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# Generalized Vitali Systems of Uniform Type

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Some Vitali systems of geometrical type consisting of more general sets than just cubes, balls or compact, convex sets are defined. Packing theorems for these are formulated, and sketches of the proofs are given. The general theory is applied on a class of nullsets, generalizing the usual Lipschitz condition for functions. A theorem is given of Jessen-Marcinkiewicz-Zygmund type for a system $\mathscr{B}_\varrho$ of axiparallel rectangles, though $\mathscr{B}_\varrho$ has the packing property.

**1. Introduction.** In 1908 G. Vitali [19] proved a packing theorem for intervals in $\mathbb{R}$, squares in $\mathbb{R}^2$ and $N$-dimensional cubes in $\mathbb{R}^N$. The depth of this discovery was realized by H. Lebesgue [7], who improved the method to systems of regular sets with respect to cubes and used the packing theorem in differentiation theory. In 1918. C. Carathéodory [3a] conjectured that squares could be replaced by rectangles in $\mathbb{R}^2$. This conjecture proved to be wrong as pointed out by S. Banach [1] in 1924. In [3b] C. Carathéodory included an example communicated to him by H. Bohr as early as in 1918. This counterexample has become the most commonly used construction ever since, just like S. Banach's method from [1] has become the most well-known proof of Vitali theorems. On the other hand S. Banach conjectured that Vitali theorems could only be proved for systems of constant regularity, introduced by H. Lebesgue [7]. A careful analysis shows that this is true *as long as Banach's method is applied*, but already J. C. Burkill [2] proved some results which indicated that this is not generally the case. As differentiation of integrals is concerned, B. Jessen, J. Marcinkiewicz and A. Zygmund [5] proved an important result, which clarifies the different roles of the geometry of the considered sets and the class of functions, which is differentiated with respect to these sets. The proof of [5] uses two ingredients: 1) A Vitali theorem combined with a modification of Bohr's example (cf. [3b]) and 2) a result of G. H. Hardy and J. E. Littlewood [4], where the maximal function is introduced. Since that time over 300 papers have been published concerning Vitali theorems and differentiation theorems, and it would be unjust not to mention that such a vast literature exists.

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The starting point here is that Banach's method (cf. [1]) need not give the best covering principle. This was demonstrated by L. Mejlbro and F. Topsøe [8] for noncentred cubes. Recently, [9]–[14], far more general results have been established and it has been possible in the proofs to some extent to distinguish between the role played by the geometry in $\mathbb{R}^N$ and the role of the Lebesgue measure. In the sequel we shall give a short review of the results in [9], [10], [11] and [15], where [15] demonstrates that we may have a Vitali theorem for our differentiation basis $\mathscr{B}_\varrho$, and yet $\mathscr{B}_\varrho$ can only differentiate the Jessen-Marcinkiewicz-Zygmund class $\text{Llog}^+ \text{L}$. The key to the understanding of this phenomenon is that differentiation theory leans heavily on some halo condition.

**2. Notation and general Vitali systems.** We shall only consider the Lebesgue measure $|\cdot|$ and the outer Lebesgue measure $|\cdot|^*$ in $\mathbb{R}^N$. Let $\text{cl}\, A$ and $\text{int}\, A$ denote the closure and the interior, resp., of a set $A$. The class of all closed sets in $\mathbb{R}^N$ is denoted by $\mathscr{F}$, and $\mathscr{K}$ denotes the class of all compact sets. The norm $\|\cdot\|$ in $\mathbb{R}^N$ is chosen as the maximum norm, and $d$ denotes the corresponding metric. Let $B[x, r]$ (and $B(x, r)$) denote the closed (open) cube of centre $x \in \mathbb{R}^N$ and radius $r \in \mathbb{R}_+$. The class of all closed cubes in $\mathbb{R}^N$ is denoted by $\mathscr{Q}$. If $Q \in \mathscr{Q}$ we let $c(Q)$ denote the centre, $e(Q)$ the edge-length and $r(Q) = \frac{1}{2}e(Q)$ the radius of $Q$. For $\alpha > 0$ and $Q \in \mathscr{Q}$ let $\alpha Q \in \mathscr{Q}$ be the blown-up cube with the factor $\alpha$, i.e. $Q$ and $\alpha Q$ have the same centre and $e(\alpha Q) = \alpha e(Q)$.

For $p \in N \setminus \{1\}$ each $Q \in \mathscr{Q}$ can be divided into $p^N$ subcubes of edge-length $e(Q/p)$. This is called the *p-adic division* of $Q$.

Following F. Topsøe [18], a *Vitali system* $\mathfrak{B}$ is a class of pairs $(A, \mathscr{S})$ with $A \subseteq \mathbb{R}^N$ and $\mathscr{S} \subseteq \mathscr{F}$, such that

VS 1: $\forall (A, \mathscr{S}) \in \mathfrak{B} \quad \forall B \subseteq A: (B, \mathscr{S}) \in \mathfrak{B}$;

VS 2: $\forall (A, \mathscr{S}) \in \mathfrak{B} \quad \forall F \in \mathscr{F}: (A \setminus F, \{S \in \mathscr{S}: S \cap F = \emptyset\}) \in \mathfrak{B}$.

In [9] we indicated the connection between this setup and the more general *blankets* of A. P. Morse [16].

Let $\mathscr{S}_0 \subseteq \mathscr{F}$ and $A \subseteq \mathbb{R}^N$. We say that $\mathscr{S}_0$ is a *packing* of $A$, if the elements of $\mathscr{S}_0$ are mutually disjoint and

$$|A \setminus \bigcup\{S: S \in \mathscr{S}_0\}|^* = 0.$$

If $\mathscr{S} \subseteq \mathscr{F}$ and $\mathscr{S}$ contains a packing $\mathscr{S}_0$ of a set $A$, we say that the pair $(A, \mathscr{S})$ has the *packing property*.

A Vitali system $\mathfrak{B}$ is said to have the *Vitali property* (or one says that the *packing theorem holds for* $\mathfrak{B}$), if every pair $(A, \mathscr{S}) \in \mathfrak{B}$ has the packing property.

A *Vitali theorem* (or a *packing theorem*) is a theorem, which states that a given Vitali system $\mathfrak{B}$ has the Vitali property.

The following classical lemma can be traced back to H. Lebesgue [7]:

**Lemma 2.1.** *A Vitali system* $\mathfrak{V}$ *has the Vitali property, if and only if there exists a positive constant* $c$, *such that whenever* $A$ *is bounded and* $(A, \mathscr{S}) \in \mathfrak{V}$ *one can select disjoint sets* $\{S_n : n \in J\} \subseteq \mathscr{S}$ *with*

$$|\bigcup\{S_n : n \in J\}| \geqq c|A|^* .$$

This lemma is used in an essential way in all the proofs of Vitali theorems in [6] and [8]–[14] together with the following elementary lemma:

**Lemma 2.2.** *Let* $f: \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous and nondecreasing function. The following two conditions are equivalent:*

i) $\int_0^R f(r)\, r^{-N-1}\, dr = +\infty$ *for one* $R > 0$ *and hence for all* $R > 0$;

ii) $\sum_{n=0}^{+\infty} q^{nN} f(q^{-n}) = +\infty$ *for one* $q > 1$ *and hence for all* $q > 1$.

**3. Uniform Vitali systems.** We shall specialize to geometrical Vitali systems of uniform type, cf. also [8]–[11].

A continuous and nondecreasing function $\varphi: [0, +\infty[ \to [0, +\infty[$ is called a *Φ function* in $\mathbb{R}^N$ and we write $\varphi \in \Phi^N$, if

$$0 \leqq \varphi(r) < (2r)^N \quad \text{for all} \quad r \geqq 0 .$$

The $\Phi$ functions were used in disguise in [8] in order to obtain a nonclassical Vitali theorem for cubes. Here we shall generalize the class of cubes to a class of more general compact sets. The elements of this class are defined by a *complexity function*, which describes the complexity of the geometry of each set.

**Definition 3.1.** *A nondecreasing function* $\eta \in C^1(\mathbb{R}_+)$, *for which* $\eta(\mathbb{R}_+) \subseteq ]0, 6^{-1}[$, *is called a complexity function, and we write* $\eta \in H$. Note especially that any constant $\eta \in ]0, 6^{-1}[$ may be considered as a member of $H$.

For every $\eta \in H$ we define a related function $\eta^*$ by

$$\eta^*(r) = r\,\eta(r), \quad r \in \mathbb{R}_+ .$$

**Definition 3.2.** *Let* $\eta \in H$ *and* $\kappa \in ]0, 3^{-N}[$. *By* $\mathscr{K}^N_{\eta,\kappa_N}$ *we shall understand the class of all* $K \in \mathscr{K}^N$, *such that for every* $Q^N \in \mathscr{Q}$ *either*

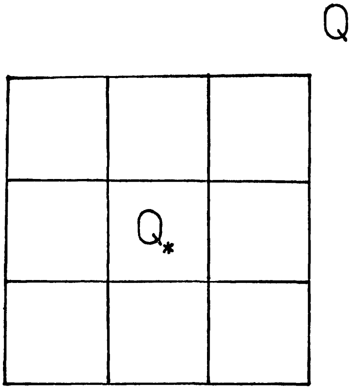
$$|Q \cap K| \geqq \kappa |Q| , \tag{3.1}$$

*or there exists a* $Q_1 \in \mathscr{Q}^N$, *such that*

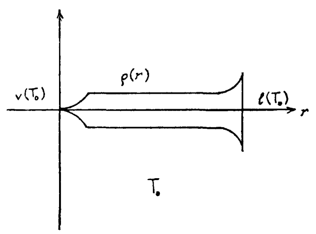
$$Q_1 \subseteq Q , \quad e(Q_1) \geqq \eta^*(e(Q)) , \quad Q_1 \cap K = \emptyset . \tag{3.2}$$

It is easily seen that

$$\mathscr{K}^N_{\eta_1,\kappa_1} \subseteq \mathscr{K}^N_{\eta_2,\kappa_2} \quad \text{for} \quad \eta_1 \geqq \eta_2 \quad \text{and} \quad \kappa_1 \geqq \kappa_2 . \tag{3.3}$$




Furthermore, $\mathscr{K}^N_{\eta,\kappa}$ contains the class of all convex and compact sets in $\mathbb{R}^N$. In fact, let $K$ be convex and compact. Let $Q \in \mathscr{Q}^N$ be chosen, such that (3.2) does not hold for $K$ and $Q$. Perform a 3-adic division of $Q$. By assumption all the $3^N$ subcubes contain points from $K$, so by the convexity the central subcube $Q_*$ is contained in $K$, and (3.1) follows.




**Example 3.1.** In [10] other elements called *trumpets* were introduced. A *trumpet* $T$ in $\mathbb{R}^N$, $N \geqq 2$, is characterized by its *vertex* $v(T)$, the direction of its *axis*, i.e. an element of the rotation group $O(N)$, its *length* $e(T)$ and its *shape function. A shape function* $\varrho$ is a continuous and nondecreasing function $\varrho$: $[0, +\infty[ \to [0, +\infty[$ such that

$$\varrho(0) = 0 \quad \text{and} \quad 0 < \varrho(r) < r \quad \text{for} \quad r > 0\,.$$

The standard trumpet $T_{0,R}$ of length $R$ and vertex 0 is defined by

$$T_{0,R} = \{x \in \mathbb{R}^N \colon 0 \leqq x_1 \leqq R\,, \quad \sum_{j=2}^{N} x_j^2 \leqq \varrho(x_1)^2\}\,.$$

A trumpet $T$ of shape $\varrho$ and length $R$ is defined by a rotation and translation of the standard trumpet $T_{0,R}$. The class of all trumpets of shape $\varrho$ in $\mathbb{R}^N$ is denoted by $\mathscr{T}^N_\varrho$.

By a fairly long, though elementary geometrical proof (cf. [10]) it is proved that $\mathscr{T}^N_\varrho \subseteq \mathscr{K}^N_{\eta,\kappa}$ for all $\eta \in H$ and all $\kappa \leqq c_{N-1}(5\sqrt{2})^{-N}$, where $c_{N-1}$ is the volume of the $(N-1)$-dimensional euclidean unit ball. Note that the bound for $\kappa$ is independent of the shape function $\varrho$. ▽

The trumpets of example 3.1 indicate that the families $\mathscr{K}^N_{\eta,\kappa}$ contain far more general elements than just compact, convex sets. The elements do not even have to be connected, so some $K \in \mathscr{K}^N_{\eta,\kappa}$ may look like an archipelago of connected components. It should be mentioned that when $\eta(r) \to 0$ for $r \to 0$ it is possible to construct elements $K \in \mathscr{K}^N_{\eta,\kappa}$ having boundaries $\partial K$ very similar to a fractal set.

**Definition 3.3.** *Let* $\varphi \in \Phi^N$ *and* $\eta \in H$ *and* $\kappa \in ]0,3^{-N}[$. *By* $\mathfrak{B}^N_{uni}[\varphi; \eta, \kappa]$ *we shall understand the class of all pairs* $(A, \mathscr{S})$, *where* $A \subseteq \mathbb{R}^N$ *and* $\mathscr{S} \subseteq \mathscr{K}^N_{\eta,\kappa}$, *such that*

$$(3.4) \qquad \forall x \in A \quad \exists r_x > 0 \quad \forall r \in ]0,r_x] \quad \exists K \in \mathscr{S} \colon K \subset B(x,r) \wedge |K| \geqq \varphi(r)\,.$$

A routine check shows that $\mathfrak{B}^N_{uni}[\varphi; \eta, \kappa]$ is a Vitali system.

**4. Vitali theorems for uniform Vitali systems.** The main theorem of this section is

**Theorem 4.1.** *Let $\varphi \in \Phi^N$ and $\eta \in H$ and $\kappa \in ]0,3^{-N}[$, and let $\eta^*(r) = r\,\eta(r)$. If*

$$\int_0^R (\varphi \circ \eta^* \circ \eta^*)(r)\, r^{-N-1} dr = +\infty \quad \text{for some} \quad R > 0\,, \tag{4.1}$$

*then $\mathfrak{B}^N_{uni}[\varphi; \eta, \kappa]$ has the Vitali property.*

By the assumption on $\varphi$ and $\eta$ it follows that if (4.1) holds for one $R > 0$, then it is satisfied for all $R > 0$. The proof of the theorem is fairly long and complicated (cf. [9] and [11]), so only a sketch is given here. First one proves that it suffices to establish the following

**Lemma 4.2.** *Let $\varphi \in \Phi^N$ and $\eta \in H$ satisfy (4.1), and let $\kappa \in ]0,3^{-N}[$. Let $A \subseteq [0, R]^N$ be dense in $[0,R]^N$ and suppose that the pair $(A, \mathscr{S})$ from $\mathfrak{B}^N_{uni}[\varphi; \eta, \kappa]$ satisfies the stronger geometrical condition*

$$\forall x \in A \quad \forall r \in ]0, R] \quad \exists K \in \mathscr{S}: K \subset B(x, r) \wedge |K| \geqq \varphi(r)\,. \tag{4.2}$$

*There exists a constant $c > 0$ and a disjointed subfamily $\mathscr{S}_1 \subseteq \mathscr{S}$ consisting of compact sets contained in $]0, R[^N$, such that*

$$|\bigcup\{S: S \in \mathscr{S}_1\}| \geqq cR^N (\geqq c|A|^*)\,. \tag{4.3}$$

The reductions leading to lemma 4.2 are using the definition of a Vitali system and a property of the outer Lebesgue measure. If $A$ is not dense in $[0, R]^N$ define $\mathscr{S}' = \mathscr{S} \cup \{K \in \mathscr{K}^N_{\eta,\kappa}: K \cap A = \emptyset\}$ and let $A'$ be given by $A \cup \bigcup\{K \in \mathscr{K}^N_{\eta,\kappa}: K \cap A = \emptyset\}$ and consider $(A' \cap [0, R]^N, \mathscr{S}') \in \mathfrak{B}^N_{uni}[\varphi; \eta, \kappa]$ instead. Then theorem 4.1 follows from lemma 4.2 and lemma 2.1.

The proof of lemma 4.2 is the core of the matter. Let $r_0 = R$ and define inductively a *division sequence* $(p_n)$ by

$$p_n = [3\alpha \cdot \eta(\alpha^{-1} r_n)^{-1}] + 1 \quad \text{and} \quad r_{n+1} = p_n^{-1} r_n\,, \quad n \in \mathbb{N}_0\,, \tag{4.4}$$

where $\alpha > 2$ is a constant and $[x]$ denotes the integer part $x \in \mathbb{R}_+$.

Let $Q^0 = [0, r_0]^N$ and perform a $p_0$-adic division of $Q^0$. If $p_0$ is odd, consider the central subcube $Q^0_*$ of edge-length $r_1$. If $p_0$ is even, consider anyone $Q^0_*$ of the $2^N$ central subcubes of edge-length $r_1$. By assumption we find $x_0 \in Q^0_* \cap A$ and choose an element $K^0 \in \mathscr{S}$, such that $K \subset Q^0$ and $|K| \geqq \phi(r_0/\alpha)$.

The subcubes from level 1 of edge-length $r_1$ are distributed into the three classes:

$$\begin{aligned} \mathfrak{A}_1 &= \{Q^1_j: K^0 \cap \operatorname{int} Q^1_j = \emptyset\}\,, && (\alpha \text{ cubes})\,,\\ \mathfrak{B}_1 &= \{Q^1_j: K^0 \cap \operatorname{int} Q^1_j \neq \emptyset\,,\ |Q^1_j \cap K^0| < \kappa|Q^1_j|\}\,, && (\beta \text{ cubes})\,,\\ \mathfrak{C}_1 &= \{Q^1_j: |Q^1_j \cap K^0| \geqq \kappa|Q^1_j|\}\,, && (\gamma \text{ cubes})\,. \end{aligned}$$

The $\gamma$ cubes satisfy the desired estimate, so they are removed from the process. The $\alpha$ cubes are treated as $Q^0$ above, only we perform a $p_1$-adic division instead, obtaining subcubes of edge-length $r_2$ from level 2, which are distributed into the three classes (after the choice of $K_1 \in \mathscr{S}$)

$$\mathfrak{A}_2 = \{Q_j^2 \colon K^1 \cap \operatorname{int} Q_j^2 = \emptyset\}, \qquad (\alpha \text{ cubes}),$$
$$\mathfrak{B}_2 = \{Q_j^2 \colon K^1 \cap \operatorname{int} Q_j^2 \neq \emptyset,\ |Q_j^2 \cap K^1| < \kappa |Q_j^2|\}, \qquad (\beta \text{ cubes}),$$
$$\mathfrak{C}_2 = \{Q_j^2 \colon |Q_j^3 \cap K^1| \geqq \kappa Q_j^2|\}, \qquad (\gamma \text{ cubes}).$$

If however $Q_j^1 \in \mathfrak{B}_1$ we cannot always be sure to choose an element $K \in \mathscr{S}$ of the right size, which also is disjoint from $K^0$, so we only perform a $p_1$-adic division of $Q_j^1$ and distribute the subcubes into the three classes $\mathfrak{A}_2$, $\mathfrak{B}_2$ and $\mathfrak{C}_2$ described above. From the definition of $(p_n)$ and definition 3.2 of $\mathscr{K}_{\eta,\kappa}^N$ follows the important fact that at least one of the $p_1^N$ subcubes of $Q_j^1$ from $\mathfrak{B}_1$ must belong to $\mathfrak{A}_2$.

In this way we proceed through all the levels, removing all $\gamma$ cubes and performing a $p_n$-adic division on all cubes from $\mathfrak{A}_n \cup \mathfrak{B}_n$. In cases of an $\alpha$ cube we choose an element $K \in \mathscr{S}$ contained in this cube and hence disjoint from all previous selected elements from $\mathscr{S}$, such that $|K| \geqq \varphi(\alpha^{-1} r_n)$.

Let $m_n = \max\{m \in \mathbb{N} \colon m_n < p_n(\alpha - 2)\alpha^{-1} - 1\}$, and let $\alpha_n$ and $\beta_n$ denote the numbers of elements in $\mathfrak{A}_n$ and $\mathfrak{B}_n$, resp.

a) Each $Q^n \in \mathfrak{A}_n$ creates at least $p_n^N - (p_n - m_n)^N$ elements in $\mathfrak{A}_{n+1}$.
b) Each $Q^n \in \mathfrak{B}_n$ creates at least one element in $\mathfrak{A}_{n+1}$.

A simple book-keeping shows that

$$\alpha_{n+1} + \beta_{n+1} \leqq p_n^N(\alpha_n + \beta_n)$$

and

$$\alpha_{n+1} \geqq \{p_n^N - (p_n - m_n)^N\}\, \alpha_n + \beta_n,$$

from which one derives that

$$\frac{\alpha_n}{\alpha_n + \beta_n} \geqq \frac{1}{p_{n-1}^N}, \quad n \in \mathbb{N}. \tag{4.5}$$

The ratio in (4.5) is the ratio of all cubes from $\mathfrak{A}_n \cup \mathfrak{B}_n$ still in the process, for which we can choose elements $K^n \in \mathscr{S}$ disjoint from the previous selected elements, where $|K^n| \geqq \varphi(\alpha^{-1} r_n)$. Letting

$$\Omega_n = \bigcup\{Q \colon Q \in \mathfrak{A}_n \cup \mathfrak{B}_n\}, \quad \Omega_0 \supseteq \Omega_1 \supseteq \ldots \supseteq \Omega_n \supseteq \ldots,$$

it suffices to prove that $|\Omega_n| \to 0$, since $Q^0 \setminus \Omega_n$ is composed of $\gamma$ cubes up to level $n$. This is done by using the selected class $\mathscr{S}' = \{K^n \colon n \in \mathbb{N}\}$ of disjointed elements from $\mathscr{S}$ as *gauges*, which can be piled up over $\Omega_n$ using (4.2) and the uniform structure. This process is totally fictive as $K \in \mathscr{S}'$ is disjoint from $\Omega_n$ for $n$ sufficiently large, but we are merely comparing the measures $|\Omega_n|$ and $|\bigcup\{K^n \colon n \in \mathbb{N}\}| \leqq 1$. We obtain that the *density* of the gauges over $\Omega = \cap\, \Omega_n$ is at least

$$\sum_{n=1}^{+\infty} r_{n-1}^{-N} \varphi(\alpha^{-1} r_n).$$

Due to (4.4), the assumption (4.1) and lemma 2.2 this density is $+\infty$, so $\Omega$ must be a nullset, and the lemma follows (□)

When $\eta \in H$ is constant we obtain $(\eta^* \circ \eta^*)(r) = \eta^2 r$, so (4.1) is reduced to

$$\int_0^R \varphi(r)\, r^{-N-1}\, dr = +\infty \quad \text{for some} \quad R > 0\,.$$

By using an example from [8] we obtain the improved result:

**Theorem 4.3.** *Let $\varphi \in \Phi^N$ and $\eta \in ]0,6^{-1}[$ and $\kappa \in ]0,3^{-N}[$. Then $\mathfrak{B}^N_{uni}[\varphi; \eta, \kappa]$ has the Vitali property, if and only if*

$$\int_0^R \varphi(r)\, r^{-N-1}\, dr = +\infty \tag{4.6}$$

*for one value of $R > 0$ and hence for all $R \in \mathbb{R}_+$.*

A simple calculation shows that for the class of trumpets $\mathcal{T}^N_\varrho$ of shape $\varrho$ one may choose as the corresponding $\Phi$ function

$$\varphi(r) = \tfrac{1}{2} c_{N-1} \int_0^{1/2r} \{\varrho(t)\}^{N-1}\, dt\,, \quad r \in \mathbb{R}_+\,,$$

cf. example 3.1. By a change of the order of integration, (4.6) is reduced to

$$\int_0^R \varrho(r)^{N-N}\, r^{-N}\, dr = +\infty \quad \text{for all} \quad R > 0\,,$$

so we easily obtain from theorem 4.3 and example 3.1:

**Corollary 4.4.** *Let $\varrho$ be a shape function, such that*

$$\int_0^R \varrho(r)^{N-N}\, r^{-N}\, dr = +\infty \quad \textit{for all} \quad R > 0\,.$$

*Let $A \subseteq \mathbb{R}^N$ and $\mathcal{S} \subseteq \mathcal{T}^N_\varrho$ satisfy*

$$\forall x \in A \quad \exists r_x > 0 \quad \forall r \in ]0, r_x] \quad \exists T \in \mathcal{S}: T \subset B(x, r) \wedge e(T) \geqq \tfrac{1}{2} r\,.$$

*There exists a disjointed subfamily $\mathcal{S}_1 \subseteq \mathcal{S}$, such that*

$$|A \setminus \bigcup\{T: T \in \mathcal{S}_1\}|^* = 0\,.$$

**5. Applications.** It was pointed out in [6] that Vitali systems of this non-classical type could be used to characterize some classes of nullsets. Here we shall mention one interesting result of this type by using trumpets.

**Theorem 5.1.** *Let $A \subseteq \mathbb{R}^N$ and let $\varrho$ be a shape function. Assume that*

$$\forall x \in A \quad \exists r > 0 \quad \exists T \in \mathcal{T}^N_\varrho: e(T) = r \wedge v(T) = \{x\} = T \cap A\,. \tag{5.1}$$

*If*

$$\int_0^R \varrho(r)^{N-1}\, r^{-N}\, \mathrm{d}r = +\infty \quad \textit{for all} \quad R > 0\,, \tag{5.2}$$

*then $A$ is a nullset.*

**Proof.** If $T \in \mathscr{T}_\varrho^N$, then also $T_\alpha \in \mathscr{T}_\varrho^N$ for $\alpha \in ]0, 1]$, where $T_\alpha$ is the truncated trumpet of $T$ of length $\alpha e(T)$. By collecting all trumpets satisfying (5.1) for some $x \in A$ we define a system $\mathscr{S}$ of trumpets, such that $(A, \mathscr{S})$ satisfies the assumptions of corollary 4.4, so we can find a disjoined family $\mathscr{S}_1 \subseteq \mathscr{S}$, such that

$$|A \setminus \bigcup\{T\colon T \in \mathscr{S}_1\}|^* = 0\,.$$

Since $\mathscr{S}_1$ is at most countable, it follows from (5.1) that

$$A \cap \bigcup\{T\colon T \in \mathscr{S}_1\} = \bigcup\{\{v(T)\}\colon T \in \mathscr{S}_1\}$$

is a nullset. Hence $A$ is a nullset. □

One interpretation of theorem 5.1 is the following. Let $\varrho$ be a shape function satisfying (5.2). Let $A \subseteq \mathbb{R}^N$ be a set, such that (almost) every point $x \in A$ is accessible from $\mathbb{R}^N \setminus A$ within a trumpet of shape $\varrho$. Then $A$ is a nullset. One example of a set having this property is the Koch curve $K$ in $\mathbb{R}^2$, where it is well-known that $K$ is a fractal set of Hausdorff dimension $\log 4/\log 3$, and hence for other reasons a nullset in $\mathbb{R}^2$.

In general one may interpret theorem 5.1 as a generalized Lipschitz condition on certain $C^0$-functions giving a sufficient condition for that the corresponding graph is a nullset.

It is well-known that Osgood's curve, cf. [17], is a Jordan curve of positive area. Assume that almost every point of Osgood's curve is accessible from the outside within a trumpet of shape $\varrho$. Then by contraposition of theorem 5.1 we obtain the following information on $\varrho$:

$$\int_0^R \frac{\varrho(r)}{r^2}\, \mathrm{d}r < +\infty \quad \text{for all} \quad R > 0\,,$$

since $N = 2$. Hence any such uniform shape function for the "gaps" in Osgood's curve must be very small near 0, and the corresponding wedges (trumpets in $\mathbb{R}^2$) extremely thin near their vertices.

The theory of trumpets presented above solves a problem posed by Flemming Topsøe at the conference Topology and Measure IV, Trassenheide, GDR, 1983. A careful examination of an earlier proof revealed that trumpets could be replaced by sets from $\mathscr{K}_{\eta,\kappa}^N$, hence the present far more general theory.

The positive Vitali theorems above for fairly general Vitali systems would naturally lead to the conjecture that the corresponding differentiation bases would differentiate more general classes of functions. It has recently been demonstrated [15] that this is not the case, because differentiation theorems also need some kind of halo condition, which is totally unnecessary for Vitali theorems. We shall end up with a result for differentiation bases, which is closely related to a result of Jessen-Marcinkiewicz-Zygmund [5], though we do have a Vitali theorem. The notation will be very close to that in [5].

Let $\varrho\colon \mathbb{R}_+ \to ]0, 1]$ be a continuous and nondecreasing function, where $\varrho(r) \to 0$ for $r \to 0$, and let $\varrho_\infty = \lim_{t\to+\infty} \varrho(t) \in ]0, 1]$. Define $\log_\varrho$ by

$$\log_\varrho t = \begin{cases} \log t & \text{for} \quad t > \varrho_\infty^{-1}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $\mathscr{B}_\varrho$ be the system of all axiparallel rectangles in $\mathbb{R}^2$, such that the smaller edge-length is $\geqq \varrho(r)\, r$, where $r$ denotes the larger edge-length. If $\varphi(r) = r^2\, \varrho(r)$, then $\varphi \in \Phi^2$, and (4.6) reduces to

$$\int_0^R \frac{\varrho(r)}{r}\, dr = +\infty \quad \text{for all} \quad R > 0\,. \tag{5.3}$$

Since rectangles are convex we have $\mathscr{B}_\varrho \subseteq \mathscr{K}^N_{\eta,\kappa}$ for all $\eta \in H$ and $\kappa \in ]0, \frac{1}{9}[$ so theorem 4.3 gives us a Vitali theorem for $(A, \mathscr{B}_\varrho) \in \mathfrak{V}^2_{uni}[\varphi; \eta, \kappa]$, when (5.3) is fulfilled. It is trivial to construct $\varrho$, such that (5.3) holds, while $\varrho(r) \to 0$ for $r \to 0$, e.g. $\varrho(r) = \{\log(1/r)\}^{-1}$ for $r$ sufficiently small and suitable otherwise.

Let $\psi\colon [0, +\infty[ \to [0, +\infty[$ be a nondecreasing function satisfying the conditions

$$\psi(0) = 0\,, \quad \liminf_{t\to+\infty} \frac{\psi(t)}{t} > 0\,,$$

and let $L_\psi$ denote the class of all measurable functions $f$, such that $\psi(|f|) \in L$. It follows immediately from [5] that the system $\mathscr{B}_\varrho$ defined above at least differentiates the class $L\log_\varrho L$. What is more interesting is the following theorem, which states that Jessen-Marcinkiewicz-Zygmund's theorem essentially still gives us the best result.

**Theorem 5.2.** *Let $\varrho$ and $\psi$ be as above, where $\varrho(r) \to 0$ for $r \to 0$. If $\mathscr{B}_\varrho$ differentiates $L_\psi$, then $\psi(r) \geqq cr \log_\varrho r$ for some constant $c > 0$.*

The proof uses the following

**Lemma 5.3.** *Let $E$ be an arbitrary bounded and measurable set, and let for $\alpha \in ]0, 1[$*

$$\sigma_\alpha(E) = \bigcup\{I \in \mathscr{B}_\varrho\colon |E \cap I| > \alpha|I|\}\,.$$

*If $\mathscr{B}_\varrho$ differentiates $L_\psi$, then one can find a constant $C > 0$, such that*

$$|\sigma_\alpha(E)| \leqq C\, \psi(1/\alpha)\, |E|\,.$$

**Lemma 5.3.** follows immediately from lemma $F$ in [5].

The rest of the proof of theorem 5.2 follows in spirit the corresponding proof in [5], though some very technical modifications are needed (cf. [15]).

The consequence of theorem 5.2 and (5.3) is that we may have a Vitali theorem in connection with $\mathscr{B}_\varrho$, and yet $\mathscr{B}_\varrho$ does not differentiate $L$. This is due to the fact that the halo $\sigma_\alpha(E)$ may be fairly large in measure compared with the measure of $E$.

A modification shows that $\log_\varrho$ may be replaced by the classical $\log^+$.

**6. Final remarks and acknowledgement.** It should be mentioned that there has been developed a corresponding theory for generalized pointwise Vitali systems, cf. [12]—[14]. The theorems in section 4 cannot be generalized to the pointwise case. In fact, M. Talagrand has given a counterexample in R (published in [6]) to such a conjecture. Instead one uses a totally different procedure which is closely related to Banach's method [1]. The estimates in the pointwise case are far more delicate than in the uniform case.

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