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On relations approximated by Continuous Functions

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Let X, Y be metric spaces. By a relation in $X \times Y$ we mean a nonempty subset of the product. A relation R is called closed if R is a closed subset of $X \times Y$.

In the papers [1], [2], [8] are given some conditions under which there exist "well-behaved" functions that approximate closed relations.

This paper studies properties of closed relations that are approximate by continuous functions in the Hausdorff metric. Properties of a special class of such closed relations are also considered in [3]. We obtain a much more inclusive result.

Let (Z, d) be a metric space. If $Z \supset E$ and $\varepsilon > 0$, let $B_\varepsilon[E]$ denote the union of all open ε-balls whose centers run over E and $B_\varepsilon[x]$ denote the open ε-ball about a point x.

If E and F are nonempty subsets of Z and for some $\varepsilon > 0$ both $B_\varepsilon[F] \supset E$ and $B_\varepsilon[E] \supset F$, then the Hausdorff distance h_d between them is given by $h_d(E, F) = \inf\{\varepsilon: B_\varepsilon[E] \supset F \text{ and } B_\varepsilon[F] \supset E\}$. Otherwise we put $h_d(E, F) = \infty$.

If we identify the sets with the same closure, then h_d is well defined on the equivalence classes so determined. Moreover, h_d defines an extended real valued metric on the class of nonempty closed subsets of Z, called the Hausdorff metric. Basic facts about this metric can be found in [7] Castaing and Valadier.

Now, let (X, d_x) and (Y, d_y) be metric spaces. We first need a metric on $X \times Y$ to induce the Hausdorff metric. For definiteness and computational simplicity, we take ϱ defined by $\varrho((x_1, y_1), (x_2, y_2)) = \max\{d_x(x_1, x_2), d_y(y_1, y_2)\}$.

Denote $C(X, Y)$ the set of all continuous functions from X to Y. Using the metric ϱ we can restrict the Hausdorff metric h_ϱ defined on the closed subsets of $X + Y$ to the graphs of functions in $C(X, Y)$. Denote this metric d_2.

Explicitly, if f and g are in $C(X, Y)$, let us represent their graphs by $G(f)$ and $G(g)$ respectively. Then $d_2(f, g)$ is defined by the formula $d_2(f, g) = \inf\{\varepsilon: B_\varepsilon[G(f)] \supset G(g) \text{ and } B_\varepsilon[G(g)] \supset G(f)\}$.

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The Hausdorff metric on $C(X, Y)$ was studied by Beer [3], Naimpally [9], Waterhouse [10] and some other authors.

Let $F(X, Y)$ be the set of all functions from X to Y. In the same way we can use d_2 to define the distance between any two functions from $F(X, Y)$: if f and g are two such functions denote the closures of their graphs by cl $G(f)$ and cl $G(g)$ respectively, and let $d_2(f, g)$ be the Hausdorff distance from cl $G(f)$ to cl $G(g)$. The function d_2 only defines a pseudometric on the space $F(X, Y)$.

The terminology and notation of J. Kelley will be used throughout. Moreover, we shall use the following notions and notations.

The closure of a subset M of a topological space X will be denoted by cl M.

Let X, Y be topological spaces. Let $\mathscr{P}(Y)$ denote the collection of all subsets of Y. A multifunction H from X to Y is a function $H: X \to \mathscr{P}(Y)$.

A multifunction H is called closed if its graph $\{(x, y): x \in X \text{ and } y \in H(x)\}$ is a closed subset of $X \times Y$. We shall denote the graph of a multifunction H by $G(H)$.

A multifunction H from X to Y is called upper semicontinuous at z in X if whenever V is an open subset of Y that contains $H(z)$ then the set $\{x: H(x) \subset V\}$ contains a neighbourhood of z. It is called upper semicontinuous if it is upper semicontinuous at every $z \in X$.

Let R be a relation in $X \times Y$. We shall use the following notation for vertical section at x of R: $R(x) = \{y:(x, y) \in R\}$. Define the multifunction H_R induced by R by $H_R(x) = R(x)$. Then $G(H_R) = R$.

N will denote the set of positive integers.

Let Y be a metric space. Let $\mathscr{X}$ be a functional defined on $\mathscr{P}(Y)$ as follows $\mathscr{X}(\emptyset) = 0$ and if A is a nonempty subset of Y, then $\mathscr{X}(A) = \inf\{\varepsilon: A \text{ has a finite } \varepsilon\text{-dense subset}\}$. In the literature $\mathscr{X}$ has been called the Hausdorff measure of noncompactness functional.

Lemma 1. (see [4]) The Hausdorff measure of non-compactness functional acts as follows:

(a) $\mathscr{X}(A) = \infty$ if and only if A is unbounded
(b) $\mathscr{X}(A) = 0$ if and only if A is totally bounded
(c) If $A \subset B$, then $\mathscr{X}(A) \leqq 2\mathscr{X}(B)$
(d) If A is totally bounded, then for each $\varepsilon > 0$, $\mathscr{X}(B_\varepsilon[A]) \leqq \varepsilon$
(e) $\mathscr{X}(\text{cl } A) = \mathscr{X}(A)$.

Theorem 1. Let X, Y be metric spaces. Let X be a locally compact space and Y be a complete metric space. Let $\{f_n\}$ be a sequence from $C(X, Y)$ such that the graphs of the terms of $\{f_n\}$ converge in the Hausdorff metric to a closed relation R in $X \times Y$. Then the multifunction H_R induced by R is upper semicontinuous and $R(x)$ is a nonempty compact set for each $x \in X$.

Proof. Put $A = \{x \in X: R(x) \neq \emptyset\}$. The set A is dense in X (see [1]). Suppose that $A \neq X$. Let $x \in X \setminus A$. There is $\delta > 0$ such that cl $B_\delta[x]$ is compact. Put $B =$

$= \bigcup\{R(a)\colon a \in A \cap B_{\delta/2}[x]\}$. We show that $\mathscr{X}(B) = 0$, where $\mathscr{X}$ is the Hausdorff measure of noncompactness functional. Let $\varepsilon > 0$. Put $\eta = \min\{\varepsilon/2, \delta/2\}$. There is $j \in N$ such that $h_\varrho(R, G(f_n)) < \eta$ for every $n \geqq j$ (1).

Let $n \geqq j$. Then $B \subset B_\eta[f_n(B_\delta[x])]$. Let $y \in B$. There is $a \in B_{\delta/2}[x]$ such that $(a, y) \in R$. By (1) there exists a point $(b, f_n(b))$ for which $\varrho((a, y), (b, f_n(b))) < \eta$. Then $y \in B_\eta[f_n(b)]$ and $b \in B_\eta[a] \subset B_{\delta/2} \subset B_\delta[x]$. Thus we have $B \subset B_\eta[f_n(B_\delta[x])]$. Since $f_n(\mathrm{cl}\, B[x])$ is compact, by (d) of Lemma 1 we have $\mathscr{X}(B_\eta[f_n(\mathrm{cl}\, B_\delta[x])] \leqq \eta \leqq \leqq \varepsilon/2$ and by (c) of Lemma 1 we have $\mathscr{X}(B) \leqq \varepsilon$. Since $\mathscr{X}(B) \leqq \varepsilon$ for any $\varepsilon > 0$, $\mathscr{X}(B) = 0$. Thus $\mathscr{X}(\mathrm{cl}\, B) = 0$. By (b) of Lemma 1,$B$ is a totally bounded set. The completeness of Y implies that cl B is compact.

There is a sequence $\{x_n\}$ of points of $A \cap B_{\delta/2}[x]$ such that $\{x_n\}$ converges to x. Let $\{y_n\}$ be a sequence of points of Y such that cl $(x_n, y_n) \in R$. Since $\{y_n\}$ is a sequence of points of B and cl B is compact there is a cluster point z of the sequence $\{y_n\}$. Then (x, z) is a cluster point of the sequence $\{(x_n, y_n)\}$, i.e. $(x, z) \in \mathrm{cl}\, R$. But $(x, z) \notin R$ contradicting to the fact that R is closed.

For each $x \in X$ there are an open neighbourhood V_x and a compact set C_x such that $\bigcup\{R(u)\colon u \in V_x\} \subset C_x$. Let $x \in X$. There is $\delta_x > 0$ such that cl $B_{\delta_x}[x]$ is compact. Put $V_x = B_{\delta_x/2}[x]$ and $C_x = \mathrm{cl} \cup \{R(v)\colon v \in V_x\}$. The proof of the compactness of C_x is similar as above.

By result of Berge (see [6]) any closed multifunction with the compact range space is upper semicontinuous. Thus H_R is upper semicontinuous on V_x for each x. It is easy to see that then H_R is upper semicontinuous. Since $R(x)$ is a closed subset of the compact set C_x for each $x \in X$, $R(x)$ is a compact set for each $x \in X$.

Corollary 1. Let X, Y be metric spaces. Let X be a locally compact metric space and Y be a complete metric space. Let $\{f_n\}$ be a sequence of functions from $C(X, Y)$ d_2-convergent to a function $f\colon X \to Y$ with a closed graph. Then f is continuous.

The following example shows that the assumption of the locally compactness in Theorem 1 and Corollary 1 is essential.

Example 1. Let Y be the set of real numbers with the usual metric. Let $n \in N$. Let $\{x_j^n\}_{j=1}^\infty$ be a sequence of points of the open interval $(1/n, 1/n - 1)$ which is convergent to $1/n$. Put $X = \{0\} \cup \bigcup_{n=1}^{\infty} \{x_j^n\colon j = 1, 2, \ldots\}$ and consider X with the usual metric. It is easy to verify that X is not a locally compact space. Define the function f by $f(x) = n$ for $x = x_j^n$ $j = 1, 2, \ldots$ and $f(0)$. Let g_n $(n = 1, 2, \ldots)$ be a bijection from the set $\{x_j^n\colon j = 1, 2, \ldots\}$ to the set $\{j \in N\colon j \geqq n\}$ and define the functions f_n $(n = 1, 2, \ldots)$ as follows:

$$f_n(x) = \begin{cases} g_n(x) & \text{for } x = x_j^n \quad j = 1, 2, \ldots \\ f(x) & \text{for } x = x_j^m \quad m < n\,, \quad j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that the sequence $\{f_n\}$ is a sequence of continuous functions d_2-convergent to the discontinuous function f with a closed graph.

Proposition 1. If a metric space Y is not complete, then there exist a compact metric space X and a sequence of continuous functions from X to Y d_2-convergent to a discontinuous function with a closed graph.

Proof. There exists a Cauchy sequence $\{y_n\}$ in Y which has no cluster point in Y. Let $\tilde{Y}$ be a completion of Y. There exists $y \in \tilde{Y}$ such that $\{y_n\}$ converges to y in $\tilde{Y}$. Put $X = \{y, y_1, y_2, \ldots, y_n, \ldots\}$ and consider X with the induced metric. Then X is compact. Define the functions $f_n\colon X \to Y$ $(n = 1, 2, \ldots)$ by $f_n(y_i) = y_i$ for $i \leqq n$ and $f_n(x) = y_1$ otherwise. It is easy to see that the functions f_n $(n = 1, 2, \ldots)$ are continuous. Now define the function $f\colon X \to Y$ as follows: $f(y_i) = y_i$ and $f(y) = y_1$. Since the sequence $\{y_n\}$ has no cluster point in Y the function f has a closed graph. But f is not continuous. (There exists an open set V in Y such that $y_1 \in V$ and $y_n \notin V$ for every $n \geqq 2$. Then $f^{-1}(V) = \{y, y_1\}$ is not open in X.)

It remains to prove that $\{f_n\}$ d_2-converges to f. Let $\varepsilon > 0$. There exists $j \in N$ such that for every $n, m \geqq j$ $d_y(y_n, y_m) < \varepsilon$ and $d_x(y, y_n) < \varepsilon/2$. We show that $G(f) \subset B_\varepsilon[G(f_n)]$ and $G(f_n) \subset B_\varepsilon[G(f)]$ for every $n \geqq j$. Let $x \in X$ and $n \geqq j$. If $x = y_i$ for $i \leqq n$ or $x = y$ then $\varrho((x, f(x)), (x, f_n(x))) = 0$. Let $x \in X$ and $x = y_i$ for $i > n$. for Then $d_x(y_i, y_n) < \varepsilon$ and thus $\varrho((y_i, f(y_i)), (y_n, f_n(y_n))) = \varrho((y_i, y_i), (y_n, y_n)) < \varepsilon$, i.e. $(x, f(x)) \in B_\varepsilon[G(f_n)]$.

Now choose $(y_i, f_n(y_i))$ for $i > n \geqq j$. Thus $f_n(y_i) = y_1$. Hence $\varrho((y_i, f_n(y_i)), (y, f(y))) = \max\{d_x(y_i, y), d_y(y_1, y_1)\} \leqq \varepsilon/2 < \varepsilon$, i.e. $(y_i, f_n(y_i)) \in B_\varepsilon[G(f)]$ and thus $G(f_n) \subset B_\varepsilon[G(f)]$.

Theorem 2. Let X be a locally connected metric space and Y be a locally compact metric space. Let R be a closed relation in $X \times Y$ such that $R(x)$ is a nonempty compact set for each $x \in X$. Let $\{f_n\}$ be a sequence from $C(X, Y)$ such that the graphs of the terms of the sequence $\{f_n\}$ converge in the Hausdorff metric to R. Then $R(x)$ is a connected set for each $x \in X$.

Proof. Fix $x \in X$. If $R(x)$ is a singleton, then $R(x)$ is connected. Otherwise, suppose that $R(x)$ contains at least two distinct points. Then x is not an isolated point of X (see [1]).

Suppose that $R(x)$ is not connected. The compactness of $R(x)$ implies that there are nonempty compact sets C, D such that $C \cap D = \phi$ and $R(x) = C \cup D$. Since Y is a locally compact metric space, there exists $\varepsilon > 0$ such that $\operatorname{cl} B_\varepsilon[C] \cap \operatorname{cl} B_\varepsilon[D] = \phi$ and $\operatorname{cl} B_\varepsilon[C]$, $\operatorname{cl} B_\varepsilon[D]$ are compact sets. Fix $u \in C$, $v \in D$. Let $\{B_n\}$ be a sequence of connected neighbourhoods of x such that $B_n \subset B_{1/n}[x]$ for each $n \in N$.

The convergence of the sequence $\{f_n\}$ to R in the Hausdorff metric implies that there are an increasing sequence of positive integers $\{k_\in\}$ and sequences $\{x_n\}$, $\{y_n\}$ of points of X such that $\varrho((x, u), (x_n, f_{k_n}(x_n))) < 1/n$, $\varrho((x, v), (y_n, f_{k_n}(y_n))) < 1/n$ and $x_n, y_n \in B_n$ for each $n \in N$.

Put $L = \{y \in Y: \inf d_y(y, c) = \varepsilon/2\}$. The connectivity of sets $f_{k_n}(B_n)$ $(n = 1, 2, \ldots)$ implies that there is $j \in N$ such that $L \cap f_{k_n}(B_n) \neq \emptyset$ for each $n \geqq j$. Let $\{v_n\}_{n=j}^{\infty}$ be a sequence of points of Y such that $v_n \in L \cap f_{k_n}(B_n)$ for each $n \geqq j$ and $\{a_n\}_{n=j}$ be a sequence of points of X such that $f_{k_n}(a_n) = v_n$ and $a_n \in B_n$ for each $n \geqq j$. Then $\{a_n\}_{n=j}^{\infty}$ converges to x.

Since L is a closed subset of the compact set $\mathrm{cl}\, B_\varepsilon[C]$, L is compact. Thus there exists a cluster point $z \in L$ of the sequence $\{v_n\}_{n=j}^{\infty}$, i.e. (x, z) is a cluster point of the sequence $\{(a_n, v_n)\}_{n=j}^{\infty}$ (2).

We show that $(x, z) \in R$. Suppose that $(x, z) \notin R$. The closedness of R implies that there is $\delta > 0$ for which $(B_\delta[x] \times B_\delta[z]) \cap R = \emptyset$. There is $l \in N$ such that $h_\varrho(R, G(f_n)) < \delta/2$ for every $n \geqq l$ (3).

By (2) there is $m \in N$ such that $k_m \geqq l$ and $(a_m, f_{k_m}(a_m)) \in B_{\delta/2}[x] \times B_{\delta/2}[z]$. By (3) there is $(a, b) \in R$ such that $\varrho((a_m, f_{k_m}(a_m)), (a, b)) < \delta/2$. But then $\varrho((a, b), (x, z)) < \delta$ and that is a contradiction. Thus $(x, z) \in R$. Then $z \in C \cup D$. But $z \in L$. Thus $R(x)$ is connected.

Theorem 3. Let X be a locally connected, locally compact metric space and Y be a locally compact complete metric space. Let $\{f_n\}$ be a sequence from $C(X, Y)$ such that the graphs of the terms of the sequence $\{f_n\}$ converge in the Hausdorff metric to a closed relation R in $X \times Y$. Then H_R is an upper semicontinuous multifunction and $R(x)$ is a nonempty compact connected set for each $x \in X$.

Proof. By Theorem 1 H_R is an upper semicontinuous closed multifunction and $R(x)$ is a nonempty compact set for each $x \in X$. By Theorem 2 $R(x)$ is a connected set for each $x \in X$.

Let $f \in F(X, Y)$. Define the limit set multifunction H_f induced by f (see [3]) as follows: $H_f(x) = \{y \in Y: (x, y) \in \overline{G(f)}\}$ for each $x \in X$ and put $U(X, Y) = \{f \in F(X, Y): H_f$ is upper semicontinuous and $H_f(x)$ is a compact connected set for every $x \in X\}$.

From Theorem 3 we can obtain the following results

Theorem 4. Let X be a locally compact, locally connected metric space and Y be a locally compact complete metric space. Then the closure of $C(X, Y)$ in $(F(X, Y), d_2)$ is a subset of $U(X, Y)$.

Proof. Let $f \in F(X, Y)$ and $\{f_n\}$ be a sequence from $C(X, Y)$ d_2-convergent to f. The graphs of the sequence $\{f_n\}$ converge in the Hausdorff metric to the closed relation $G(H_f)$. By Theorem 3 $H_{G(H_f)}$ is upper semicontinuous and $G(H_f)(x)$ is a compact connected set for each $x \in X$. Since $H_{G(H_f)} = H_f$ and $G(H_f)(x) = H_f(x)$ for every $x \in X$ we have the assertion of Theorem.

If Y is the set of real numbers, Theorem 4 is proved in [3].

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