 1, 2, \ldots$. Since $D = \bigcap_{n=1}^{\infty} D_n$ is just the set of well-posed problems it has to be shown that each D_n is open and dense in $\mathscr{F}(X) \times C_b(X)$. The denseness of D_n is a consequence of Lemma 1.4. It remains to show that D_n are open for every n.

Let n be fixed and $(A_0, f_0) \in D_n$. There exists $\varepsilon_0 > 0$ such that $\operatorname{diam}(L_{A_0,f_0}(\varepsilon_0)) < 1/n$. Let $\delta > 0$ be such that $\delta < \varepsilon_0/3$, $\delta/(1 - \delta) < \varepsilon_0/3$ and such that for every (A, f) from $B[A_0; \delta] \times B[f_0; \delta]$ the next inequality holds

$$M(A, f) \leqq M(A_0, f_0) + \varepsilon_0/3\,. \tag{1.6}$$

Now if $(A, f) \in B[A_0; \delta] \times B[f_0; \delta]$ is arbitrary and x belongs to the set $L_{A,f}(\varepsilon_0/3)$ then

1) $f(x) \leqq \inf_A f + \varepsilon_0/3$ and
2) $p(x, A) \leqq \varepsilon_0/3$.

But $p_H(A, A_0) \leqq \delta \leqq \varepsilon_0/3$, hence $p(x, A_0) \leqq \varepsilon_0/3 + \varepsilon_0/3 < \varepsilon_0$. From the other hand $f_0(x) < f(x) + \varepsilon_0/3$ (because of the choice of δ) and by 1) and (1.6) we obtain that $f_0(x) \leqq \inf_{A_0} f_0 + \varepsilon_0$ which together with $p(x, A_0) \leqq \varepsilon_0$ shows that $x \in L_{A_0,f_0}(\varepsilon_0)$. Therefore $L_{A,f}(\varepsilon_0/3) \subset L_{A_0,f0}(\varepsilon_0)$ and consequently $\inf\{\text{diam}\,(L_{A,f}(\varepsilon): \varepsilon > 0\} < 1/n$. The proof is completed.

Immediately from Proposition 1.5 and Theorem 1.6 we have the following

Corollary 1.9. There exists a dense and G_δ-subset D of $\mathscr{F}(X) \times C_b(X)$ such that for every $(A, f) \in D$ the corresponding minimization problem is Tykhonov well-posed.

For the class of unconstrained minimization problems (X, f), $f \in \mathscr{B}(X)$ and X is closed (resp. open) subset of a complete metric space an analogous result to Theorem 1.8 has been proved by Lucchetti and Patrone [10] (resp. by De Blasi and Myjak [3]).

When X belongs to a large class of compacts (including non-metrizable ones a result as Theorem 1.8 is proved by Kenderov [8]. The metrizable part of Kenderov's results is a consequence of Theorem 1.8.

2. Convex case

Let us consider now a class of convex optimization problems. Till the end X stands for a non-empty closed and convex subset of a real Banach space $(E, \|\cdot\|)$. Denote by $\text{Conv}_b(X)$ those $f \in C_b(X)$ which are convex. $\text{Conv}_b(X)$ is a complete metric space under the metric d. Let $\mathscr{K}(X) = \{K \subset X\colon K \neq \emptyset,\ K$ – closed and convex$\}$. When X is bounded $(\mathscr{K}(X), p_H)$ is a complete metric space.

Of course if in some class of optimization problems the "majority" of the problems are well-posed it does not automatically follow that the same is true for some subclass of optimization problems. From the other hand if we have some important class of optimization problems it is interesting to know whether arbitrary near (in some sense) to every problem of that class there is a problem which is well-posed. The next theorem shows that in the class $\mathscr{K}(X) \times \text{Conv}_b(X)$ this is true.

Theorem 2.1. Let X be bounded. Then there exists a dense and G_δ-subset D of $\mathscr{K}(X) \times \text{Conv}_b(X)$ such that for every couple (K, f) from D the corresponding minimization problem is well-posed.

Before giving the proof of Theorem 2.1 let us introduce a notion of continuity of (multivalued) mappings due to Kenderov [7]. Let Y and Z be topological spaces and $T\colon Y \to Z$ be a (multivalued) mapping. T is said to be almost lower semicontinnuous (a.l.s.c.) at a point $y_0 \in Y$ if for every open set W in Z such that $W \cap Ty_0 \neq \emptyset$ it follows that $T^{-1}W = \{y \in Y: Ty \cap W \neq \emptyset\}$ is dense in some open neighborhood of y_0. T is said to be a.l.s.c. on Y if it is a.l.s.c. at every point $y \in Y$. If Z is separable and Y is a Baire space (i.e. the intersection of a countable family of open and dense subsets of Y is dense in Y) a result of Kenderov [7], asserts that every (multivalued) mapping $T: Y \to Z$ is a.l.s.c. at the points of a dense and G_δ-subset of Y.

Proof of Theorem 2.1. Let us consider the function $G: \mathscr{K}(X) \times \mathrm{Conv}_b(X) \to R$ defined by $G(K, f) = \inf\{\mathrm{diam}\,(L_{A,f}(\varepsilon)^) \ \varepsilon > 0\}$ The Kenderov's result mentioned above gives the existence of a dense G_δ-subset D_1 of $\mathscr{K}(X) \times \mathrm{Conv}_b(X)$ such that G is a.l.s.c. at the points of D_1. Further $M(\cdot,\cdot)$ is upper semicontinuous on the complete metric space $\mathscr{K}(X) \times \mathrm{Conv}_b(X)$ and consequently (Fort [4]) it is continuous at the points of a dense and G_δ-subset D_2 of $\mathscr{K}(X) \times \mathrm{Conv}_b(X)$. Let $D = D_1 \cap D_2$ – a dense and G_δ in $\mathscr{K}(X) \times \mathrm{Conv}_b(X)$. We claim that for every $(K, f) \in D$ the corresponding minimization problem is well-posed.

Suppose the contrary: there exists $(K_0, f_0) \in D$ such that $\inf\{\mathrm{diam}\,(L_{K_0,f_0}(\varepsilon): \varepsilon > 0\} = 2\delta_0$. Let $V = (\delta_0, 3\delta_0)$ and U is an arbitrary open subset of $\mathscr{K}(X) \times \mathrm{Conv}_b(X)$ which contains (K_0, f_0). We may assume that $U = B(K_0; \delta) \times B(f_0; \delta)$ for some $\delta \in (0, 1)$. We will find an open $U' \neq \emptyset$ such that $U' \subset U$ and $G(U') \cap V = \emptyset$ which will contradict to the fact that G is a.l.s.c. at (K_0, f_0).

Let us fix $N > 0$ such that $\mathrm{diam}\,(X)/N < \delta/4$. Let $\mu_0 = \delta_0/2N$ and $x_0 \in K_0$ have the property

(2.1) $$f_0(x_0) \leqq \inf_{K_0} f_0 + \mu_0/6\,.$$

The function f_0 is continuous whence there is $\delta_1' \in (0, \delta_0/24)$ such that

(2.2) $$|f_0(x) - f_0(x_0)| \leqq \mu_0/6 \quad \text{for every} \quad x \quad \text{from} \quad B[x_0; \delta_1']\,.$$

Now let us choose a positive δ_1 with the properties:

(2.3) 1) $\delta_1 < \min\{\delta, \delta_1'\}$
2) $|M(K, f_0) - M(K_0, f_0)| \leqq \mu_0/6$ for every $K \in B[K_0; \delta_1]$.

2) is possible because of continuity of M at (K_0, f_0).

Consider the function $f_1(x) = f_0(x) + h_0(x)$, $x \in X$, where $h_0(x) = \|x - x_0\|/N$. Obviously $f_1 \in \mathrm{Conv}_b(X)$ and from the choice of N it follows that $d(f_1, f_0) \leqq \mathrm{diam}\,(X)/N \leqq \delta/4$. And at the end let us choose $\delta_2 \in (0, 1)$ for which $\delta_2/(1 - \delta_2) < \mu_0/6$. Then $\delta_2 < \mu_0/6 < \mu_0 = \delta_0/(2N) < \mathrm{diam}\,(X)/N < \delta/4$ whence it follows that $B(f_1; \delta_2) \subset B(f_0; \delta)$. Let $U' = B(K_0; \delta_1/2) \times B(f_1; \delta_2)$. From the choice of δ_1 and δ_2 it follows that $U' \subset U$. Let $(K, f) \in U'$ be an arbitrary couple and let us consider the function

$$h(x) = f(x) - f_0(x)\,, \quad x \in X\,.$$

From $d(f, f_1) < \delta_2$ we may conclude that

(2.4) $$|h(x) - h_0(x)| \leqq \delta_2/(1 - \delta_2) \leqq \mu_0/6 \quad \text{for every} \quad x \in X\,.$$

Now we claim that for arbitrary small $\varepsilon > 0$ $L_{K,f}(\varepsilon)$ is contained in the ball $B[x_0; \delta_0/2]$ Indeed let $\varepsilon < \delta_1/2$ and x be an arbitrary point from $K_\varepsilon \setminus B[x_0; \delta_0/2]$. Since K is closed and convex the same is true for K_ε. From the other hand $p_H(K_\varepsilon, K) \leqq \varepsilon < \delta_1/2$ whence (having in mind that $K \in B(K_0; \delta_1/2)$) it follows that $p_H(K_\varepsilon. K_0) \leqq \delta_1$. Then by (2.1), (2.3), (2.4) and the choice of x we have:

$$f(x) = f_0(x) + h(x) \geqq f_0(x) + h_0(x) - \mu_0/6 \geqq \inf_{K_\varepsilon} f_0 + \\ + \|x - x_0\|/N - \mu_0/6 \geqq \inf_{K_0} f_0 + \delta_0/(2N) - \mu_0/6 - \mu_0/6 \geqq \\ f_0(x_0) + \mu_0 - (2\mu_0)/6 - \mu_0/6 = f_0(x_0) + 3\mu_0/6\,.$$

Let us recall that $\delta_1 < \delta_1'$ and since $p_H(K, K_0) < \delta_1/2$ then there exists $x_1 \in K$ such that $\|x_1 - x_0\| < \delta_1'$. Also the choice of δ_1' shows that $\delta_1'/N \leqq \delta_0/(24N) = \mu_0/12$. Then the above chain of inequalities may be continued in the next way (see (2.2))

$$f(x) \geqq f_0(x_1) + (2\mu_0)/6 = \\ f_0(x_1) + \|x_1 - x_0\|/N - \|x_1 - x_0\|/N + (2\mu_0)/6 \geqq f_0(x_1) + \\ + \|x_1 - x_0\|/N - \delta_1'/N + (2\mu_0)/6 \geqq f_0(x_1) + h_0(x_1) + \\ + (2\mu_0)/6 - \mu_0/12 \geqq f_0(x_1) + h(x_1) + \mu_0/6 - \mu_0/12 = \\ f(x_1) + \mu_0/12 \geqq \inf_K f + \mu_0/12\,.$$

Hence $f(x) \geqq \inf_K f + \mu_0/12$ for $\varepsilon < \delta_1/2$ and $x \in K_\varepsilon \setminus B[x_0; \delta_0/2]$. Therefore if in addition $\varepsilon < \mu_0/12$ then $L_{K,f}(\varepsilon) \subset B[x_0; \delta_0/2]$ which shows that $G(K,f) = \inf\{\operatorname{diam}(L_{K,f}(\varepsilon)\colon \varepsilon > 0\} \leqq \delta_0$ i.e. $G(K,f) \cap V = \emptyset$. Since (K,f) was arbitrary couple from U' it follows that $G(U') \cap V = \emptyset$ – the contradiction is achieved and the proof of Theorem 2.1 is completed.

For the class of unconstrained minimization problems (X,f), f is convex and l.s.c., X being bounded, an analogous result concerning Tykhonov well-posedness was proved by De Blasi and Myjak [3]. Here we have the next immediate corollary of a Theorem 2.1.

Corollary 2.2. Let X be bounded. Then there exists a dense and G_δ-subset of $\mathscr{K}(X) \times \mathrm{Conv}_b(X)$ such that for every $(K,f) \in F$ the corresponding minimization problem is well-posed in the sense of Tykhonov.

Let us observe that Theorem 2.1 does not hold when X is unbounded (see Remark 5.1 in [3]).

Remark 2.3. We would like to mention that in Theorem 1.8 and Theorem 2.1 (and in their corollaries) the hypotheses about A and f are such that there may not even exist a solution to the corresponding minimization problem. But it turns out that for the "majority" of the problems there exists a solution, this solution is unique and depends continuously upon the data.

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