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T_2-Separation Axioms on Frames

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In the theory of frames (or "pointless topologies"), several authors have tried to find a suitable form of separation axioms. Our purpose is to describe a T_2-axiom in the form usual in the case of regular frames. T_2-frames coincide for topological spaces with Hausdorff spaces but they are described independently on points. We also investigate almost compact frames and H-closed extensions of frames (see [5], 6.1, h, for spaces).

All unexplained facts concerning frames can be found in Johnstone [2] or in [5]. Recall that a *frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge s: s \in S\}$$

holds for all $a \in L$, $S \subseteq L$.

Frames can be viewed as generalized topological spaces. Frames which are isomorphic to the frame $O(T)$ of all open sets of a suitable topological space T are called topologies or spatial frames. Frames form a category denoted by *Fmr* (frame morphism are maps $f: K \to L$ of frames K, L preserving arbitrary joins and finite meets).

We shall investigate subcategories of *Frm* denoted by Frm_i which are defined by some separation axiom. We shall discuss three basic questions for these subcategories:

1. Is the meet of Frm_i with the category of sober spaces equal to the subcategory Top_i of *Top* (on the objects) defined by the same separation axiom as Frm_i?
2. Is the category Frm_i closed with respect to sums and homomorphic images?
3. Does there exist for any frame $L \in Frm_i$ a topological space T, $O(T) \in Top_i$ such that L is a homomorphic image of $O(T)$? Is Frm_i the monocoreflective hull of Top_i?

Consider some examples:

i = 1: In [5], frames in which *primes* (i.e. $\bigwedge$-*irreducible* and $\neq 1$ elements) are dual atoms (called T_1-frames), are investigated. The category of all T_1-frames is the smallest monocoreflective subcategory in *Frm* containing all T_1-spaces. Every

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T_1-frame is a homomorphic image of a T_1-space and every spatial T_1-frame is a T_1-space.

i = 3: The main separation axiom for frames is ***regularity*** which is defined in the following way:

A frame L is ***regular*** if

$$a = \bigvee(b \in L: b \prec a)$$

holds for all $a \in L$, where $b \prec a$ means $b^* \vee a = 1$. The positive answer of questions 1 and 2 is well known (see [2]). The last question was answered negatively by I. Kříž.

i = 2: We will now investigate some candidates for *Hausdorffness* on frames.

a) We say that an element $a \in L$, $a \neq 1$ is *semiprime* if $x \wedge y = 0$ implies $x \leqq a$ or $y \leqq a$ for any $x, y \in L$. Clearly, any prime element is semiprime.

We say that L is an *S-frame* if semiprime elements of L are dual atoms (see [5]). Every S-frame is a T_1-frame and every spatial S-frame is a Hausdorff space. The subcategory S of S-frames is closed under sums and homomorphic images.

b) The following definition of a *Hausdorff frame* was introduced by P. T. Johnstone and Sun Shu-Hao in [3].

We say that a frame L is *Hausdorff* if for any $a, b \in L$ such that $1 \neq a \nleqq b$ there exists an element $l \in L$ with $l^* \nleqq a$, $l \nleqq b$. The subcategory *HauFrm* of Hausdorff frames is closed under sums and homomorphic images, every Hausdorff frame which is spatial is a Hausdorff space. Any Hausdorff frame is an S-frame.

c) We shall say that L is a *T_2-frame* (see [4]) if for any $a, b \in L$ such that $1 \neq a \nleqq b$ there exists an element $l \in L$ with $l \leqq a$, $l^* \nleqq a$ and $l \nleqq b$. Equivalently, L is a T_2-frame if

$$a = \bigvee\{l \in L: l \leqq a, l^* \nleqq a\}$$

for any $a \in L$. Now, we can consequently define T_2-frames in the form usual for regular frames. Clearly, every T_2-frame is Hausdorff. The subcategory $\mathbf{Frm_2}$ is closed under sums and homomorphic images. Evidently, spatial T_2-frames are exactly Hausdorff spaces.

d) We say that a frame L is a *T_2'-frame* if for any $a \in L$ there exists an ideal $A \subseteq \subseteq \{l \in L: l \leqq a, l^* \nleqq a\}$ such that $a = \bigvee A$. Obviously, every T_2'-frame is a T_2-frame and any spatial T_2'-frame is a Hausdorff space. The subcategory $\mathbf{Frm_2'}$ of T_2'-frames is closed under homomorphic images but we do not know whether T_2'-frames are closed under sums.

Now, we have the following:

$$\mathbf{Reg} \subsetneqq \mathbf{Frm_2'} \subseteq \mathbf{Frm_2} \subseteq \mathbf{HauFrm} \subsetneqq \mathbf{S} \subsetneqq \mathbf{Frm_1} \subsetneqq \mathbf{Frm}\,.$$

We now know that every Hausdorff topological space which is not regular has no T_2-compactification but we can consider a T_2-extension of the Hausdorff space with some properties of compactification. It is natural to investigate this question for T_2-frames.

We say that a frame L is *almost compact* if it holds:

$\bigvee(x_i: i \in I) = 1$ implies there exists $K \subseteq I$ finite such that $(\bigvee(x_k: k \in K))^{**} = 1$. Clearly, every spatial almost compact frame is an almost compact space. We can prove that a sum of almost compact frames is almost compact. Compact frames are almost compact and any almost compact frame has at least one semiprime element.

Proposition 1. *There exists a compact normal T_2'-frame which is not regular (i.e. is not spatial).*

Proof. Let I be the closed interval $[0, 1]$ with the usual topology. We put $K(O(I)) = \{(x, y): x \leqq y, x \in O(I), y$ is a *regular open set* in $I\}$. We can easily verify that $K(O(I))$ is a compact normal T_2'-frame which is not conjunctive.

Proposition 2. *An almost compact frame which is a homomorphic image of a Hausdorff topology is a topology.*

Proof. The proof is easy.

Corollary 3. *A compact normal T_2'-frame exists which is not a homomorphic image of a Hausdorff topology.*

Now, let us describe the construction of an *H-closed extension* of a frame L. A maximal filter F on L is called a *β-filter* if

$$\bigvee(a^*: a \in F) = 1$$

holds. If $\{F_j: j \in J\}$ is the set of all β-filters on L then the subframe $L_\beta \subseteq L \times 2^J$, generated by the elements $\{(l, \emptyset): l \in L\} \cup \{(a, \{j\}): a \in F_j\}$, is called an *H-closed extension* of L. Clearly, any H-closed extension is almost compact.

Now, we have the following.

Proposition 4. *An H-closed extension of a Hausdorff frame (T_2-frame) is a Hausdorff frame (T_2-frame).*

Proposition 5. *Let L be a frame. Then $(a, I) \in L_\beta$ is semiprime (prime, dual atom resp.) iff $a = 1$, $I = J - \{j\}$ for some $j \in J$ or a is semiprime (prime, dual atom resp.), $I = J$.*

Corollary 6. *There exists an almost compact T_2-frame which is not dually atomic.*

Corollary 7. *Homomorphic images of Hausdorff topolcgies form a subcategory in Frm which is not closed under H-extensions.*

It would be interesting to know whether some of preceding classes is the mono-coreflective hull of Hausdorff spaces. However, we have not been able to answer this question.

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