Characterization of Tournaments by Coned 3-Cycles

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In [2] we studied the tournaments whose fundamental group is not trivial, giving a structural characterization of them. Here we obtain a new characterization of them by using coned 3-cycles.

Introduction

A tournament T_{2n+1} is *highly regular* if the vertices can be labelled as $v_1, v_2, \ldots, v_{2n+1}$ in such a way that $v_i \to v_j$, for all indices $i = 1, 2, \ldots, 2n + 1$ and for all indices $j \equiv i + 1, i + 2, \ldots, i + n \pmod{2n + 1}$.

The vertices of a subtournament A are *equivalent*, if for any $q \in T - A$, either $q \to A$ or $A \to q$. If the vertices of T_n can be partitioned into disjoint subtournaments $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$ of equivalent vertices and R_m denotes the tournament on the m vertices $w_1, w_2, \ldots, w_m$ in which $w_i \to w_j$ if and only if $S^{(i)} \to S^{(j)}$, then $T_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ is the *composition* of the m *components* $S^{(1)}, S^{(2)}, \ldots, S^{(m)}$ with the *quotient* R_m.

A tournament T_n is *simple* if $T_n = R_m(S^{(1)}, S^{(2)}, \ldots, S^{(m)})$ implies that either $m = 1$ or $m = n$. The *simple quotient related to* T is the simple tournament univocally determined in the class of the quotients of a tournament T. If T is irreducible and R_m is the simple quotient related to T, then the components too are univocally determined and are the maximal equivalent sets of vertices of T, whereas, if T is reducible, there are two maximal, in general non-disjoint, equivalent sets in T (see [4]).

In [2] we considered the *complex* K_T *associated with a tournament* T as the simplicial complex whose vertex set is T and whose simplexes are spanned by the transitive subtournaments of T and we called T *simply disconnected* if and only if the fundamental group of the polyhedron $|K_T|$ is not trivial. In this way we proved the following structural characterization: "*T is simply disconnected if and only if the simple quotient of T is highly regular*". If T is simply disconnected, a *3-cycle-loop* (i.e. a loop of $|K_T|$ made up of the edges of a 3-cycle C of T) is nullhomotopic if C

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is *shrinkable* (i.e. is included in an equivalent set) and it is non-nullhomotopic if C is *non-coned* (i.e. for each $v \in T$, neither $v \to C$ nor $C \to v$), thus in T there exist only shrinkable and non-coned 3-cycles. Otherwise, if T is *simply connected*, each 3-cycle-loop is nullhomotopic but three kinds of 3-cycles can exist: shrinkable, coned and non-coned ones.

The main result in this paper is the converse of the previous argument which gives a combinatorial characterization of simply disconnected tournaments by using 3-cycles:

Theorem. *A tournament is simply disconnected if and only if:*
- *there exists a non-coned* 3-*cycle;*
- *all the coned* 3-*cycles are shrinkable.*

Moreover, by Theorem 8, we characterize the tournaments whose 3-cycles are all nonconed. In this way we obtain a class of tournaments studied by Moon in [3].

In this paper we use only combinatorial arguments even if some proofs would be easier by homotopical arguments. Thus, here, a simply disconnected tournament is regarded as a tournament with a highly regular quotient.

Definitions and preliminary results

Definition 1. *A subtournament T' of a tournament T is said to be* coned *by a vertex v (i.e. v* cones *T') if there exists $v \in T - T'$ such that either $v \to T'$ or $T' \to v$. If no vertex of $T - T'$ cones T', T' is said to be* non-coned.

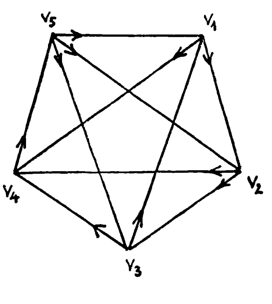
Definition 2. *A subtournament T' of a tournament T is said to be* shrinkable *if there exists an equivalent proper subset of vertices of T which includes the vertices of T'. Otherwise T' is said to be* unshrinkable.

Remark 1. Since every equivalent set is included in a maximal one, a subtournament T' is shrinkable if and only if it is included in a maximal component of T.

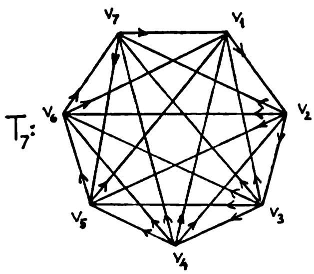
Remark 2. Each shrinkable subtournament is also coned.

If we restrict the previous definitions to the 3-cycles, these can be partitioned into non-coned, shrinkable (coned) and coned but unshrinkable 3-cycles, as can be seen in the following examples:

T_5:



In the simply disconnected tournament T_5 the 3-cycle $\langle v_1, v_2, v_3 \rangle$ is shrinkable and all the other 3-cycles are non-coned.



In the simply connected tournament T_7, $\langle v_1, v_2, v_3 \rangle$ is a non-coned 3-cycle, $\langle v_2, v_3, v_4 \rangle$ is a coned unshrinkable 3-cycle and $\langle v_5, v_6, v_7 \rangle$ is a shrinkable 3-cycle.

Proposition 1. *Let R be a non-trivial quotient of a tournament T and p the canonical projection from T to R. A 3-cycle γ is non-coned in T if and only if its projection $p(\gamma)$ is non-coned in R.*

Proof. In fact, a non-coned 3-cycle of T can not be shrinkable. ∎

By using the three different kinds of 3-cycles, we can characterize some classes of tournaments and we have the following preliminary results:

Proposition 2. *A tournament T is transitive if and only if there are no 3-cycles in T.* ∎

Proposition 3. *A tournament is irreducible if and only if there exists an unshrinkable 3-cycle in T.* ∎

Proposition 4. *Each 3-cycle of a simple tournament is unshrinkable.* ∎

Remark. The tournaments whose 3-cycles are unshrinkable are the compositions of transitive components with a simple quotient.

Proposition 5. *Each 3-cycle of a highly regular tournament is non-coned.*

Proof. Denote with T_{2m+1} the tournament and with $\gamma = \langle x, y, z \rangle$ a 3-cycle of T_{2m+1}. Label the vertices of T_{2m+1} in the standard cyclical order beginning from $x = v_1$. Then $v_h = y$ and $v_k = z$, where $h < k \leqq 2m + 1$. Since T_{2m+1} is highly regular:

$$\begin{aligned} &\text{from } h \leqq m+1 \text{ then } v_1 \to v_i \to v_h, \quad \forall v_i / 1 < i < h; \\ &\text{from } k - h \leqq m \text{ then } v_h \to v_i \to v_k, \quad \forall v_i / h < i < k; \\ &\text{from } k \geqq m+2 \text{ then } v_k \to v_i \to v_1, \quad \forall v_i / k < i \leqq 2m+1. \end{aligned}$$

Thus no vertex of T_{2m+1} cones γ. ∎

Remark. The tournaments whose 3-cycles are non-coned will be characterized by Theorem 8.

In order to obtain a characterization of these tournaments we need the following

Lemma 6. *Let T be a tournament whose coned 3-cycles are all shrinkable and w a vertex of T. Then a non-coned 3-cycle of $T' = T - w$ is also a non-coned 3-cycle of T.*

Proof. Let R_h be the simple quotient related to T, where the components are denoted by $S^{(1)}, S^{(2)}, \ldots, S^{(h)}$ and let $\tilde{R}_k$ be the simple quotient related to T', where the components are denoted by $\tilde{S}^{(1)}, \tilde{S}^{(1)}, \ldots, \tilde{S}^{(k)})$. Suppose that a non-coned 3-cycle γ of T' is coned by w. It follows that γ is shrinkable in T and is therefore included in a component, e.g., $\gamma \subseteq S^{(1)}$. Since T' is irreducible by Proposition 3, the partition $\{S^{(1)} - w, S^{(2)} - w, \ldots, S^{(h)} - w\}$ must be a cover of T' finer than the one $\{\tilde{S}^{(1)}, \tilde{S}^{(2)}, \ldots, \tilde{S}^{(k)}\}$, formed by the maximal equivalent sets of T. But this is impossible because $\gamma \subseteq S^{(1)} - w$ and $S^{(1)} - w$ is included in one among $\tilde{S}^{(1)}, \tilde{S}^{(2)}, \ldots, \tilde{S}^{(k)}$, whereas the vertices of γ must belong to three different $\tilde{S}^{(p)}, \tilde{S}^{(q)}, S^{(r)}$. ∎

Remark. Under the assumptions of Lemma 6, T' irreducible implies T irreducible.

Theorem 7. *The simple quotient related to a tournament is highly regular if and only if:*

a) *there exists a non-coned 3-cycle;*

b) *all the coned 3-cycles are shrinkable.*

Proof. Let $T_n = R_{2m+1}(S^{(1)}, S^{(2)}, \ldots, S^{(2m+1)})$ be, where R_{2m+1} is a non-trivial highly regular tournament. If a 3-cycle γ is unshrinkable, its vertices must be included in three different components. Since R_{2m+1} is highly regular, we can prove that γ is non-coned following the proof of Proposition 5. Moreover, by using Propositions 3, 5 and 1, there exists at least one non-coned 3-cycle in T_n.

We prove the converse by induction on the order n of T_n.

For $n = 3$ only the 3-cycle satisfies a) and b) and it is highly regular.

Assume that for each tournament T_k of order k, which satisfies a) and b), the simple quotient is highly regular. Consider a tournament T_{k+1}, which satisfies a) and b) and a non-coned 3-cycle $\gamma = \langle x, y, z \rangle$ of T_{k+1}. Choose a vertex $w \in T_{k+1} - \gamma$ and put $T_k = T_{k+1} - w$. Thus γ is also non-coned in T_k, whereas each coned 3-cycle σ in T_k is also coned in T_{k+1}. Then σ is shrinkable in T_{k+1} by b), i.e. it is included in a proper component A of T_{k+1}. Therefore σ is also shrinkable in T_k, since $\sigma \in A - w$. Consequently a) and b) are true for T_k, and by inductive hypothesis, $T_k = R_{2h+1}(S^{(1)}, S^{(2)}, \ldots, S^{(2h+1)})$ where the quotient R_{2h+1} is a non-trivial highly regular tournament.

Now, for each $i = 1, 2, \ldots, 2h + 1$, in $S^{(i)}$ consider the complementary subsets:

$$S^{\rightarrow(i)} = \{v \in S^{(i)} / v \rightarrow w\} \quad \text{and} \quad {}^{\leftarrow}S^{(i)} = \{v \in S^{(i)} / w \rightarrow v\}\,.$$

We prove that only for one index $i = 1, 2, \ldots, 2_h + 1$, at the most, is the partition $\{S^{\rightarrow(i)}, {}^{\leftarrow}S^{(i)}\}$ of $S^{(i)}$ not trivial. Otherwise assume that $S^{\rightarrow(p)} \neq \phi \neq {}^{\leftarrow}S^{(p)}$ and $S^{\rightarrow(q)} \neq \phi \neq {}^{\leftarrow}S^{(q)}$ with $p \neq q$. Moreover let $S^{(p)} \rightarrow S^{(q)}$. Since R_{2h+1} is a nontrivial highly regular tournament, there exists $r = 1, 2, \ldots, 2h + 1$ such that $S^{(r)} \rightarrow S^{(p)} \rightarrow$ $\rightarrow S^{(q)} \rightarrow S^{(r)}$. Now, choose $v_r \in S^{(r)}$ and suppose $w \rightarrow v_r$. Suitable elements v_p (resp. v_q) can be choosen in $S^{(p)}$ (resp. $S^{(q)}$) such that $w \rightarrow \langle v_p, v_h, v_k\rangle$. (When $v_r \rightarrow w$, a similar argument holds). But this is a contradiction of Lemma 6.

Now there are two possibilities:

1) for each $i = 1, 2, \ldots, 2h + 1$, either $S^{\rightarrow(i)} = \phi$ or ${}^{\leftarrow}S^{(i)} = \phi$;
2) there is precisely one index i, such that $S^{\rightarrow(i)} \neq \phi \neq {}^{\leftarrow}S^{(i)}$.

1) T_{k+1} is irreducible and then w does not cone T_k (see Remark to Lemma 6). By making a rotation on the indices of R_{2h+1}, we can suppose $w \rightarrow S^{(h+1)}$ and $S^{(h+2)} \rightarrow w$. By considering 3-cycles in R_{2h+1} and by using Lemma 6, in both cases $w \rightarrow S^{(1)}$ and $S^{(1)} \rightarrow w$, we obtain that w is a successor of $S^{(2h+3)}, S^{(h+4)}, \ldots$ $\ldots, S^{(2h+1)}$ and is a predecessor of $S^{(2)}, S^{(3)}, \ldots, S^{(h)}$. Hence $T_{k+1} =$ $= R_{2h+1}(S^{(1)} \cup \{w\}, S^{(2)}, \ldots, S^{(2h+1)})$ and the assertion is proved.
2) By making a rotation on the indices of R_{2h+1}, suppose $S^{\rightarrow(1)} \neq \phi \neq {}^{\leftarrow}S^{(1)}$. If $w \rightarrow S^{(h+1)}$, we obtain $T_{k+1} = R_{2h+1}(S^{(1)} \cup \{w\}, S^{(2)}, \ldots, S^{(2h+1)})$ as before. If $S^{(h+1)} \rightarrow w$, we obtain, as above, that w is a predecessor of $S^{(h+2)}, S^{(h+3)}, \ldots$ $\ldots, S^{(2h+1)}$ and a successor of $S^{(2)}, S^{(3)}, \ldots, S^{(h)}$.

Moreover, we have ${}^{\leftarrow}S^{(1)} \rightarrow S^{\rightarrow(1)}$. Otherwise, let $v_1 \in S^{\rightarrow(1)}$ and $v_1' \in {}^{\leftarrow}S^{(1)}$ be such that $v_1 \rightarrow v_1'$. Choose a vertex v_{2h+1} in $S^{(2h+1)}$, then the 3-cycle $\delta = \langle v_1, w, v_{2h+1}\rangle$ is coned by v_1' and then is shrinkable in T_{k+1} by b). Let $\tilde{T}_t$ be the simple quotient related to T_{k+1}, where the components are denoted by $\tilde{S}^{(1)}, \tilde{S}^{(2)}, \ldots, \tilde{S}^{(t)}$. Thus v_1 and v_{2h+1} are included in the same component, e.g. $\tilde{S}^{(1)}$. Following the proof of Lemma 6, the partition $\{\tilde{S}^{(1)} - w, \tilde{S}^{(2)} - w, \ldots, \tilde{S}^{(t)} - w\}$ must be a cover of T_k finer than the one $\{S^{(1)}, S^{(2)}, \ldots, S^{(2h+1)}\}$. But this is impossible because v_1 and v_{2h+1} belong to $\tilde{S}^{(1)}$, whereas $v_1 \in S^{(1)}$ and $v_{2h+1} \in S^{(2h+1)}$.

Hence $T_{k+1} = R_{2h+3}(S^{\rightarrow(1)}, S^{(2)}, \ldots, S^{(h+1)}), \{w\}, S^{(h+2)}, S^{(h+3)}, \ldots, S^{(h+1}, {}^{\leftarrow}S^{(1)})$, where R_{2h+3} is highly regular.

Therefore the theorem is proved. ■

Remark. The tournaments whose coned 3-cycles are all shrinkable are either the reducible tournaments of the ones with a non-trivial highly regular quotient.

Theorem 8. *The following conditions are equivalent for any tournament* T_n:
a) *every subtournament of* T_n *is either irreducible or transitive;*
b) *every subtournament of* T_n *of order* 4 *is either irreducible or transitive;*
c) *every* 3-*cycle of* T_n *is non-coned;*
d) $T_n = R_{2m+1}\ (S^{(1)}, S^{(2)}, \ldots, S^{(2m+1)})$ *is the composition of* $2m + 1$ *transitive components* $S^{(1)}, S^{(2)}, \ldots, S^{(2m+1)}$ *with a highly regular quotient* R_{2m+1}.

Proof.

a) $\Rightarrow$ b): obvious.

b) $\Rightarrow$ c). If $\gamma = \langle x, y, z \rangle$ is a 3-cycle coned by a vertex v, $\langle x, y, z, v \rangle$ is a reducible non-transitive subtournament T_4 of T_n.

c) $\Rightarrow$ *d*). If there is no 3-cycle in T_n, T_n is transitive. Thus $T_n = R_1(T_n)$, where R_1 is the trivial (highly regular) tournament. If γ is a 3-cycle of T_n, γ is non-coned. By Theorem 7 $T_n = R_{2m+1}(S^{(1)}, S^{(2)}, \ldots, S^{(2m+1)})$, where R_{2m+1} is highly regular and non-trivial. Moreover, for each $i = 1, 2, \ldots, 2m + 1$, $S^{(i)}$ is transitive, since no 3-cycle is included in $S^{(i)}$, as it would be coned.

d) $\Rightarrow$ a). If $R_{2m+1} = R_1$, T_n is transitive and a) holds.
If R_{2m+1} is not trivial, consider any vertex $w \in T_n$ and put $T_{n-1} = T_n - w$. w is included in a component, e.g. $w \in S^{(2m+1)}$.

Two cases are possible:

1) $S^{(2m+1)} - w \neq \phi$. Then $T_{n-1} = R_{2m+1}(S^{(1)}, S^{(2)}, \ldots, S^{(2m+1)} - w)$. Therefore T_{n-1} is irreducible and also the component $S^{(2m+1)} - w$ is transitive.

2) $S^{(2m+1)} - w = \phi$.
   - If $m = 1$, $T_{n-1} = R_2(S^{(1)}, S^{(2)})$ and is transitive, since $S^{(1)}$ and $S^{(2)}$ are transitive and $S^{(1)} \to S^{(2)}$;
   - if $m > 1$, $T_{n-1} = R_{2m-1}(S^{(1)}, S^{(2)}, \ldots, S^{(m-1)}, S^{(m)} \cup S^{(m+1)}, S^{(m+2)}, \ldots, S^{(2m)})$ where R_{2m-1} is highly regular and also the component $S^{(m)} \cup S^{(m+1)}$ is transitive, since $S^{(m)}$ and $S^{(m+1)}$ are transitive and $S^{(m)} \to S^{(m+1)}$.

In this way, it follows that all the subtournaments of order $n - 1$ are irreducible or transitive and satisfy condition d). Consequently, by using the same argument we obtain the previous result also for subtournaments of orders $n - 2, n - 3, \ldots$ $\ldots, 4$ of T_n ∎

Remark. In 1965, Beineke and Harary (see [1]) showed that highly regular tournaments satisfy condition a). In 1979, Moon (see [3]) called tournaments with *property L* the ones satisfying condition a) and proved a) $\Leftrightarrow$ d), giving a structural characterization of these tournaments. Here we generalize Moon's result by b), since it is sufficient to check only the subtournaments of order 4. Moreover, condition b) can not be improved because both the tournaments of order 3 satisfy condition b).

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