Michael Cambern; Peter Greim

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Mappings of Continuous Functions on Hyperstonean Spaces

MICHAEL CAMBERN*), Santa Barbara, U.S.A., PETER GREIM**), Charleston, U.S.A.

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If X is a compact Hausdorff space and E^* a Banach dual, denote by $C(X, (E^*, \sigma^*))$ the Banach space of continuous functions on X to E^* when the latter space is given its weak * topology. Here we establish Banach-Stone theorems for spaces of this type, paralleling those for spaces of norm-continuous functions. If X_1, X_2 are hyperstonean and E_1^*, E_2^* have one-dimensional centralizers then the existence of an isometry T of $C(X_1, (E_1^*, \sigma^*))$ onto $C(X_2, (E_2^*, \sigma^*))$ implies that X_1 and X_2 are homeomorphic. When in addition, the E_i^* are separable, then they are isometrically isomorphic. In this case we also obtain an explicit description of T. Our description is (necessarily) more complicated than the one obtainable in the norm-continuous case, but we also give a necessary and sufficient condition on the E_i^* which permits the description to be simplified.

1. Introduction

Throughout this paper the letter E will stand for a Banach space, while X and Y will denote compact Hausdorff spaces. The scalar field associated with E is denoted by $\mathbb{K}$. We will write $E_1 \cong E_2$ to indicate that the Banach spaces E_1 and E_2 are isometric. $\mathscr{B}(E_1, E_2)$ denotes the space of continuous linear operators from E_1 to E_2. And, for a family of Banach spaces $\{E_\gamma: \gamma \in \Gamma\}$, the notation $\prod_{\gamma\in\Gamma}^\infty E_\gamma$ stands for the space of all functions F defined on Γ such that $F(\gamma) \in E_\gamma$ for each $\gamma \in \Gamma$ and $\|F\|_\infty = \sup_{\gamma\in\Gamma} \|F(\gamma)\|$ is finite.

Given X, we denote by $C(X, E)$ the Banach space of continuous functions F on X to E provided with the supremum norm. And if E^* is a dual space, $C(X, (E^*, \sigma^*))$ stands for the Banach space of continuous functions F on X to E^* when the latter space if provided with its weak * topology, again normed by $\|F\|_\infty = \sup_{x\in X} \|F(x)\|$. We will write $C(X)$ for $C(X, \mathbb{K})$.

The well known Banach-Stone theorem states that if $C(X)$ and $C(Y)$ are isometric then X and Y are homeomorphic. A number of authors, beginning with M. Jerison [14], have considered the problem of determining geometric properties of E which allow generalizations of this theorem to the context of isometries between spaces of norm-continuous vector functions $C(X, E)$ and $C(Y, E)$. Jerison's initial positive

*) Department of Mathematics University of California Santa Barbara, CA 93106, U.S.A.
**) Department of Mathematics The Citadel Charleston, S.C. 29409, U.S.A.

results were obtained for spaces E which were strictly convex, or whose T-sets satisfied a certain geometric property. Subsequently, other mathematicians widened the class of Banach spaces E for which these results can be shown to be valid. And the results obtained are of two distinct types depending upon the nature of E; some are wholly topological in nature — the existence of an isometry implies that X and Y are homeomorphic — while other results go on to describe the isometry in terms of the homeomorphism. The most exhaustive compilation of developments in this area can be found in the monograph by E. Behrends [1], which also contains Behrends' own more definitive results for spaces E with one-dimensional centralizer.

The first paper to deal with the analogous problem for spaces of weak * continuous functions is [3], where it is shown that if X and Y are extremally disconnected compact Hausdorff spaces and E^* is uniformly convex, then the existence of an isometry between $C(X, (E^*, \sigma^*))$ and $C(Y, (E^*, \sigma^*))$ implies that X and Y are homeomorphic. Spaces of the form $C(X, (E^*, \sigma^*))$, where X is extremally disconnected — in fact hyperstonean — arise naturally in various mathematical contexts. In [4] it is shown that if E^* has the Randon-Nikodym property, then for any compact Hausdorff space Y the bidual of $C(Y, E)$ is of the form $C(X, (E^{**}, \sigma^*))$, where X is a certain hyperstonean space dependent on Y. More generally, spaces of the form $C(X, (E^*, \sigma^*))$ with X hyperstonean arise as duals of spaces of vector measures, and are the natural candidates for the duals of Bochner L^1 spaces, [5]. In this article we show that with the assumption that X and Y are hyperstonean, the topological result of [3] can be extended to the case of Banach duals E^* with one-dimensional centralizer. When, in addition, the duals are separable, an explicit description of the isometry is obtained. The description is somewhat more complicated than in the case of norm-continuous functions, and we show by examples that these complications are a feature of the problem, and not simply of our method of proof. We obtain a necessary and sufficient condition on the space E^* which allows our description to be simplified considerably. The condition is, rather surprisingly, the strong uniqueness of the predual of E^* We first establish some notation and terminology.

For a positive measure space (Ω, Σ, μ) and $1 \leqq p \leqq \infty$, the Bochner space $L^p(\Omega, \Sigma, \mu, E)$ will be denoted by $L^p(\mu, E)$ when there is no danger of confusing the underlying measurable space involved, and by $L^p(\mu)$ when $E = \mathbb{K}$. We refer to [6] for the definition and properties of these spaces.

If X is an extremally disconnected compact Hausdorff space we will call a nonnegative, extended real-valued Borel measure μ on X a *category measure* if

(i) every nonempty clopen set has positive measure,
(ii) every nowhere dense Borel set has measure zero, and
(iii) every nonempty clopen set contains a nonempty clopen set with finite measure.

(In [2] and [5] measures having these properties are referred to as "perfect.") An extremally disconnected compact Hausdorff space in which a category measure is defined will be called *hyperstonean*. This is equivalent to the definition of hyperstonean space obtained via the use of normal measures, [2, p. 26]. Given a hyperstonean space X with category measure μ, property (iii), together with an application of Zorn's lemma, can be used to show that

(iv) X is the Stone-Čech compactification of the disjoint union of clopen subsets X_γ, $X = \beta(\bigcup_{\gamma\in\Gamma} X_\gamma)$ with $\mu(X_\gamma) < \infty$ for all γ, and for all Borel subsets B of X, $\mu(B) = \sum_{\gamma\in\Gamma} \mu(B \cap X_\gamma)$.

Since for hyperstonean X every Borel set B has a unique representation $B = C \Delta D$ with C clopen and D nowhere dense, [2, pp. 1–2] and [17, p. 160], it follows that the null sets for a category measure are precisely the nowhere dense Borel sets.

An *L-projection* of E is a projection $P: E \to E$ satisfying $\|e\| = \|Pe\| + \|e - Pe\|$ for all $e \in E$. The *Cunningham algebra* $C_1(E)$ is the operator algebra generated by all L-projections. The centralizer of E^* is $Z(E^*) = \{T^*: T \in C_1(E)\}$, [1, 5.7 (iii) and 5.9]. References for the centralizer and Cunningham algebra of a Banach space are [1] and [2].

2. Centralizers, and decompositions of isometries

Theorem 2.1: Let X be a hyperstonean space and E^* a Banach dual with $Z(E^*) = \mathbb{K}$. Then $Z(C(X, (E^*, \sigma^*))) \cong C(X)$.

Proof: Let μ be a category measure on X. We first assume that μ is σ-finite so that the results of [10] apply. It is known that $C(X, (E^*, \sigma^*)) \cong L^1(\mu, E)^*$ [5, Theorem 1], hence $Z(C(X, (E^*, \sigma^*))) \cong C_1(L^1(\mu, E))$. This latter space is isometric to $L^\infty(\mu)$ because of [10, Corollary 4.3] and our assumption that $Z(E^*)$, (and hence $C_1(E)$), is $\mathbb{K}$. But $L^\infty(\mu) \cong C(X)$ [2, p. 31], and thus $Z(C(X, (E^*, \sigma^*))) \cong C(X)$.

If now μ is not assumed to be σ-finite we know that $X = \beta(\bigcup_{\gamma\in\Gamma} X_\gamma)$, where X_γ are pairwise disjoint clopen sets and $\mu \mid X_\gamma$ is finite for all γ. It is then straightforward to see that $C(X, (E^*, \sigma^*))$ is isometric to $\prod_{\gamma\in\Gamma}^\infty C(X_\gamma, (E^*, \sigma^*))$. The injection of $C(X, (E^*, \sigma^*))$ into $\prod_{\gamma\in\Gamma}^\infty C(X_\gamma, (E^*, \sigma^*))$ is obvious, whereas given an element of the latter space, it can be considered as a bounded, weak * continuous function defined on $\bigcup_{\gamma\in\Gamma}^\infty X_\gamma$ to E^*, so that, by the weak * compactness of the unit ball in E^*, it extends to a weak * continuous function on $X = \beta(\bigcup_{\gamma\in\Gamma} X_\gamma)$ to E^*. Thus $Z(C(X, (E^*, \sigma^*))) \cong$ $\cong Z(\prod_{\gamma\in\Gamma}^\infty C(X_\gamma, (E^*, \sigma^*)))$ which (e.g., using the characterization of multipliers as M-bounded operators [1, p. 54]) is easily seen to be $\prod_{\gamma\in\Gamma}^\infty Z(C(X_\gamma, (E^*, \sigma^*)))$. Finally this space is, by the previous paragraph, $\prod_{\gamma\in\Gamma}^\infty C(X_\gamma) \cong C(\beta(\prod_{\gamma\in\Gamma}^\infty X_\gamma)) = C(X)$.

We shall say that an isometry $R: C(X_1, (E^*, \sigma^*)) \to C(X_2, (E^*, \sigma^*))$ is "induced" by a homeomorphism Φ from X_2 onto X_1 if $RF = F \circ \Phi$ for all $F \in C(X_1, (E^*, \sigma^*))$. And an isometry $S: C(X, (E_1^*, \sigma^*)) \to C(X, (E_2^*, \sigma^*))$ is a $C(X)$-module isomorphism if, of course, $S(fF) = f \,.\, SF$ for all $f \in C(X)$ and $F \in C(X, (E_1^*, \sigma^*))$. As a consequence of Theorem 2.1 we then have

Theorem 2.2: Let X_i be hyperstonean and E_i^* Banach duals with $Z(E_i^*) = \mathbb{K}$, $i = 1, 2$. If $T: C(X_1, (E_1^*, \sigma^*)) \to C(X_2, (E_2^*, \sigma^*))$ is a surjective isometry, then X_1 and X_2 are homeomorphic. Furthermore, T admits a decomposition $T = S \circ R$ into surjective isometries $R: C(X_1, (E_1^*, \sigma^*)) \to C(X_2, (E_1^*, \sigma^*))$ and $S: C(X_2, (E_1^*, \sigma^*)) \to C(X_2, (E_2^*, \sigma^*))$, where R is induced by a suitable homeomorphism $\Phi: X_2 \to X_1$ and S is a $C(X_2)$-module isomorphism.

Proof: Identifying $f \in C(X_1)$ with the corresponding element of the centralizer we have that $f \to T \circ f \circ T^{-1} = \hat{f}$ is an isometry of $C(X_1)$ onto $C(X_2)$ carrying the function 1 onto 1 and thus by the classical Banach-Stone theorem $\hat{f} = f \circ \Phi$ for a suitable homeomorphism $\Phi: X_2 \to X_1$. Put $RF = F \circ \Phi$ for $F \in C(X_1, (E_1^*, \sigma^*))$. Then R is obviously a surjective isometry, and so is $S = T \circ R^{-1}$. Furthermore, for $\hat{f} \in C(X_2)$ and $G \in C(X_2, (E_1^*, \sigma^*))$, $S(\hat{f}G) = T(R^{-1}(\hat{f}G)) = T((\hat{f} \circ \Phi^{-1})(G \circ \Phi^{-1})) = T(f(R^{-1}G)) = (T \circ f)(R^{-1}G) = \hat{f} \circ T \circ R^{-1}(G) = \hat{f} \circ SG$.

3. Representations and weak * continuity of isometries

It is a consequence of Theorem 2.2 that if we seek representations for T, it suffices to find representations for $C(X_2)$-module isomorphisms S and then replace F by $F \circ \Phi$.

Thus let $S: C(X, (E_1^*, \sigma^*)) \to C(X, (E_2^*, \sigma^*))$ be an isometry as in Theorem 2.2 We would like to find a representation

(1) $$(SF)(x) = U(x)\, F(x)$$

where the $U(x)$ are surjective isometries of E_1^* onto E_2^*. If such operators $U(x)$ exist, then it is clear what form they must take: replace F in (1) by a constant function $\underset{\sim}{e}^*$ and obtain

(2) $$U(x)\, \underset{\sim}{e}^* = (S\underset{\sim}{e}^*)(x)\,, \quad e^* \in E_1^*\,.$$

Thus let us define $U(x)$ by (2). It is obvious that the $U(x)$ are linear and $\|U(x)\| \leqq 1$. It remains to verify that, with this definition of $U(x)$, the right hand side of (1) defines an operator $\tilde{S}$ on $C(X, (E_1^*, \sigma^*))$, and that $\tilde{S} = S$.

At this point, let us restrict the range spaces to duals with the Radon-Nikodym property (RNP). Then $C(X, (E_i^*, \sigma^*))$ is $L^\infty(\mu, E_i^*)$ under the natural embedding, where μ is some category measure on X. The inverse of this embedding is a vector-valued lifting with the properties established in [12, Proposition 1] and Theorem 2 of that paper tells us that, for every $F \in C(X, (E_1^*, \sigma^*))$, (1) holds almost everywhere – i.e. on an open dense subset O_F of X. We note that, even though [12] deals only with σ-finite measures, we can apply it to each of the clopen sets X_γ, (see property (iv) of a category measure), find dense open subsets O_γ of X_γ and then observe that $X - \bigcup_{\gamma \in \Gamma} O_\gamma$ has measure zero and is therefore nowhere sense.

If the E_i^* are separable, (which, because of their RNP is equivalent to the separability of the E_i, [6, p. 218]), then the fact that countable unions of μ-null sets are μ-null shows that $U(x)$ is a surjective isometry for almost all x. (Compare the reasoning in [17].) Combining the results from [12] and [13] and the arguments above we have parts (a) and (b) of the following theorem.

Theorem 3.1: Let X_i be hyperstonean, E_i^* Banach duals with RNP and $Z(E_i^*) = \mathbb{K}$, $(i = 1, 2)$, and $T: C(X_1. (E_1^*, \sigma^*)) \to C(X_2, (E_2^*, \sigma))$ a surjective isometry.

(a) Then there exists a homeomorphism Φ of X_2 onto X_1 and there exist operators $U(x): E_1^* \to E_2^*$ with $\|U(x)\| \leqq 1$, $x \in X_2$, such that for every $F \in C(X_1, (E_1^*, \sigma^*))$,

$$(TF)(x) = U(x)(F \circ \Phi)(x) \tag{3}$$

on an open dense set $O_F \subseteq X_2$,

(b) If the spaces E_i are separable, then all the $U(x)$ can be chosen to be surjective isometries.

(c) If the spaces E_i are separable then, on a suitable open dense subset O of X_2, the function $U(\cdot)$ is continuous with respect to the strong operator topology on $\mathscr{B}(E_1^*, E_2^*)$.

Proof of (c): We may assume, without loss of generality, that $X_1 = X_2 = X$ and that Φ is the identity on X. Let μ be a category measure on X. Arguing as in the second paragraph preceding the statement of the theorem we may also assume that μ is finite. Now we have to show that, on a suitable dense open set $O \subseteq X$, the mappings $x \to U(x) e^*$ are norm-continuous for all $e^* \in E_1^*$. Since $\|U(x)\| = 1$ for all x, we may restrict our attention to all e^* in a countable dense subset of E_1^*, say $\{e_n^*: n \in \mathbb{N}\}$. Each of the functions $U(\cdot)\, e_n^*$ coincides with $T\underset{\sim}{e}_n^*$ on an open dense subset $O_{n,1}$ of X, and $T\underset{\sim}{e}_n^*$ is norm continuous on an open dense subset $O_{n,2}$ of X [10, Lemma 2.3]. Thus $U(\cdot)\, e_n^*$ is norm continuous on $O_n = O_{n,1} \cap O_{n,2}$. The set $N = \bigcup_{n=1}^{\infty} (X - O_n)$ is thus of first category, and since X is hyperstonean N is, in fact, nowhere dense [7, p. 160]. Thus we may take $O = X - \overline{N}$.

Theorem 3.1. leaves several questions open.

(A) What if the E_i^* do not have RNP? The same arguments used above show that the restriction of T to the subspace of locally measurable functions (i.e. functions essentially separably valued on each set of finite measure) is still represented by (3). However, for other functions F we don't even know that $U(\cdot)\, F(\cdot)$ is weak * continuous on an open dense set.

(B) Is every such isometry T the adjoint of an isometry between the canonical preduals? We will address that question in Theorem 3.3.

(C) The lifting on $L^\infty(\mu, E_2^*)$ being weak * continuous, can we expect that, as in the case of isometries of spaces $C(X, E)$ of norm-continuous vector functions [1], equation (1) holds on all of X rather than merely on an open dense set? Easy examples show that we can not. It might happen that $F(x) = 0$ and $(SF)(x) \neq 0$. Consider the isometry of $C(\beta\mathbb{N}, (l^2, \sigma^*))$ onto itself defined by $(SF)(n) = U(n)F(n)$ where the $U(n)$ are the l^2 – isometries obtained by interchanging the first and n^{th} coordinates. Then for $F \in C(\beta\mathbb{N}, (e^2, \sigma^*))$ given by $F(n) = e_n$ (the n^{th} unit vector in l^2) we have $F(x) = 0$ for $x \in \beta\mathbb{N} - \mathbb{N}$, but $SF = \underset{\sim}{e}_1$ and thus $(SF)(x) \neq 0$. This example also shows that in part (c) of Theorem 3.1 we cannot expect $O = X_2$: the set $\{U(n)e_1: n \in \mathbb{N}\}$ is not relatively compact.

(D) Can we, at least, choose the exceptional set in (3) to be independent of F? Here again the answer is negative and the problem, rather surprisingly, involves the question of strong unicity of preduals.

Example 3.2. Let $\mathbb{C}$ denote the unit circle in the plane, ν Lebesgue measure on $\mathbb{C}$, and let $X_1 = X_2$ be the maximal ideal space X of $L^\infty(\nu)$. Again, X is hyperstonean [15, pp. 95–96]. Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^*$ its one-point compactification with the point at infinity denoted by 0. Consider the Banach spaces $c_0 = C_0(\mathbb{N})$ and $c = C(\mathbb{N}^*)$. By the Riesz representation theorem the elements of c_0^* and c^* can be considered as regular Borel measures, and element of c_0^* being given by $\sum_{j \geqq 1} \alpha_j \mu_j$ and one of c^* by $\beta_0\mu_0 + \sum_{j \geqq 1} \beta_j \mu_j$. Let U be the isometry of c_0^* onto c^* defined by $U(\sum_{j \geqq 1} \alpha_j \mu_j) = \alpha_1\mu_1 + \sum_{j \geqq 1} \alpha_{j+1}\mu_j$.

Let μ be any category measure on X. Then since c_0^* is separable, the Pettis Measurability Theorem shows that each $F \in C(X, (c_0^*, \sigma^*))$ is measurable and thus by Lemma 2.3 of [10] there exists an open dense subset $V(F)$ of X upon which $F(\cdot)$ is norm-continuous. Hence $UF(\cdot)$ is norm-continuous on $V(F)$ taking values in c^*, and hence there exists a unique weak * continuous extension $UF(\cdot)^\wedge$ of this function to all of X. The map $T: C(X, (c_0^*, \sigma^*)) \to C(X, (c^*, \sigma^*))$ defined by $TF = UF(\cdot)^\wedge$ is thus isometric and is easily seen to be onto. We will show that for each point $x \in X$ there exists an $F \in C(X, (c_0^*, \sigma^*))$ with $F(x) = 0$ but $\|(TF)(x)\| = 1$. Consequently, however the O_F are chosen, we have $\cap\{O_F: F \in C(X, (c_0^*, \sigma^*))\} = \emptyset$.

To this end let I denote the natural injection of $C(\mathbb{C})$ into $L^\infty(\nu)$ which sends a function to its equivalence class. Since I is multiplicative, the adjoint I^* takes unit point masses on X to unit point masses on $\mathbb{C}$ and we let ω denote the continuous map of X onto $\mathbb{C}$ defined by $\omega(x) = e^{it}$ if $I^*\mu_x$ is the unit point mass at e^{it}. Then fix $x_0 \in X$ and let $e^{it_0} = \omega(x_0)$. Let $t_n = t + 1/n$ and $t_{-n} = t - 1/n$ $(n \in \mathbb{N})$ and define functions $\tilde{F}$, $\tilde{G}$ on $\mathbb{C}$ as follows. Set

$\tilde{F}(e^{is}) = \mu_1$ on the arc taken counterclockwise from e^{it_1} to $e^{it_{-1}}$

$\tilde{F}(e^{it_n}) = \mu_n = \tilde{F}(e^{it_{-n}})$, $(n \in \mathbb{N})$,

$\tilde{F}(e^{it_0}) = 0$,

and let $\tilde{F}(e^{is})$ be affine in s on the intervals $[t_{n+1}, t_n]$ and $[t_{-n}, t_{-n-1}]$, $(n \in \mathbb{N})$. $\tilde{F}$ is norm-continuous at all points except e^{it_0}, where it is $\sigma(c_0^*, c_0)$-continuous. (Observe that $\{\mu_n\}$ converges to zero weak*.) Similarly let

$$\tilde{G}(e^{is}) = \mu_0 \text{ on the arc taken counterclockwise from } e^{it_1} \text{ to } e^{it_{-1}},$$
$$\tilde{G}(e^{it_n}) = \mu_{n-1} = \tilde{G}(e^{it_{-n}}), (n \in \mathbb{N}),$$
$$\tilde{G}(e^{it_0}) = \mu_0$$

and let $\tilde{G}(e^{is})$ be affine in s on the intervals $[t_{n+1}, t_n]$ and $[t_{-n}, t_{-n-1}]$, $(n \in \mathbb{N})$. Again, $\tilde{G}$ is norm-continuous at all points except e^{it_0}, where it is $\sigma(c^*, c)$-continuous. Hence the functions F and G defined by $F = \tilde{F} \circ \omega$, $G = \tilde{G} \cdot \omega$ belong, respectively, to $C(X, (c_0^*, \sigma^*))$ and $C(X, (c^*, \sigma^*))$. Since for all points e^{is} of the circle other than e^{it_0} we have $G(e^{is}) = \underset{\sim}{U}F(e^{is})$, it follows that $G(\cdot) = \underset{\sim}{U}F(\cdot)$ on X outside $\omega^{-1}(\{e^{it_0}\})$, and the latter set is easily seen to be nowhere dense. But as $TF(\cdot)$ is equal to $\underset{\sim}{U}(F\cdot)$ on $V(F)$ we thus have $TF(\cdot) = G(\cdot)$ outside of a nowhere dense subset of X. Hence $TF = G$ in $C(X, (c^*, \sigma^*))$ and as $F(x_0) = 0$ but $\|TF(x_0)\| = \|G(x_0)\| = 1$, we are done.

We can, in fact, precisely identify those Banach duals for which the sort of pathology exhibited in Example 3.2 can be found. We say that E_1^* has a *strongly unique predual* if every isometry T between E_1^* and another dual E_2^* is $\sigma(E_1^*, E_1) - \sigma(E_2^*, E_2)$ continuous [9, p. 92]. It follows from the Krein-Šmulian theorem [8, p. 429] and the weak* metrizability of the unit balls that this is the case if and only if every $\sigma(E_1^*, E_1)$-null sequence in the unit ball of E_1^* is mapped into a $\sigma(E_2^*, E_2)$-null sequence.

Using this fact it is straightforward to generalize Example 3.2 from l^1 to any separable Banach dual E_1^* having one-dimensional centralizer but without strongly unique predual. (Replace the sequence $\{\mu_n\}$ by a suitable $\sigma(E_1^*, E_1)$-null sequence in the unit ball.) We may even replace the maximal ideal space X of $L^\infty(\nu)$ by any hyperstonean Y whose set of isolated points is not dense: if Y_s denotes the closure of this latter set, then $Y - Y_s$ is perfect, and thus admits a continuous surjection onto X [16, Corollary 4.9(a)], and mapping Y_s onto an arbitrary point of X we obtain a continuous surjection τ: $Y \to X$. Then one can take $\omega \circ \tau$ instead of ω in the construction used in example 3.2.

In other words, strong uniqueness of E_1^*'s predual is necessary if one wants to represent all isometries T between E_1^*-valued weak * continuous function spaces in the form $(TF)(x) = U(x) F \circ \Phi(x)$ on O_F, with O_F independent of F. On the other hand, this condition is also sufficient as the next theorem shows. Let us call X "not purely atomic" if it is not the closure of the set of its isolated points.

Example 3.3: Let E_1^*, E_2^* be separable Banach duals with one-dimensional centralizers, and let X_1, X_2 be not-purely-atomic hyperstonean spaces with category measures μ_1, μ_2 respectively. Then the following are equivalent:

(a) Every surjective isometry U: $E_1^* \to E_2^*$ is the adjoint of an isometry U_*: $E_2 \to E_1$.

(b) Every surjective isometry T: $C(X_1, (E_1^*, \sigma^*)) \to C(X_2, (E_2^*, \sigma^*))$ is the adjoint of an isometry T_*: $L^1(\mu_2, E_2) \to L^1(\mu_1, E_1)$.

(c) For every surjective isometry T: $C(X_1, (E_1^*, \sigma^*)) \to C(X_2, (E_2^*, \sigma^*))$ the sets O_F in Theorem 3.1 can be chosen independently of F.

(Observe that the condition in (a) is independent of the choice of the X_i's.)

Proof: (a) $\Rightarrow$ (*b*): Let T be such an isometry. Then, in the notation of Theorem 2.2, $T = S \circ R$, with R: $C(X_1, (E_1^*, \sigma^*)) \to C(X_2, (E_1^*, \sigma^*))$ induced by a homeomorphism Φ: $X_2 \to X_1$. Obviously R is the adjoint of the isometry R_*: $L^1(\mu_2, E_1) \to \to L^1(\mu_1, E_1)$ defined by $R_*(H) = f \, . \, (H \circ \Phi^{-1})$ where f is the Radon-Nikodym derivative of $\mu_2 \circ \phi^{-1}$ with respect to μ_1. Therefore it is enough to show that S is an adjoint; i.e. we may assume that $X_1 = X_2 = X$, $\mu_1 = \mu_2 = \mu$ and Φ is the identity, hence $S = T$.

Let O_F and $x \to U(x)$ be as in Theorem 3.1. Define T_*: $L^1(\mu, E_2) \to L^1(\mu, E_1)$ by $T_*H(x) = U(x)_* H(x)$, where $(U(x)_*)^* = U(x)$. T_*H has σ-finite support, is separably valued and weakly measurable. (Observe that $\langle T_*H(x), e^*\rangle = \langle H(x), U(x)\, e^*\rangle$, and $x \to U(x)\, e^*$ coincides with a weak * continuous function on an open dense set.) Thus T_*H is Bochner measurable; of course it is in $L^1(\mu, E_1)$. For all $H \in L^1(\mu, E_2)$ and $F \in C(X, (E_1^*, \sigma^*))$ we have

$$\langle H, TF\rangle = \int_{O_F} \langle H(x), U(x)\, F(x)\rangle \, d\mu(x) = \int_{O_F} \langle U(x)_* \, H(x), F(x)\rangle \, d\mu(x) = $$
$$= \langle T_*H, F\rangle \, ,$$

and thus $T = (T_*)^*$.

(b) $\Rightarrow$ (c): Again, we may assume that $X_1 = X_2 = X$, $\mu_1 = \mu_2 = \mu$, and Φ is the identity on X. Then $T = (T_*)^*$ with T_*: $L^1(\mu, E_2) \to L^1(\mu, E_1)$ by assumption, and T_* commutes with the characteristic projections χ_C, C clopen $\subseteq X$, since T does. According to [11, Theorem 1 and p. 217] we have $T_*H(x) = V(x)\, H(x)$ a.e. for all $H \in L^1(\mu, E_2)$, where $x \to V(x)$ is a strongly measurable function with values in the set of isometries of E_2 onto E_1. We want to show that there is an open dense set $O \subseteq X$ and that for all $x \in O$ and all $F \in C(X, (E_1^*, \sigma))$ we have

$$TF(x) = V(x)^* \, F(x) \, .$$

(and thus, in the notation of Theorem 3.1, we can take the $U(x)$ to be the $V(x)^*$).

If $X = (\beta \bigcup_{\gamma \in \Gamma} X_\gamma)$ where the X_γ are pairwise disjoint clopen sets with finite measure, then it is enough to find such a dense open set O_γ for each γ. Thus let us assume, for simplicity of notation, that $\mu(X) < \infty$. Let $\{e_n\}$ be a dense sequence in E_2 and choose an L^1 function H_n representing $T_* \underset{\sim}{e}_n$, $n \in \mathbb{N}$. Using the facts that measurable functions are norm-continuous on an open dense set [10, Lemma 2.3] and, again, that sets of first category are nowhere dense, we can find an open set $O_0 \subseteq X$ such that, for each $n \in \mathbb{N}$, H_n is continuous on O_0 and $H_n(x) = V(x) \, \underset{\sim}{e}_n$ on O_0.

The function $x \to V(x)^* \, F(x)$ is weak * continuous on O_0 for all $F \in C(X, (E_1^*, \sigma^*))$. For as the function is bounded, it is enough to verify the continuity of the functions

$x \to \langle e_n, V(x)^* F(x)\rangle = \langle H_n(x), F(x)\rangle$, and these latter functions are continuous due to the norm-continuity of the H_n and the weak * continuity of the bounded function F. In order to see that this weak * continuous function coincides with $TF|_{O_0}$ it suffices to show that for all $n \in \mathbb{N}$ we have

$$\langle e_n, TF(x)\rangle = \langle H_n(x), F(x)\rangle \quad \text{on} \quad O_0 .$$

Since both of the latter functions are continuous, it is enough to observe that for all clopen $C \subseteq O_0$ we have

$$\int_C \langle e_n, TF(x)\rangle \, d\mu(x) = \langle \chi_C \underset{\sim}{e}_n, TF\rangle = \langle T_* \chi_C \underset{\sim}{e}_n, F\rangle = \langle \chi_C H_n, F\rangle = = \int_C \langle H_n(x), F(x)\rangle \, d\mu(x) .$$

The continuity (in the strong operator topology) of $U(x) = V(x)^*$ then follows as in the proof of Theorem 3.1(c).

Since we do not know whether, for E^* with strongly unique predual, $C(X, (E^*, \sigma^*)))$ can have preduals other than those of the form $L^1(\mu, E)$, the equivalence of (a) and (b) is only a partial answer to the following question, suggested by the fact that, for hyperstonean X, $C(X)$ has a strongly unique predual.

Question 3.4: If E^* has a (strongly) unique predual and X is hyperstonean, does $C(X, (E^*, \sigma^*))$ have a strongly) unique predual? We note that it seems to be still unknown whether strong uniqueness of the of the predual is implied by uniqueness of the predual (any two preduals are isometrically isomorphis.)

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References

[1] Behrends E., M-structure and the Banach-Stone theorem. Lecture Notes in Mathematics 736, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

[2] Behrends E. et al., L^p-structure in real Banach spaces, Lecture Notes in Mathematics 613, Springer-Verlag, Berlin—Heidelberg—New York, 1977.

[3] Cambern, M, A Banach-Stone theorem for spaces of weak* continuous functions. Proc. Royal Soc. Edinburgh 101 A (1985), 203—206.

[4] Cambern M., and Greim, P., The bidual of $C(X.E)$, Proc. Amer. Math. Soc., 85 (1982), 53—58.

[5] Cambern M., and Greim, P., The dual of a space of vector measures. Math. Z., 180 (1982), 373—378.

[6] Diestel J., and Uhl, J. J., Jr., Vector measures, Math. Surveys 15, Amer. Math. Soc., Providence, R. I., 1977.

[7] Dixmier J., Sur certains espaces considérés par M. H. Stone, Summa Brasil. Math., 2 (1951), 151—182.

[8] DUNFORD N., and SCHWARTZ J. T., Linear operators, Part I, Interscience, New York, 1958.
[9] GODEFROY G., Espaces de Banach: existence et unicité de certains préduaux, Ann. Inst. Fourier (Grenoble), 28 (1978), 87—105.
[10] GREIM P., Banach spaces with the L^p-Banach-Stooe property. Trans. Amer. Math. Soc., 287 (1985), 819—828.
[11] GREIM P., Isometries and L!-structure of separably valued Bochner L^p-spaces, in: Measure Theory and Its Applications, Proc. Conf. Sherbrooke 1982, Lecture Notes in Mathematics 1033, Springer-Verlag, Berlin—Heidelberg—New York, 1983, 209—218.
[12] GREIM P., Banach-Stone theorems for non-separably valued Bochner L^∞-spaces, Rend. Circ. Mat. Palermo (22), Suppl. 2(1982), 123—129.
[13] GREIM P., The centralizer of Bochner L^∞-spaces, Math. Ann. 260 (1982), 463—468.
[14] JERISON M., The space of bounded maps into a Banach space, Ann. of Math. (2) 52 (1950), 309—327.
[15] LACEY H. E., The isometrical theory of classical Banach spaces, Springer-Verlag, Berlin—Heidelberg—New York, 1974.
[16] SHARIR *M*, Extremal structure in operator spaces, Trans. Amer. Math. Soc., 186 (1973), 91—111.
[17] SOUROUR A. R., The isometries of $L^p(\Omega, X)$, J. Funct. Anal. 30 (1978), 276—285.
[18] WALKER R., The Stone-Čech compactification, Springer-Verlag, Berlin—Heidelberg—New York, 1974.