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Some Remarks on Almost Radiality in Function Spaces

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Recently it was proved, by Gerlits, Nagy and Szentmiklóssy, that the space $C_p(X)$ of continuous real functions on X, with the topology of pointwise convergence, is radial if and only if it is Fréchet and that there exists a space X for which $C_p(X)$ is pseudoradial but not Fréchet, To find the precise border between the properties of being pseudoradial and Fréchet for $C_p(X)$. we introduce the classes of u-pseudoradial and u-almost radial spaces. If $S = (f_\alpha)_{\alpha<\lambda}$ is a λ-sequence, a function φ: $X \to \lambda$ is called an S-function if $f_\alpha(x) = f_{\varphi(x)}(x)$ for every $\alpha \geqq \varphi(x)$ and every $x \in X$. (S,f) is said to be an $ut\lambda$-sequence if it is an ω-sequence or it is a $t\lambda$-sequence $(\lambda > \omega)$ and has a continuous S-function. A space $C_p(X)$ is called u-almost radial if for any non-closed A in it, there is an $ut\lambda$-sequence $(S\ f)$ such that S is a λ-sequence in A and $f \in \bar{A} - A$. Various properties of u-pseudoradial and of u-almost radial spaces are proved. In particular, that, if ξ is an ordinal number, then $C_p(\xi)$ is pseudoradial if and only if it is u-almost radial. This implies that there exist u-almost radial spaces $C_p(X)$ which are not Fréchet.

1. Introduction

All spaces mentioned in this paper are completely regular Hausdorff spaces.

In [G] J. Gerlits proved that for spaces $C_p(X)$ of continuous real valued functions on a topological space X endowed with the topology of pointwise convergence, the notion of Fréchet and of sequential space coincide. Recently, J. Gerlits showed in [GNS] that also the notion of radial space coincide with the notion of Fréchet space for $C_p(X)$ spaces, and Gerlits, Nagy and Szentmiklóssy showed in [GNS] that the notions of pseudoradial and Fréchet spaces are different for $C_p(X)$.

In [AIT] a new class of spaces, namely the class of almost radial spaces, was

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introduced. It contains the class of radial and is contained in the class of pseudoradial spaces. A natural question arises after Gerlits theorem from [GNS], whether the notions of almost radial and of Fréchet space coincide in $C_p(X)$. In this paper it is shown that these properties are different in $C_p(X)$. Moreover, a special subclass of the class of all almost radial spaces of the type $C_p(X)$ is introduced, namely the class of *u-almost radial spaces*, which contains the class of radial spaces of the type $C_p(X)$ (i.e. the class of spaces $C_p(X)$ which are Fréchet spaces) but is different from it. The class of u-almost radial spaces is defined using the notion of $u\lambda$-sequence which is introduced here as a generalization of the notion of uniformly convergent sequence of real-valued functions for "long" sequences of real-valued functions (namely, for λ-sequences).

For spaces of the type $C_p(X)$ a new subclass of the pseudoradial spaces is introduced: the class of *u-pseudoradial spaces*. It is observed that if X is metacompact, then $C_p(X)$ is u-pseudoradial if and only if it is Fréchet. On the other hand, every u-almost radial space is u-pseudoradial and hence the radial spaces $C_p(X)$ are a proper subclass of the u-pseudoradial spaces $C_p(X)$.

A. V. Arhangel'skii posed the problem of characterizing internally the class of spaces X for which $C_p(X)$ is pseudoradial. Here a rather complicated characterization of this class is given, which can be regarded as a first step towards the solution of Arhangelskii's question.

2. Preliminary results and definitions

2.1. Definition. For any cardinal number λ a λ-sequnce $S = (x_\alpha)_{\alpha<\lambda}$ in a topological space X is a function ftom λ to X.

2.2. Definition. [AIT], [DIT] Let $S = (x_\alpha)_{\alpha<\lambda}$ be a λ-sequence in X and $x \in X$. The pair (S, x) is called a $t\lambda$-sequence if λ is an initial and regular ordinal number, x is a limit point of S, $x_\alpha = x_\beta$, for $\alpha \neq \beta$, $\alpha, \beta < \lambda$ and $x \notin \operatorname{cl}\{x_\alpha \in S: \alpha < \beta\}$ for any $\beta < \lambda$.

2.3 Notations. Let X be a topological space and $A \subset X$. We shall use the following notations:

$\operatorname{Lim} A = \{x \in X$: there exists a λ-sequence S of points in A, such that x is a limit point of S;;

$t\text{-}\operatorname{Lim} A = \{x \in X\}$ there exists a λ-sequence S of points in A, such that (S, x) is a $t\lambda$-sequence$\}$;

If τ is a cardinal number, then

$\operatorname{Lim}_\tau A = \{x \in X$: there is a λ-sequence S of points of A such that $\lambda \leqq \tau$ and x is a limit point of $S\}$.

2.4 Definition. [H] A topological space X is said to be pseudoradial if every subset A of X, for which $\operatorname{Lim} A \subset A$ holds, is closed in X. A topological space X is radial if for every subset A of X $\operatorname{Lim} A = \bar{A}$ holds.

2.5 Definition. [AIT] A topological space X is said to be almost radial if every subset A of X, for which t-$\operatorname{Lim} A \subset A$ holds, is closed in X.

2.6 Theorem. (Gerlits [G, GNS)] The following conditions are equivalent:

(a) $C_p(X)$ is a Fréchet space;
(b) $C_p(X)$ is a sequential space;
(c) $C_p(X)$ is a radial space.

Let us recall that an internal characterization of spaces X for which $C_p(X)$ is Fréchet is given in [GN].

2.7 Theorem (Gerlits, Nagy, Szentmiklóssy [GNS]) Let ξ be an ordinal number, also considered as the topological space of all ordinal numbers less than ξ with the usual order topology. Then $C_p(\xi)$ is Fréchet if and only if $\operatorname{cf}(\xi) \leqq \omega$. $C_p(\xi)$ is pseudoradial and non-Fréchet if and only if ξ is regular and $\lambda^\omega < \xi$ for every $\lambda < \xi$ (i.e. ξ is ω_1-inaccessible).

2.8 Theorem. [GNS] If X is metacompact and $C_p(X)$ is pseudoradial then X is Lindelöf.

We will denote by $\mathbb{R}$ the real line with its natural topology, by $\mathbb{Q}$ the rational numbers in $\mathbb{R}$ and by $\mathbb{Q}^+$ the non-negative rationals. If ξ is an ordinal number, we will use the same notation for the topological space of all ordinal numbers less than ξ, with the usual order topology. We denote by c_0 the function $c_0: X \to \mathbb{R}$ such that $c_0(x) = 0$ for every $x \in X$.

Recall that a family Ω of subsets of a set X is called an ω-cover of X (see [GN]) if for every finite subset F of X there is an element U of Ω such that $F \subset U$. Finally, if $\Omega = \{U_\alpha: \alpha < \lambda\}$ is a family of subsets of a set X, then we denote $\operatorname{Lim} \Omega = \bigcup\{\bigcap\{U_\beta: \alpha \leqq \beta < \lambda\}: \alpha < \lambda\}$.

3. Results

3.1 Definition. Let λ be a regular uncountable cardinal number and $S = (f_\alpha)_{\alpha<\lambda}$ be a λ-sequence in $C_p(X)$. A function $\varphi: X \to \lambda$ is said to be an S-function if $f_\alpha(x) = f_{\varphi(x)}(x)$ for every $\alpha \geqq \varphi(x)$ and for every $x \in X$.

3.2 Remark. It is well known that if λ is a regular uncountable cardinal and $(a_\alpha)_{\alpha<\lambda}$ is a convergent λ-sequence in $\mathbb{R}$, then there exists $\alpha_0 < \lambda$ such that $a_\alpha = a_{\alpha_0}$ for every $\alpha \geqq \alpha_0$ $(\alpha < \lambda)$. Hence, for every convergent λ-sequence $S = (f_\alpha)_{\alpha<\lambda}$ in $C_p(X)$ there is an S-function $\varphi: X \to \lambda$, namely $\varphi(x) = \min\{\alpha < \lambda: f_\beta(x) = f_\alpha(x)$ for every $\beta \geqq \alpha\}$.

3.3 Definition. Let λ be a regular uncountable cardinal. A λ-sequence ($t\lambda$-sequence) $S = (f_\alpha)_{\alpha<\lambda}$ in $C_p(X)$ will be said to be a *uλ-sequence* (*utλ-sequence*) if there is a *continuous* S-function $\varphi: X \to \lambda$. It will be convenient to call every usual (countable) sequence $S = (f_n)_{n\in\omega}$ a *uω-sequence* (a *utω-sequence*).

3.4 Remark. Let λ be a regular uncountable cardinal and $S = (f_\alpha)_{\alpha<\lambda}$ be a convergent λ-sequence in $C_p(X)$. Let f be the limit function. It is natural to say that S is *uniformly convergent* to f if for every $\varepsilon > 0$ there is an $\alpha_0 < \lambda$ such that $|f(x) - f_\alpha(x)| < \varepsilon$ for every $\alpha \geqq \alpha_0$ and every $x \in X$. But this means that there exists $\bar{\alpha} < \lambda$, such that $f_\alpha = f$ for every $\alpha \geqq \bar{\alpha}$.

Hence, putting $\varphi(x) = \bar{\alpha}$ for every $x \in X$, we obtain a continuous S-function $\varphi: X \to \lambda$. So, every uniformly convergent λ-sequence is an $u\lambda$-sequence. It is easy to see that the convergence is not true.

The following proposition (whose obvious proof is omitted) is probably known.

3.5 Proposition. Let λ be a regular uncountable cardinal and $S = (f_\alpha)_{\alpha<\lambda}$ be a λ-sequence in $C_p(X)$. Then S is convergent in $C_p(X)$ (i.e. there is a continuous function f such that $\lim_{\alpha\to\lambda} f_\alpha(x) = f(x)$ for every $x \in X$) if and only if there is an S-function $\varphi: X \to \lambda$ such that for every $n \in \omega$ there exists an open neighborhood $U_{n,x}$ of x such that $|f_{\varphi(x)}(x) - f_{\varphi(y)}(y)| < 1/n$ for every $y \in U_{n,x}$.

It is very easy to prove (directly or using Proposition 3.5) the following proposition, which again (see Remark 3.4) exhibits the analogy between $u\lambda$-sequences and uniformly convergent sequences.

3.6 Proposition. Let λ be a regular uncountable cardinal and $S = (f_\alpha)_{\alpha<\lambda}$ be an $u\lambda$-sequence in $C_p(X)$. Then S is convergent in $C_p(X)$.

3.7 Example. There exists a space X and a convergent λ-sequence $S = (f_\alpha)_{\alpha<\lambda}$ in $C_p(X)$, such that λ is a regular uncountable cardinal and there is no $u\lambda'$-sequence in the set $\{f_\alpha: \alpha < \lambda\}$ ($\lambda' \leqq \lambda$) convergent to the limit function of S.

Proof. Let λ be a regular uncountable cardinal and X be the Alexandroff's long line of size λ (see [E]). Let $f_\alpha: X \to \mathbb{R}$, for $\alpha < \lambda$, be defined by $f_\alpha(x) = 0$ if $x \leqq \alpha$, $f_\alpha(x) = 1$ if $x \geqq \alpha + 1$, and f_α be linearly increasing from 0 to 1 for $\alpha \leqq x \leqq \alpha + 1$. Then $S = (f_\alpha)_{\alpha<\lambda}$ is a convergent to c_0 λ-sequence. Since X is connected, every $u\lambda'$-sequence $S' = (g_\alpha)_{\alpha<\lambda'}$ (where λ' is a regular uncountable cardinal) is almost trivial (i.e. there exists $\alpha_0 < \lambda'$, such that $g_\alpha = g_{\alpha_0}$ for every $\alpha \geqq \alpha_0 (\alpha < \lambda')$. Hence c_0 cannot be limit function of any $u\lambda'$-sequence in the set $\{f_\alpha: \alpha < \lambda\}$. ∎

3.8 Definition. Let X be a space. The space $Y = C_p(X)$ is called u-pseudoradial (respectively, *u-almost radial*) if for every non-closed subset A of Y there exist a $u\lambda$-sequence (respectively, a $ut\lambda$-sequence) $S = (f_\alpha)_{\alpha<\lambda}$ in A, which converges in Y to some $f \in \bar{A} \setminus A$.

3.9 Remark. Obviously, every u-pseudoradial (respectively, u-almost radial) space is pseudoradial (respectively, almost radial), and every u-almost radial space is u-pseudoradial. Also every Fréchet space is u-almost radial and hence, by Gerlits' theorem (see Theorem 2.6 here) every radial space is u-almost radial. (Of course, here by space we mean a space of type $C_p(X)$).

3.10 Proposition. Let X be a Lindelöf space. Then $C_p(X)$ is u-pseudoradial if and only if it is Fréchet.

Proof. Let λ be a regular uncountable cardinal and let $S = (f_\alpha)_{\alpha<\lambda}$ be a $u\lambda$-sequence in $C_p(X)$. Let $\varphi\colon X \to \lambda$ be a continuous S-function which exists by the definition of $u\lambda$-sequence. Then for every $x \in X$ there is an open neighborhood U_x of x such that $\varphi(y) \leqq \varphi(x)$ for any $y \in U_x$. Since X is Lindelöf, there exists a subcover $\{U_{x_n}\colon n \in \omega\}$ of the open cover $\{U_x\colon x \in X\}$ of X. Let $\alpha_0 = \sup\{\varphi(x_n)\colon n \in \omega\}$. Then $\alpha_0 < \lambda$ and $\varphi(x) \leqq \alpha_0$ for every $x \in X$. This means $f_\alpha = f_\beta$ for $\alpha, \beta \geqq \alpha_0$, i.e. the λ-sequence S is almost trivial.

Hence, if $C_p(X)$ is u-pseudoradial, then $C_p(X)$ is Fréchet. The converse is clear.

3.11 Corollary. Let X be metacompact. Then $C_p(X)$ is u-pseudoradial if and only if it is Fréchet.

Proof. Let $C_p(X)$ be u-pseudoradial. Then it is pseudoradial and, since X is metacompact, from Gerlits' theorem (see Theorem 2.8 here) it follows that X is Lindelöf. Now apply Proposition 3.10. ■

3.12 Proposition. Let ξ be a regular cardinal. Then $C_p(\xi)$ is pseudoradial if and only if it is almost radial.

Proof. Let $C_p(\xi)$ be pseudoradial. If $cf(\xi) \leqq \omega$, then, by Theorem 2.7, it is Fréchet and hence almost radial. So we can suppose that $cf(\xi) > \omega$. Let λ be a regular uncountable cardinal, $\lambda \leqq \xi$ and $S = (f_\alpha)_{\alpha<\beta}$ be a convergent to c_0 λ-sequence in $C_p(\xi)$.

Since $cf(\xi) > \omega$, for every $\alpha < \lambda$ there exists $c_\alpha \in \mathbb{R}$ and $x_\alpha \in \xi$ such that $f_\alpha \mid [x_\alpha, \xi) \equiv c_\alpha$.

Let us suppose first that $\lim_{\alpha\to\lambda} c_\alpha = 0$.

Since $cf(\xi) > \omega$. we have that $\beta\xi = \xi + 1$. Hence every $f_\alpha \in S$ has an extension $\beta f_\alpha\colon \xi + 1 \to \mathbb{R}$. Obviously, $\beta f_\alpha(\xi) = c_\alpha$.

Since $\lim c_\alpha = 0$, we obtain that the λ-sequence $S^\beta = (\beta f_\alpha)_{\alpha<\lambda}$ is convergent to c_0 in $C_p(\xi + 1)$. But, since $cf(\xi + 1) = 1 \leqq \omega$, we obtain, from Theorem 2.7, that $C_p(\xi + 1)$ is a Fréchet space. Hence there is a sequence $(\beta f_{\alpha_n})_{n\in\omega}$ converges to c_0 in $C_p(\xi + 1)$. But then $(f_{\alpha_n})_{n\in\omega}$ converges to c_0 in $C_p(\xi)$. So the function c_0 can be obtained as a limit function of an usual sequence in the set $\{f_\alpha\colon \alpha < \lambda\}$.

Let now be false that $\lim_{\alpha\to\lambda} c_\alpha = 0$.

Then we can suppose, eventually taking some λ-subsequence of S, that $c_\alpha \neq 0$ for every $\alpha < \lambda$. Indeed, if there is a cofinal λ-subsequence $(c_{\alpha_\beta})_{\beta<\lambda}$ of the λ-sequence $(c_\alpha)_{\alpha<\lambda}$, such that $c_{\alpha_\beta} = 0$ for every $\beta < \lambda$, then the λ-sequence $S' = (f_{\alpha_\beta})_{\beta<\lambda}$ is convergent to c_0 λ-sequence in $C_p(\xi)$ and $\lim c_{\alpha_\beta} = 0$. Hence we can argue as in the previous case. So, we can suppose that there exists α_0 such that $c_\alpha \neq 0$ for any $\alpha \geqq \alpha_0$. Now, we can obviously take a λ-sequence converging to c_0 in $C_p(\xi)$ for which $c_\alpha \neq 0$ for every $\alpha < \lambda$.

Let, for every $k \in \omega$, $A_k = \{\alpha < \lambda: |c_\alpha| \geqq 1/k\}$.

Then $\bigcup\{A_k: k \in \omega\} = \lambda$ and, since λ is a regular uncountable cardinal, it follows that there exists $k_0 \in \omega$ such that $|A_{k_0}| = \lambda$.

So, passing eventually to some cofinal subsequence of S, we can suppose that $|c_\alpha| \geqq 1/k_0$ for every $\alpha < \lambda$.

Let $U(k_0, f_\alpha) = \{x \in \xi: |f_\alpha(x)| < 1/k_0\}$, for $\alpha < \lambda$. Since $|c_\alpha| \geqq 1/k_0$ for every $\alpha < \lambda$, we obtain that $U(k_0, f_\alpha) \subset [0, x_\alpha)$ for every $\alpha < \lambda$.

Hence $|U(k_0, f_\alpha)| < \xi$, for every $\alpha < \lambda$.

Let now $\lambda' < \lambda$. Then, since $\lambda \leqq \xi$ and ξ is a regular cardinal, we have

$$|\bigcup\{(k_0, f_\alpha): \alpha < \lambda'\}| < \xi .$$

Hence $\{U(k_0, f_\alpha)\, \alpha < \lambda'\}$ is not a cover of ξ and, consequently, not an ω-cover of ξ. This implies that $c_0 \notin \mathrm{cl}\{f_\alpha: \alpha < \lambda'\}$ for every $\lambda' < \lambda$, since the following fact is easily verified:

Claim. Let $A = \{f_\alpha: \alpha < \tau\} \subset C_p(X)$ and $U(k, \alpha) = \{x \in X: |f_\alpha(x)| < 1/k\}$, for $\alpha < \tau$, $k \in \omega$. Then $c_0 \in \bar{A}$ if and only if $\Omega_k = \{U(k, \alpha): \alpha < \tau\}$ is an ω-cover of X for every $k \in \omega$.

So, every convergent to c_0 λ-sequence S in $C_p(\xi)$ contains a cofinal convergent to c_0 $t\lambda$-subsequence $\tilde{S}$.

Hence, we have proved that for every $A \subset C_p(\xi)$, $\mathrm{Lim}\, A = t\text{-}\mathrm{Lim}\, A$ holds. Consequently, $C_p(\xi)$ is almost radial. ∎

3.13 Corollary. Let ξ be an ordinal number. Then $C_p(\xi)$ is almost radial if and only if $C_p(\xi)$ is pseudoradial.

Proof. If $\mathrm{cf}(\xi) \leqq \omega$ then $C_p(\xi)$ is Fréchet (Theorem 2.7) and all is clear. If $\mathrm{cf}(\xi) > \omega$ then, again by Theorem 2.7, $C_p(\xi)$ pseudoradial implies that ξ is a regular cardinal number. Now we can apply Proposition 3.12. ∎

3.14 Lemma. Let ξ be a regular cardinal and $A \subset C_p(\xi)$. Then

$$\mathrm{Lim}\, A = \mathrm{Lim}_{\aleph_0} \cup \mathrm{Lim}_\xi\, A .$$

Proof. The proof follows from arguments similar to those given in Proposition 3.12.

3.15 Proposition. Let ξ be an ordinal number. Then $C_p(\xi)$ is pseudoradial if and only if it is u-pseudoradial.

Proof. Let $C_p(\xi)$ be pseudoradial. If it is Fréchet, then obviously it is u-pseudoradial. So, let us suppose that $C_p(\xi)$ is not Fréchet. Then ξ is an uncountable regular cardinal (even ω_1-inaccessible), by Theorem 2.7.

Let us introduce the following operation $\tilde{}$ on S-functions.
If $\varphi\colon \xi \to \xi$ is an S-function, we define $\tilde{\varphi}\colon \xi \to \xi$ by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x < \varphi(x) \\ x + 1 & \text{if } x \geqq \varphi(x) \text{ and } x \text{ is a non-limit ordinal} \\ x & \text{if } x \geqq \varphi(x) \text{ and } x \text{ is a limit ordinal.} \end{cases}$$

Obviously, $\tilde{\varphi}$ is again an S-function, but $\tilde{\varphi}(x) \geqq x$ holds for every $x \in \xi$.
Let $S = (f_\alpha)_{\alpha<\xi} \to c_0$. We will prove that there is a cofinal ξ-subsequence S' of S, which has an S-function $\varphi'\colon \xi \to \xi$ of the following type:

$$\varphi'(x) = \begin{cases} x & \text{if } x \text{ is a limit ordinal} \\ x + 1 & \text{if } x \text{ is a non-limit ordinal.} \end{cases}$$

For convenience, we will use the following notations, which distinguish between different occurrences of ξ. Let $\xi = [0, \xi)$; $\xi^{(\alpha)} \equiv \xi$ for any $\alpha \in \xi$, $\xi^{(-1)} \equiv \xi$; π_α: $\xi^{(\alpha)} \to \xi$ is the "identity" function for $\alpha \in \xi$ and $\alpha = -1$. Put $\kappa_{-1} = \pi_{-1}$. If α is a limit ordinal, define $\bar{\xi}^{(\alpha)} \equiv \xi$ and $\bar{\pi}_\alpha\colon \bar{\xi}^{(\alpha)} \to \xi$, the "identity" function.
Consider the sequence S above, where each $f_\alpha\colon \xi \to \mathbb{R}$ is a continuous function. Let $\varphi_{-1}\colon \xi \to \xi^{(-1)}$ be the following function:

$$\varphi_{-1}(x) = \pi_{-1}^{-1}(\min\{\alpha < \xi\colon f_\beta(x) = 0\,,\ \forall \beta \geqq \alpha\})\,.$$

Obviously this is a well-defined function and it is an S-function.
Consider $\tilde{\varphi}_{-1}\colon \xi \to \xi^{(-1)}$.

We shall define, by transfinite induction, for any $\alpha \in \xi$, points $y_\alpha \in \xi$ and, for α non-limit $z_\alpha \in \xi$; functions $\kappa_\alpha\colon \xi^{(\alpha)} \to \xi$ and $\varphi_\alpha\colon \xi \to \xi^{(\alpha)}$ (for $-1 \leqq \alpha < \xi$); functions μ_α (for $\alpha \in \xi$), which, when α is non-limit, go from $\xi^{(\alpha)}$ to $\xi^{(\alpha-1)}$, while, when α is a limit ordinal, go from $\xi^{(\alpha)}$ to $\bar{\xi}^{(\alpha)}$, finally, for $\alpha \in \xi$, ξ-sequences $S^{(\alpha)} = (f_\delta^{(\alpha)})_{\delta<\xi}(\alpha)$ of continuous functions $f_\delta^{(\alpha)}\colon \xi \to \mathbb{R}$. Suppose that they are all defined for every $\alpha \leqq \beta$, where $-1 \leqq \beta < \xi$.
Let $\alpha = \beta + 1$. Define

$$y_{\beta+1} = y_\alpha = \min\{x \in \xi\colon \tilde{\varphi}_\beta(x) \neq \pi_\beta^{-1}(x) \text{ for } x \text{ limit, or } \tilde{\varphi}_\beta(x) \neq \pi_\beta^{-1}(x) + 1, \text{ for } x \text{ non-limit}\}\,.$$

Define $\mu_\alpha\colon \xi^{(\beta+1)} \to \xi^{(\beta)}$ by

$$\mu_{\beta+1}(\gamma) = \begin{cases} \pi_\beta^{-1}\pi_{\beta+1}(\gamma) & \text{if } \gamma < \pi_{\beta+1}^{-1}(y_{\beta+1}) \\ \pi_\beta^{-1}(y_{\beta+1}) & \text{if } \gamma = \pi_{\beta+1}^{-1}(y_{\beta+1}) \text{ and } y_{\beta+1} \text{ is non-limit} \\ \tilde{\varphi}_\beta(y_{\beta+1}) + \pi_\beta^{-1}(\lambda) & \text{if } \gamma = \pi_{\beta+1}^{-1}(y_{\beta+1}) + 1 + \pi_{\beta+1}^{-1}(\lambda),\ y_{\beta+1} \\ & \quad \text{is non-limit, } \lambda \in \xi \\ \tilde{\varphi}_\beta(y_{\beta+1}) + \pi_\beta^{-1}(\lambda) & \text{if } \gamma = \pi_{\beta+1}^{-1}(y_{\beta+1}) + \pi_{\beta+1}^{-1}(\lambda),\ y_{\beta+1} \\ & \quad \text{is limit, } \lambda \in \xi\,. \end{cases}$$

Let $\kappa_{\beta+1} = \kappa_\beta \mu_{\beta+1}$. So $\kappa_{\beta+1}\colon \xi^{(\beta+1)} \to \xi$.

Define $S^{(\beta+1)} = (f_\delta^{(\beta+1)})_{\delta<\xi}(\beta+1)$ by $f_\delta^{(\beta+1)} = f_{\kappa_{\beta+1}}(\delta)$.

Define $\varphi_{\beta+1}\colon \xi \to \xi^{(\beta+1)}$ by

$$\varphi_{\beta+1}(x) = \begin{cases} \mu_{\beta+1}^{-1}\,\tilde{\varphi}_\beta(x) & \text{if } (x < y_{\beta+1}) \text{ or } (x \geqq y_{\beta+1} \text{ and } \tilde{\varphi}_\beta(x) > \tilde{\varphi}_\beta(y_{\beta+1})) \\ \pi_{\beta+1}^{-1}(y_{\beta+1}) & \text{if } x \geqq y_{\beta+1},\ \tilde{\varphi}_\beta(x) \leqq \tilde{\varphi}_\beta(y_{\beta+1}) \text{ and } y_{\beta+1} \text{ limit} \\ \pi_{\beta+1}^{-1}(y_{\beta+1}) + 1 & \text{if } x \geqq y_{\beta+1},\ \tilde{\varphi}_\beta(x) \leqq \tilde{\varphi}_\beta(y_{\beta+1}) \text{ and } y_{\beta+1} \\ \quad \text{non-limit.} \end{cases}$$

Define $z_{\beta+1} = \kappa_\beta\,\tilde{\varphi}_\beta(y_\beta)$.

Let α be a limit ordinal. Define $\varphi'_\alpha\colon \xi \to \tilde{\xi}^{(\alpha)}$ by

$$\varphi'_\alpha(x) = \min\{\tilde{\pi}_\alpha^{-1}\pi_\beta\,\varphi_\beta(x)\colon -1 \leqq \beta < \alpha\}.$$

Take $\tilde{\varphi}'_\alpha\colon \xi \to \tilde{\xi}^{(\alpha)}$.

Define $y_\alpha = \min\{x \in \xi\colon \tilde{\varphi}'_\alpha(x) \neq \tilde{\pi}_\alpha^{-1}(x)$ for x limit, or $\tilde{\varphi}'_\alpha(x) \neq \tilde{\pi}_\alpha^{-1}(x) + 1$ for x non-limit$\}$.

Define $y'_\alpha = \sup\{y_\beta\colon \beta < \alpha\}$ and $\bar{y}_\alpha = \sup\{\kappa_\beta\,\pi_\beta^{-1}(y_\beta)\colon \beta < \alpha\}$.

Define $\kappa'_\alpha\colon \tilde{\xi}^{(\alpha)} \to \xi$ by

$$\kappa'_\alpha(\gamma) = \begin{cases} \tilde{\pi}_\alpha(\gamma) & \text{if } \tilde{\pi}_{\alpha-1}^{-1}(0) \leqq \gamma < \tilde{\pi}_{\alpha-1}^{-1}(y_0) \\ \kappa_\beta(\pi_\beta^{-1}\,\pi_\alpha(\gamma)) & \text{if } \tilde{\pi}_\alpha(y_\beta) \leqq \gamma < \tilde{\pi}_\alpha(y_{\beta+1}), \text{ for } \beta < \alpha, \beta \text{ non-limit} \\ \kappa_\beta(\pi_\beta^{-1}\,\pi_\alpha(\gamma)) & \text{if } \tilde{\pi}_\alpha^{-1}(y'_\beta) \leqq \gamma < \tilde{\pi}_\alpha^{-1}(y_{\beta+1}), \text{ for } \beta < \alpha, \beta \text{ limit} \\ \bar{y}_\alpha + \lambda & \text{if } \gamma = \tilde{\pi}_\alpha^{-1}(y'_\alpha) + \tilde{\pi}_\alpha^{-1}(\lambda), \lambda \in \xi. \end{cases}$$

Define now $\mu_\alpha\colon \xi^{(\alpha)} \to \tilde{\xi}^{(\alpha)}$ by

$$\mu_\alpha(\gamma) = \begin{cases} \tilde{\pi}_\alpha^{-1}\,\pi_\alpha(\gamma) & \text{if } \gamma < \pi_\alpha^{-1}(y_\alpha) \\ \tilde{\pi}_\alpha^{-1}(y_\alpha) & \text{if } \gamma = \pi_\alpha^{-1}(y_\alpha) \text{ and } y_\alpha \text{ is non-limit} \\ \tilde{\varphi}'_\alpha(y_\alpha) + \pi_\alpha^{-1}(\lambda) & \text{if } \gamma = \pi_\alpha^{-1}(y_\alpha) + 1 + \pi_\alpha^{-1}(\lambda), \lambda \in \xi, y_\alpha \text{ non-limit} \\ \tilde{\varphi}'_\alpha(y_\alpha) + \pi_\alpha^{-1}(\lambda) & \text{if } \gamma = \pi_\alpha^{-1}(y_\alpha) + \pi_\alpha^{-1}(\lambda), \lambda \in \xi, y_\alpha \text{ limit.} \end{cases}$$

Let $\kappa_\alpha\colon \xi^{(\alpha)} \to \xi$ be given by $\kappa_\alpha = \kappa'_\alpha\mu_\alpha$ and $S^{(\alpha)} = (f_\gamma^{(\alpha)})_{\gamma\in\xi(\alpha)}$ where $f_\gamma^{(\alpha)} = f_{\kappa_\alpha(\gamma)}$.

Define $\varphi_\alpha\colon \xi \to \xi^{(\alpha)}$ by

$$\varphi_\alpha(x) = \begin{cases} \mu_\alpha^{-1}\,\varphi'_\alpha(x) & \text{if } (x < y_\alpha) \text{ or } (x \geqq y_\alpha \text{ and } \tilde{\varphi}'_\alpha(x) > \tilde{\varphi}'_\alpha(y_\alpha)) \\ \pi_\alpha^{-1}(y_\alpha) + 1 & \text{if } x \geqq y_\alpha, \tilde{\varphi}'_\alpha(x) \leqq \tilde{\varphi}'_\alpha(y_\alpha) \text{ and } y_\alpha \text{ non-limit} \\ \pi_\alpha^{-1}(y_\alpha) & \text{if } x \geqq y_\alpha, \tilde{\varphi}'_\alpha(x) \leqq \tilde{\varphi}'_\alpha(y_\alpha) \text{ and } y_\alpha \text{ is limit.} \end{cases}$$

Take $\tilde{\varphi}_\alpha: \xi \to \xi^{(\alpha)}$.

Let us now remark that, if at some step $\alpha > 0$ the point y_α is 0 (as the minimum of an empty set), then we stop the induction because all what we need is already done. So, the above written formulas hold on the assumption that $y_\alpha \neq 0$ for every $\alpha > 0$. Then induction can be done for every $\alpha < \xi$, since ξ is a regular cardinal number.

From the construction, it easily follows that the functions μ_α, κ_α, κ'_α are one-to-one. Moreover, the following hold:

$$y_0 < y_1 < y_2 < \ldots < y'_\omega \leqq y_\omega < y_{\omega+1} < \ldots; z_0 < z_1 < \ldots < z_n < \ldots$$
$$\ldots < z_{\omega+1} < \ldots; S \supsetneqq S^{(0)} \supsetneqq S^{(1)} \supsetneqq \ldots; (f_{z_\beta})_{\beta \leqq \alpha} \subset S^{(\alpha)};$$

$\tilde{\varphi}_\alpha$ is an S-function for $S^{(\alpha)}$ and

$$(\tilde{\varphi}_\alpha \mid [0, y_\alpha])(x) = \begin{cases} \pi_\alpha^{-1}(x) & \text{if } x \text{ is limit} \\ \pi_\alpha^{-1}(x) + 1 & \text{if } x \text{ is non-limit} \end{cases}$$

Define now $\kappa': \xi \to \xi$ by

$$\kappa'(\gamma) = \begin{cases} \gamma & \text{if } 0 \leqq \gamma < y_0 \\ \kappa_\beta \, \pi_\beta^{-1}(\gamma) & \text{if } y_\beta \leqq \gamma < y_{\beta+1} \text{ and } \beta \text{ is non-limit} \\ \kappa_\beta \, \pi_\beta^{-1}(\gamma) & \text{if } y'_\beta \leqq \gamma < y_{\beta+1} \text{ abd } \beta \text{ is limit.} \end{cases}$$

Finally, define $S' = (f'_\gamma)_{\gamma<\xi}$ by $f'_\gamma = f_{\kappa'(\gamma)}$, and $\varphi': \xi \to \xi$ by

$$\varphi'(x) = \min\{\pi_\beta \, \tilde{\varphi}_\beta(x): -1 \leqq \beta < \xi\}.$$

Then $S' \supset (f_{z_\alpha})_{\alpha<\xi}$, α non-limit,

$$\varphi'(x) = \begin{cases} x & \text{if } x \text{ is limit} \\ x + 1 & \text{if } x \text{ is non-limit}, \end{cases}$$

and φ' is an S-function for S'. But φ' is obviously a continuous function. Hence S' is a $u\xi$-subsequence of S which converges to c_0 in $C_p(\xi)$.

Now, using Lemma 3.14, we obtain that $C_p(\xi)$ is u-pseudoradial.

The converse implication is obvious. ∎

3.16 Theorem. Let ξ be an ordinal number. Then $C_p(\xi)$ is pseudoradial if and only if it is u-almost radial.

Proof. Let $C_p(\xi)$ be pseudoradial. Then, by Corollary 3.13, it is almost radial. In the proof of Proposition 3.15, it was shown that every convergent ξ-sequence in $C_p(\xi)$ has an $u\xi$-subsequence. Hence, if we start with a $t\xi$-sequence S, then we obtain, arguing as in the proof of Proposition 3.15, an $ut\xi$-subsequence of S. This implies that $C_p(\xi)$ is u-almost radial (using once more Lemma 3.14). ∎

3.17 Remark. $C_p(X)$ is pseudoradial (respectively, almost radial) if and only if the space X has the following property: for every family of open systems $\{\omega_r =$

$= \{U_\alpha^r : \alpha < \lambda\} : r \in \mathbb{Q}\}$ (where λ is a cardinal number) such that, for every $\alpha < \lambda$, $\bigcup\{U_\alpha^r : r \in \mathbb{Q}\} = X$, $\bar{U}_\alpha^r \subset U_\alpha^s$ for $r < s$, $\bigcup\{U_\alpha^r : r < t\} = U_\alpha^t$, $(\bigcap\{U_\alpha^r : r \in \mathbb{Q}^+\}) \setminus U_\alpha^0 \neq X$ and $\omega_r' = \{U_\alpha^r \setminus \bar{U}_\alpha^{-r} : \alpha < \lambda\}$ is an ω-cover for every $r \in \mathbb{Q}^+$, there exist a $\lambda' \leqq \lambda$, a function $S: \lambda' \to \lambda$ and an open cover $\omega = \{U^r : r \in \mathbb{Q}\}$ such that $\bigcup\{U^r : r < t\} = U^t$, $\bar{U}^r \subset U^s$ for $r < s$, $\omega \neq \{U_\alpha^r : r \in \mathbb{Q}\}$ for every $\alpha < \lambda$, and, for every $k \in \omega$, $\operatorname{Lim}\{V_\beta^k : \beta < \lambda'\} = X$ (respectively, and for every $\beta_0 < \lambda'$, there exists $k \in \omega$ such that $\tilde{\omega}_{\beta_0}^k = \{V_\beta^k : \beta < \beta_0\}$ is not an ω-cover of X), where for every $\beta < \lambda$ and every $k \in \omega$, $V_\beta^k = \bigcup\{(U_{S(\beta)}^r \cap U^p) \setminus (\bar{U}_{S(\beta)}^s \cup \bar{U}^q) : r, s, p, q \in \mathbb{Q}, r > s, p > q, r - q < 1/k, p - s < 1/k\}$.

The proof is direct and technical, although almost straightforward and for this reason we omit it.

References

[AIT] Arhangel'skii A. V., Isler R., Tironi G.: Pseudo-radial spaces and another generalization of sequential spaces. Convergence Structures 1984 (Bechyně), Math. Research, Band 24, Akademie-Verlag, Berlin 1985 33–37.

[DIT] Dimov G., Isler R., Tironi G.: On functions preserving almost radiality and their relations to radial and pseudo-radial spaces. Preprint.

[E] Engelking R.: General Topology. PWN, Warszawa, 1977.

[G] Gerlits J.: Some properties of C(X), II. Topology and its Applications 15 (1983) 255 to 262.

[GN] Gerlits J., Nagy Zs.: Some properties of C(X), I. Topology and its Applications 14 (1982) 151–161.

[GNS] Gerlits J., Nagy Zs., Szentmiklóssy Z.: Some convergence properties in function spaces. Preprint.

[H] Herrlich H.: Quotienten geordneter Raüme und Folgenkonvergenz. Fund. Math. 61 (1967) 79–81.