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Minimal Rearrangements of Sobolev Functions

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1. Introduction

It is well known that if u: $\mathbf{R}^n \to \mathbf{R}$ is a nonnegative function with compact support, then for $1 \leqq p < \infty$,

$$\left[\int_{\mathbf{R}^n} |\nabla u^*|^p\right]^{1/p} \leqq \left[\int_{\mathbf{R}^n} |\nabla u|^p\right]^{1/p} \tag{1}$$

where u^* denotes the spherical symmetric rearrangement of u; c.f. [PS]. u^* is defined by

$$u^*(x) = \sup\{t: \mu(t) > \alpha(n)\,|x|^n\} \tag{2}$$

where $\alpha(n)$ is the volume of the unit n-ball in $\mathbf{R}^n$ and $\mu(t) < \infty$ is the Lebesgue measure of the set $E_t = \{x: u(x) > t\}$. Note that $\mu(t) = |E_t^*|$ where $E_t^* = \{x: u^*(x) > t\}$ and $|E_t^*|$ denotes the Lebesgue measure of E_t^*. The purpose of this paper is to show that if μ is absolutely continuous and equality holds in (1), then u is almost everywhere equal to a translate of u^*. We also construct examples which show that this may not be true if μ is not absolutely continuous.

More generally, we establish the following result: For $1 \leqq p < \infty$ let A: $[0, \infty) \to [0, \infty)$ be of class C^2 and such that

$$A(0) = 0\,, \quad \text{and} \tag{3}$$

$$A^{1/p} \quad \text{is convex}\,. \tag{4}$$

Note that (4) implies that A is convex. Further, (3) and the convexity of A imply that

$$A \text{ is increasing}\,. \tag{5}$$

Let u: $\mathbf{R}^n \to \mathbf{R}$ be nonnegative and measurable with compact support. Assume

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that the gradient ∇u of u in the sense of distributions is a measurable function with

$$(6) \qquad \int_{\mathbf{R}^n} A(|\nabla u|) < \infty .$$

Note that (4) implies that $u \in W^{1,p}(\mathbf{R}^n)$. Denote $M = \operatorname{ess\,sup} u = \operatorname{ess\,sup} u^* \leqq \infty$ and $C^* = \{x : \nabla u^*(x) = 0\}$.

1.1. Theorem. *If u satisfies (6), then ∇u^* is a measurable function and*

$$(7) \qquad \int_{\mathbf{R}^n} A(|\nabla u^*|) \leqq \int_{\mathbf{R}^n} A(|\nabla u|) .$$

Moreover, if $1 < p < \infty$, $|C^* \cap u^{*-1}(0, M)| = 0$, A is strictly increasing, and equality holds in (7), then there is a translate of u^* which is almost everywhere equal to u.

For the case $A(\xi) = |\xi|^p$ this result was recently discussed by Friedman and McLeod [FM] under the additional assumption that u is of class C^n, and with the hypothesis $|C^* \cap u^{*-1}(0, M)| = 0$ replaced by the weaker hypothesis that μ is continuous on the interval $(0, M)$. ($|C^* \cap u^{*-1}(0, M)| = 0$ is equivalent to absolute continuity of μ on $(0, M)$; see Lemma 2.3.) However, the proof in [FM] contains an error which can be only repaired using the apparently stronger assumption $|C \cap u^{-1}(0, M)| = 0$, $C = \{x : \nabla u(x) = 0\}$. Indeed, in Section 4, we give an example of a C^∞ function u whose distribution function μ is merely continuous and for which equality holds in (7) for every $p \geqq 1$ and with A strictly increasing, but yet no translate of u^* is equal to u almost everywhere.

If A is strictly increasing and u satisfying (6) is such that equality holds in (7), then we prove that E_t is equivalent to an n-ball for every t. The example mentioned above shows that without the assumption that μ is absolutely continuous, these n-balls need not be concentric. Since the sets E_t^* corresponding to u^* are also n-balls, each E_t^* is a translate of E_t. This correspondence induces a mapping $T: \mathbf{R}^n \to \mathbf{R}^n$ of the form

$$T(x) = x + \tau(|x|)$$

such that $u^* = u \circ T$. If μ is continuous, τ is lipschitzian. Defining $\tilde{u}$ by $\tilde{u}(|x|) = u^*(x)$ we prove that if $p > 1$, then a necessary and sufficient condition for equality in (7) is that $\tilde{u}'\tau' = 0$ almost everywhere. Note that $|C^* \cap u^{*-1}(0, M)| = 0$ is equivalent to $\tilde{u}'(r) \neq 0$ for almost all r such that $0 < \tilde{u}(r) < M$. It therefore follows that if equality holds in (7) and $|C^* \cap u^{*-1}(0, M)| = 0$, then $\tau' = 0$ almost everywhere and so $\tau \equiv 0$ because τ is lipschitian. In case $p = 1$ the condition $\tilde{u}'\tau' = 0$ is only sufficient for equality in (7). For $p > 1$, our analysis also shows that if $I \subset (0, M)$ is an interval for which $\nabla u \neq 0$ on $u^{-1}(I)$, then $u^{-1}\{t\}$ and $u^{-1}\{s\}$ are concentric $(n-1)$-spheres whenever $s, t \in I$. This result was established by Uribe [U] under the assumption that $u \in C^n$.

In case $p = 1$ it may not be true that u is almost everywhere equal to a translate of u^*. For example, if u is any (non-spherically symmetric) lipschitzian function whose level sets are $(n-1)$-spheres, the coarea formula (see Proposition 2.1 below) implies that

$$\int_{\mathbf{R}^n} |\nabla u| = \int_{\mathbf{R}^n} \mathscr{H}^{n-1}(u^{-1}\{t\})\,dt = \int_{\mathbf{R}^n} \mathscr{H}^{n-1}(u^{*-1}\{t\})\,dt = \int_{\mathbf{R}^n} |\nabla u^*|\,.$$

Conversely, we show that if $0 \leqq u \in W^{1,1}(\mathbf{R}^n)$ with compact support satisfies

$$\int_{\mathbf{R}^n} |\nabla u^*| = \int_{\mathbf{R}^n} |\nabla u|\,,$$

then each E_t is an open ball.

Finally, we remark that it is possible to prove the analog of Theorem 1.1 for Sobolev functions on the n-sphere; the proof will appear in [BZ].

The proof of Theorem 1.1 uses techniques similar to some in [FM], [TA] and [U] but the generality of our hypothesis requires a delicate analysis that ultimately rests on developments that are now basic to geometric measure theory. We also employ ideas from [AT]. In the present paper we will only outline the main ideas of the proof. Complete details are in [BZ]. We express our appreciation for assistance from Jiri Dadok in the discovery of the map T.

2. Notation and Preliminaries

We will denote by χ_A the characteristic function of a set A and by $|A|$ the Lebesgue outer measure of A. The Lebesgue density of A at $x \in \mathbf{R}^n$ is defined by

$$D(A, x) = \lim_{r \to 0} \frac{|A \cap B(x, r)|}{|B(x, r)|}\,.$$

Here $B(x, r)$ denotes the closed ball of radius r centered at x. We denote by $W^{1,p}(\mathbf{R}^n)$, $1 \leqq p < \infty$, the Sobolev space of functions in $L^p(\mathbf{R}^n)$ whose distributional derivatives are also in $L^p(\mathbf{R}^n)$. The L^p-norm of u is denoted by $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$.

To begin our analysis, we use H. Federer's theory of functions of bounded variation [F, 4.5.9] to obtain a general version of the coarea formula which is valid for Sobolev functions. For this purpose we will adopt the following convention concerning the pointwise definition of u: For $x \in \mathbf{R}^n$ we define

$$u(x) = \frac{\mu(x) + \lambda(x)}{2} \tag{8}$$

where $\lambda(x)$ and $\mu(x)$ denote the lower and upper approximate limits of u at x, respectively. That is,

$$\mu(u) = \inf\{t: D(E_t, x) = 0\}\,,$$

and $\lambda(x)$ is defined similarly.

2.1. Proposition. *If $u \in W^{1,p}(\mathbf{R}^n)$ and f is a nonnegative Borel function, then with convention* (8) *in force,*

$$\int_{\mathbf{R}^n} |\nabla u| f = \int_{\mathbf{R}} \int_{u^{-1}\{s\}} f \, d\mathscr{H}^{n-1} ds . \tag{9}$$

Here $\mathscr{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure and on the left side we employ the convention $0 . \infty = 0$.

Another fundamental result from geometric measure theory which we will employ is a general version of the isoperimetric inequality. To state it, we first recall a characterization of a bounded, measurable set E of finite perimeter, or in the terminology of [F], a bounded, measurable set E with the property that $\partial(\mathbf{R}^n \llcorner E)$ is representable by integration. In particular, this means that the current defined by E is an n-dimensional integral current. First the measure theoretic boundary of E is defined to be

$$\partial^* E = \{x : 0 < D(E, x) < 1\} .$$

Then E has finite perimeter if and only if $\mathscr{H}_u^{-1}(\partial^* E) < \infty$; c.f. [F, 4.5.6, 4.5.11].

2.2. Proposition. *Let $E \subset \mathbf{R}^n$ be a bounded, measurable set with finite perimeter, and E^* be an n-ball such that $|E^*| = |E|$. Then*

$$\mathscr{H}^{n-1}(\partial E^*) \leqq \mathscr{H}^{n-1}(\partial^* E)$$

with equality holding if and only if E is equivalent to an n-ball.

An elementary proof can be found in [B 2].

2.3. Lemma. *If $u \in W^{1,p}(\mathbf{R}^n)$, $1 \leqq p < \infty$, then on $[0, M]$ the following are true:*

(i) *μ is one-one.*

(ii) $\tilde{u} \circ \left(\dfrac{\mu}{\alpha(n)}\right)^{1/n} = \text{identity}$.

(iii) *For almost all t,*

$$\infty > -\mu'(t) \geqq \int_{u^{-1}\{t\}} |\nabla u|^{-1} \, d\mathscr{H}^{n-1} . \tag{10}$$

(iv) *$\mu' < 0$ almost everywhere.*

(v) *μ is absolutely continuous if and only $|C^* \cap u^{*-1}(0, M)| = 0$.*

Proof. It follows from [F, 4.5.9] that E_τ has finite perimeter for almost all τ. Further, for almost all τ,

$$\mathscr{H}^{n-1}(\partial^* E_\tau \sim u^{-1}\{\tau\}) = 0 . \tag{11}$$

We obtain (i) using this, the coarea formula (9) and the isoperimetric theorem.

The second assertion in the Lemma follows immediately from the fact that μ is one-one.

We next use the coarea formula (9) to obtain for $0 \leqq t \leqq M$

$$\mu(t) = |C \cap u^{-1}(t, M)| + \int_t^M \int_{u^{-1}\{\tau\}} |\nabla u|^{-1} \, d\mathscr{H}^{n-1} d\tau \, , \tag{12}$$

which clearly implies (10).

Denote by B the set of $t \in [0, M]$ such that $|\nabla u| = \infty$ almost everywhere on $u^{-1}\{t\}$. One infers from (10) that (iv) is implied by $|B| = 0$, which follows from application of the coarea formula.

Turning to (v) we see from Lemma 2.4 (the proof of which does not depend on (v)) that $u^* \in W^{1,p}(\mathbf{R}^n)$. Applying (12) with $u = u^*$ and recalling that μ is the distribution function of u^* we infer that $|C^* \cap u^{*-1}(0, M)| = 0$ implies that μ is absolutely continuous on $(0, M)$. On the other hand, if μ is absolutely continuous on $(0, M)$, then (12) implies that $|C^* \cap u^{*-1}(t, M)|$ is an absolutely continuous function of $t \in (0, M)$. In particular, if $S \subset (0, M)$ with $|S| = 0$, then $|C^* \cap u^{*-1}(S)| = 0$. The coarea formula implies that $C^* \cap u^{*-1}(0, M) = C_0 \cup C_1$ where $|u^*(C_0)| = 0$ and $\mathscr{H}^{n-1}(u^{*-1}\{t\}) = 0$ for $t \in u^*(C_1)$. Thus it follows that $|C_0| = 0$. Finally, $\mathscr{H}^{n-1}(u^{*-1}\{t\}) = 0$ only for $t \geqq M$. ∎

2.4. Lemma. *If $u \in W^{1,p}(\mathbf{R}^n)$, $1 \leqq p < \infty$, then $u^* \in W^{1,p}(\mathbf{R}^n)$.*

Proof. First consider the case $p > 1$. Let $\{u_k\}$ be a sequence of smooth regularizers of u and recall that $u_k \to u$ in $W^{1,p}(\mathbf{R}^n)$. It is well known that u_k^* is lipschitzian and that $\| |\nabla u^*| \|_p \leqq \| |\nabla u| \|_p$; c.f. [TA]. Therefore, the norms $\|u_k^*\|_{1,p}$ are bounded and this implies the existence of a subsequence $\{u_{k_j}^*\}$ converging weakly to $v \in W^{1,p}(\mathbf{R}^n)$. This, in turn, implies that $u_{k_j}^* \to v$ in $L^p(\mathbf{R}^n)$. One shows that $u^* = v$ to complete the proof.

Now consider $p = 1$. $\tilde{u}$ is a decreasing function and therefore of bounded variation. Moreover, Lemma 2.3 (i) and (ii) imply that $\tilde{u}$ is continuous, hence it will be sufficient to show that $|\tilde{u}(N)| = 0$ whenever $|N| = 0$. This follows with the aid of Lemma 2.3.

We will employ the following useful characterization of $W^{1,p}(\mathbf{R}^n)$; c.f. [MO, 3.1.2].

2.5. Proposition. *$u \in W^{1,p}(\mathbf{R}^n)$ $(p \geqq 1)$ if and only if u is equivalent to a function $\bar{u} \in L^p(\mathbf{R}^n)$ such that $\bar{u}$ is absolutely continuous (as a function of one variable) on each closed interval in almost every line parallel to the coordinate axes and $|\nabla \bar{u}| \in L^p(\mathbf{R}^n)$.*

Here $\nabla \bar{u}$ is the classical gradient; clearly, $\nabla u = \nabla \bar{u}$ almost everywhere.

Applying Lemma 2.4 and the Proposition to u^* where $u \in W^{1,p}(\mathbf{R}^n)$, we obtain the following (where as in the Introduction, $\tilde{u}(|x|) = u^*(x)$):

2.6. Corollary. *$\tilde{u}$ is locally absolutely continuous on $(0, \infty)$.*

It follows from Proposition 2.5 that u has partial derivatives almost everywhere and, consequently, u has an approximate total differential almost everywhere [SA, page 300]. That is, for almost every $x_0 \in \mathbf{R}^n$, there is a measurable set A with $D(A, x_0) = 1$ and a linear mad $du(x_0)\colon \mathbf{R}^n \to \mathbf{R}$ such that

$$\lim_{\substack{x \to x_0 \\ x \in A}} \frac{|u(x) - u(x_0) - du(x_0)(x - x_0)|}{|x - x_0|} = 0 .$$

3. Structure of an extremal

3.1. Lemma. *Let u satisfy (6). Then for almost all $t \in (0, M)$,*

$$\text{(13)} \qquad \int_{u^{-1}\{t\}} A(|\nabla u|)\, |\nabla u|^{-1}\, d\mathcal{H}^{n-1} \geqq A(|\nabla u^*|)\, |\nabla u^*|^{-1}\, \mathcal{H}^{n-1}(u^{*-1}\{t\}) .$$

In case A is strictly increasing equality in (13) implies that

$$\text{(14)} \qquad \mathcal{H}^{n-1}(\partial^* E_t) = \mathcal{H}^{n-1}(u^{-1}\{t\}) = \mathcal{H}^{n-1}(u^{*-1}\{t\})$$

and, in case $p > 1$, $\mathcal{H}^{n-1}$ almost everywhere on $u^{-1}\{t\}$,

$$\text{(15)} \qquad |\nabla u| = |\nabla u^*| = \textit{constant on } u^{*-1}\{t\} .$$

Proof. Fixing t such that (10) holds we set $C_t = u^{-1}\{t\}$ and $C_t^* = u^{*-1}\{t\}$, and apply Hölder's inequality to obtain

$$\int_{C_t} A^{1/p}(|\nabla u|)\, |\nabla u|^{-1}\, d\mathcal{H}^{n-1} = \int_{C_t} [A(|\nabla u|)\, |\nabla u|^{-1}]^{1/p}\, (|\nabla u|^{-1})^{1-1/p}\, d\mathcal{H}^{n-1} \leqq$$
$$\leqq \left(\int_{C_t} A(|\nabla u|)\, |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right)^{1/p} \left(\int_{C_t} |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right)^{1-1/p}$$

hence

$$\text{(16)} \qquad \int_{C_t} A(|\nabla u|)\, |\nabla u|^{-1}\, d\mathcal{H}^{n-1} \geqq [-\mu'(t)]^{1-p} \cdot \left(\int_{C_t} A^{1/p}(|\nabla u|)\, |\nabla u|^{-1}\, d\mathcal{H}^{n-1}\right)^p .$$

In case $p > 1$, equality clearly holds if and only if equality holds in (10) and $A(|\nabla u|) =$ $= \textit{constant}$ $\mathcal{H}^{n-1}$ almost everywhere on C_t. If A is strictly increasing, the latter statement is equivalent to $|\nabla u| = \textit{constant}$ $\mathcal{H}^{n-1}$ almost everywhere on C_t.

Referring to Lemma 2.3 we see that μ is one-one on $[0, M]$ and

$$\tilde{u}[\alpha(n)^{-1}\, \mu(t)]^{1/n} = t \quad \text{for} \quad 0 < t < M ,$$

hence

$$\text{(17)} \qquad \mu^{-1}(\alpha(n)\, r^n) = \tilde{u}(r) , \quad r = [\alpha(n)^{-1}\, \mu(t)]^{1/n} .$$

Since $\tilde{u}$ is locally absolutely continuous, (17) implies that μ^{-1} carries null sets to null sets and so we infer using the chain rule that for almost all $t \in (0, M)$,

$$1 = \tilde{u}'(r) \frac{d}{dt} [\alpha(n)^{-1}\, \mu(t)]^{1/n} = \tilde{u}'(r)\, \mu'(t) / \mathcal{H}^{n-1}(C_t^*) .$$

Denoting by $|\nabla u^*|$ the constant value of $|\nabla u^*|$ on C_t^* we thus have

$$|\nabla u^*|^{-1}\,\mathscr{H}^{n-1}(C_t^*) = -\tilde{u}'(r)^{-1}\,\mathscr{H}^{n-1}(C_t^*) = -\mu'(t)\,;$$

we conclude using (10) that

$$\int_{C_t} |\nabla u|^{-1}\,d\mathscr{H}^{n-1} \leqq -\mu'(t) = |\nabla u^*|^{-1}\,\mathscr{H}^{n-1}(C_t^*)\,. \tag{18}$$

We also infer using (11) and the isoperimetric theorem that

$$\mathscr{H}^{n-1}(C_t) \geqq \mathscr{H}^{n-1}(\partial^* E_t) \geqq H^{n-1}(C_t^*)\,. \tag{19}$$

We next define $\phi(\zeta) = \zeta A^{1/p}(\zeta^{-1})$, infer from (3) and (4) that ϕ is decreasing on $(0, \infty)$ and use (4) to verify that ϕ is convex. Using Jensen's inequality, ideas from [AT] and [B 1] and (5), (18) and (19) we obtain

$$\begin{aligned} &\int_{C_t} A^{1/p}(|\nabla u|)\,|\nabla u|^{-1}\,d\mathscr{H}^{n-1} = \int_{C_t} \phi(|\nabla u|^{-1})\,d\mathscr{H}^{n-1} \geqq \\ &\geqq \mathscr{H}^{n-1}(C_t^*)\,\phi(|\nabla u^*|^{-1}) = \mathscr{H}^{n-1}(C_t^*)\,A^{1/p}(|\nabla u^*|)\,|\nabla u^*|^{-1}\,. \end{aligned} \tag{20}$$

(13) now follows from (16) and (18). (Note that (20) reduces to (19) in case $A(\xi) = |\xi|^p$.) Now assume A is strictly increasing. If equality holds in (20), then equality holds in (19), whence follows (14). Furthermore, if $p > 1$, equality in (13) implies equality in (10) hence in (18), and thus we conclude (15) using (14). ■

Integrating (13) and using the coarea formula (9) we conclude

$$\int_{\mathbf{R}^n} A(|\nabla u|) \geqq \int_{\mathbf{R}^n} \int_{u^{-1}\{t\}} A(|\nabla u|)\,|\nabla u|^{-1}\,d\mathscr{H}^{n-1}dt \geqq \int_{\mathbf{R}^n} A(|\nabla u^*|)\,,$$

which verifies the first assertion of Theorem 1.1

For the remainder of this section we assume that equality holds in (7).

3.2. Lemma *If A is strictly increasing and u satisfying (6) is such that equality holds in (7), then E_t is equivalent to an open ball U_t for each $t \in [0, M)$. Further, $\{x\colon u(x) \geqq M\}$ is equivalent to $\bigcap_{t<M} \bar{U}_M = B_M$. is a closed ball (possibly a single point), which we assume to be centered at* 0.

Proof. The isoperimetric theorem implies that E_t is equivalent to an open ball U_t for each t in the set G of $t \in (0, M)$ such that (14) holds. The remainder of the proof is straightforward. ■

With A and u as in Lemma 3.2 we will now derive a fundamental relationship ((21) below) between u and u^*. To this end, let $c(t)$ denote the center of U_t for $t \in [0, M)$ and set $c(M) = 0$. Now define $\tau\colon \mathbf{R} \to \mathbf{R}^n$ and $T\colon \mathbf{R}^n \to \mathbf{R}^n$ by

$$\tau = c \circ \tilde{u}\,, \quad T(x) = x + \tau(|x|)\,.$$

Note that $\tau(r) = 0$ for $|r| \leqq r_0 =$ radius B_M, hence $T \mid B_M$ is the identity.

3.3. Lemma. *Assume that μ is continuous on $(0, M)$.*

(i) *For $0 \leqq t < M$, T translates ∂E_t^* to ∂U_t, and $T(E_t^*) = U_t \cup \sigma_t$ where $\sigma_t \subset \partial U_t$ contains at most one point.*

(ii) *τ and T are lipschitzian with* $\operatorname{Lip}(\tau) = 1$ and $\operatorname{Lip}(T) = 2$.

(iii) *There exists $\Omega \subset \mathbf{R}^n$ such that $|\mathbf{R}^n \sim \Omega| = 0$ and $T \mid \Omega$ is one-one.*

(iv) *For almost all $x \in \mathbf{R}^n$,*

$$u^*(x) = u \circ T(x) . \tag{21}$$

(v) *For almost all $x \in \mathbf{R}^n$, T is differentiable at x and $dT(x)$ is one-one.*

(vi) *For $S \subset \mathbf{R}^n$, $|S| = 0$ if and only if $|T(S)| = 0$.*

(vii) *For almost all $x \in \mathbf{R}^n$,*

$$\nabla u(x) \cdot v = \nabla u(T(x)) \cdot dT(x)(v) , \quad v \in \mathbf{R}^n . \tag{22}$$

Proof. (i) Note that for $0 \leqq t_0 \leqq t_1 < M$, ∂U_{t_1} lies within (but possibly tangent to) ∂U_{t_0}. Thus since μ and u^* are continuous by assumption and Corollary 2.6, for $0 \leqq t_0 < M$,

$$E_{t_0}^* = B_M \cup \Big(\bigcup_{t > t_0} \partial E_t^* \Big) , \quad U_{t_0} = B_M \cup \Big(\bigcup_{t > t_0} \partial U_t \Big) \sim \sigma_{t_0} . \tag{23}$$

(ii) Fix $x_1, x_2 \in \mathbf{R}^n$. By (23) we can assume $x_i \in \partial E_{t_i}^*$, where $0 \leqq t_2 < t_1 < M$. Denote $r_i = |x_i|$. Then $u^*(r_i) = t_i$,

$$|T(x_2) - T(x_1)| \leqq |x_2 - x_1| + |\tau(r_2) - \tau(r_1)| ,$$

and

$$|\tau(r_2) - \tau(r_1)| = |c(t_2) - c(t_1)| \leqq r_2 - r_1 \leqq |x_2 - x_1| .$$

(iii) follows from (i).

(iv) follows through use of (iii) and Lemma 2.3.

(v) follows from (i) and the definition of T.

(vi) By the area formula [F, 3.2.3] and (iii) we have $\int_B JT = |T(B)|$ for each measurable subset $B \subset \mathbf{R}^n$, where $JT = |\det dT|$ is the jacobian of T. Further, (v) implies that $JT(x) > 0$ for almost all $x \in E_0^*$, hence (vi) follows directly.

(vii) It will suffice to verify (22) for $x \in E_0^*$. Now (vi) implies that if $A_1, A_2, \ldots$ are measurable subsets of U_0 such that

$$\Big|U_0 \sim \bigcup_{i=1}^{\infty} A_i\Big| = 0 ,$$

then

$$0 = \Big|T^{-1}\Big(U_0 \sim \bigcup_{i=1}^{\infty} A_i\Big)\Big| = \Big|E_0^* \sim \bigcup_{i=1}^{\infty} T^{-1}(A_i)\Big| .$$

We thus fix $\varepsilon > 0$ and use [CZ, Theorem 13] to find a measurable subset A of U_0 and $u_1 \in C^1(\mathbf{R}^n)$ such that $u \mid A = u_1 \mid A$ and $|U_0 \sim A| < \varepsilon$. We may assume $D(A, y) = 1$ for $y \in A$. From the remark following Proposition 2.5 we infer that we may also assume that $\nabla u = \nabla u_1$ on A. Denote $u_1^* = u_1 \circ T$. The classical chain rule implies that for almost all $x \in T^{-1}(A)$,

$$\nabla u_1^*(x) = \nabla u_1(T(x)) \cdot dT(x) = \nabla u(T(x)) \cdot dT(x) .$$

(Here $\nabla u_1^*(x)$ is the classical gradient.) On the other hand, $u^* = u_1^*$ on $T^{-1}(A)$ implies that each $x \in T^{-1}(A)$ where $\nabla u_1^*(x)$ exists and $D(T^{-1}(A), x) = 1$, u^* is approximately differentiable and $\nabla u_1^*(x) = \nabla u^*(x)$. We conclude that (22) holds at almost all $x \in T^{-1}(A)$. ∎

Assuming $p > 1$ and $|C^* \cap u^{*-1}(0, M)| = 0$ (which is equivalent to absolute continuity of μ by Lemma 2.3), we will show in Section 4 that $\tau = 0$, hence T is the identity. In view of (21) this will complete the second part of the proof of Theorem 1.1.

4. Characterization of an extremal

We recall here that $\tilde{u}(|x|) = u^*(x)$.

4.1. Theorem. *Let u satisfy (6) with μ continuous on $(0, M)$. Assume A is strictly increasing. If $p > 1$ and*

$$\int_{\mathbf{R}^n} A(|\nabla u|) = \int_{\mathbf{R}^n} A(|\nabla u^*|) ,$$

then $\tilde{u}'\tau' = 0$ almost everywhere.

Proof. Note that $\nabla u^*(x) = \tilde{u}'(|x|)\,|x|^{-1}\,x$ for almost all $x \neq 0$. Using the coarea formula (9) with u replaced by u^*, we infer from (15) and (22) that for almost all $r_0 < r < r_1$ such that $\tilde{u}'(r) \neq 0$, for $\mathscr{H}^{n-1}$ almost all x with $|x| = r$, the chain rule for $u^* = u \circ T$ holds at x and

$$(24) \qquad |\nabla u(T(x))| = |\nabla u^*(x)| \neq 0 .$$

For such an r, assuming $\tau'(r) \neq 0$ we choose x so that also $\cos\theta < -1/2$ where $\theta \in (\pi/2, \pi]$ satisfies

$$0 > \cos\theta = \frac{x}{|x|} \cdot \frac{\tau'(r)}{|\tau'(r)|} .$$

Denoting $v = \nabla u^*(x)/|\nabla u^*(x)| = -x/|x|$ we compute

$$|\nabla u^*(x)| = \nabla^*(x) \,.\, v = \nabla u(T(x)) \,.\, dT(x)\,(v) = \nabla u(T(x)) \,.\, w ,$$

where

$$w = v + \left(v \cdot \frac{x}{|x|}\right)\tau'(r) = -\left[\frac{x}{|x|} + \tau'(r)\right].$$

Setting $a = |\tau'(r)| \cos\theta$ we have

$$\tau'(r) = \frac{ax}{|x|} + v_0 , \quad x \,.\, v_0 = 0 ,$$

hence

$$|w|^2 = (1 + a)^2 + |v_0|^2 = (1 + a)^2 + |\tau'(r)|^2 - a^2 < 1$$

because $|\tau'(r)| \leqq 1 < -2\cos\theta$. Consequently,

$$\nabla u(T(x)) \,.\, w < |\nabla u(T(x))| ,$$

which contradicts (24). ∎

4.2. Remark. The converse to Theorem 4.1 is also true. Assume u satisfies (6), and assume there exists a lipschitzian mapping T, $T(x) = x + \tau(|x|)$, such that $u^* = u \circ T$ almost everywhere. If $p \geqq 1$ and $\tilde{u}'\tau' = 0$ almost everywhere, then equality holds in (7).

4.3. Proof of Theorem 1.1. The first part of the Theorem was proved following the proof of Lemma 3.1. For the proof of the second part of the Theorem, note that in view of Lemma 2.3(v), μ is continuous on $(0, M)$ and $\tilde{u}'(r) \neq 0$ for almost all $r_0 =$ radius $B_M < r < r_1 =$ radius E_0^*. Consequently, Theorem 4.1 implies that $\tau' = 0$ almost everywhere on (r_0, r_1), and this in turn implies that $\tau = 0$ because τ is lipschitzian (hence absolutely continuous) and $\tau(r_0) = 0$. In view of (21) this completes the proof. ∎

4.4. Examples. Here we construct radially decreasing functions w and translating functions τ such that $\tilde{w}'\tau' = 0$ almost everywhere and $\tau' \neq 0$ on some set of positive measure. Thus τ is not constant and so $u = w \circ T^{-1}$ is not equivalent to a translate of $u^* = w$. (Here, as always, $w(x) = \tilde{w}(|x|)$). On the other hand, equality will hold in (7) for such functions because $\tilde{w}'\tau' = 0$. (Recall Remark 4.2.)

First we consider a nonnegative, decreasing, lipschitzian $\tilde{w}$: $[0, \infty) \to \mathbf{R}$ with compact support. If $\tilde{w}' = 0$ on a set which is equivalent to an interval I with $|I| > 0$, then $\tilde{w} \mid I$ is constant because $\tilde{w}$ is absolutely continuous. It follows that the level set $\{x\colon w(x) = \tilde{w}(|x|) = t_0\}$, $\{t_0\} = \tilde{w}(I)$, contains an annulus $\{x\colon \varrho_1 < |x| < \varrho_2\}$. Fix $0 < c < \varrho_2 - \varrho_1$ and define

$$u(y) = \begin{cases} w(y)\,, & |y| \leqq \varrho_1\,, \\ t_0\,, & \varrho_1 < |y| < \varrho_2\,, \\ w(y - ce_1)\,, & |y| \geqq \varrho_2\,. \end{cases}$$

Clearly, u and $u^* = w$ have equal distribution functions and (49) holds. Note that the distribution μ is not continuous at t_0.

Next consider a nowhere dense, closed $C_0 \subset (0, 1)$ such that $|C_0| > 0$ and

$$|C_0 \cap I| < |I| \tag{25}$$

for every interval $I \subset [0, 1]$ with $|I| > 0$. Define

$$\tilde{w}(r) = |[r, 1] \sim C_0| \,.$$

$\tilde{w}$ is strictly decreasing by (25). Moreover, $\tilde{w}$ is lipschitzian and $\tilde{w}'(r) = 0$ for almost all $r \in C_0$. Define

$$\tau(r) = \tfrac{1}{2}|[0, r] \cap C_0|\, e_1 \, .$$

Clearly $\tilde{w}'\tau' = 0$ almost everywhere. The map $T(x) = x + \tau(|x|)$ is one-one with $\operatorname{Lip}(T^{-1}) = 2$, hence $u = w \circ T^{-1}$ is lipschitzian. w is the symmetrization of u; however, $u^* = w$ is not equivalent to u because $|\tau(1)| = \frac{1}{2}|C_0| \neq |\tau(0)|$. Equality holds in (7) by Remark 4.2.

We next construct a smooth extremal u such that $u \neq u^*$. (However, note that the above method of constructing u from $w = u^*$ will not work because τ cannot be C^1.) It is well known that there exists $f \in C^\infty(\mathbf{R})$ such that $f \geqq 0$, $\operatorname{spt} f \subset [0, 1]$ and $C_0 = (0, 1) \cap \{x : f(r) = 0\}$; cf. [K, page 28]. By following f with $g \in C^\infty(\mathbf{R})$ such that $g \mid (0, \infty) > 0$ and $\operatorname{spt} g = [0, \infty]$, we obtain a function satisfying the conditions on f and such that all derivatives of $g \circ f$ vanish on C_0. Define

$$w_+(r) = \sigma \int_r^1 g \circ f \, ,$$

where $\sigma \in \mathbf{R}$ is chosen so that $w_+(0) = 1$. w_+ is clearly strictly decreasing on $[0, 1]$, hence $w_+(C_0)$ is closed, nowhere dense and of measure 0.

Let c be a continuous increasing function on $[0, 1]$ such that $c(1) = 0$, $c(0) = -1$, and c is constant on each component of $[0, 1] \sim w_+(C_0)$. (c is a generalized Cantor function. In case C_0 is obtained by shrinking then translating one of the standard Cantor sets of positive measure obtained by removing intervals of length $\varepsilon/3^k$ from $[0, 1]$ where $0 < \varepsilon < 1$, one can take $c = -h \circ u_+^{-1}$ where h is the Cantor function constructed using C_0 such that $h(0) = 0$, $h(1) = 1$.) Observing that $w_+^{-1} - c$ is strictly decreasing on $[0, 1]$, we define $\tilde{w} : [0, 2] \to \mathbf{R}$ so that

$$\tilde{w}^{-1} = w_+^{-1} - c \, , \tag{26}$$

and denote

$$\tau_1 = c \circ \tilde{w} \, . \tag{27}$$

(26) implies that $\operatorname{Lip}(\tau_1) = 1$. Moreover, it also follows from (26) that $\tilde{w}$ is C^1 with $\{r : \tilde{w}'(r) = 0\} = \tilde{w}^{-1}(w_+(C_0))$. (See the following paragraph.) Differentiating (27) we have for almost all r such that $\tilde{w}'(r) \neq 0$, $\tau'(r) = c' \circ \tilde{w}(r)\, \tilde{w}'(r) = 0$ because $\tilde{w}(r) \notin w_+(C_0)$. Consequently,

$$\tau_1' \tilde{w}' = 0 \quad \text{almost everywhere} \, . \tag{28}$$

We also conclude that $|\{r : \tilde{w}'(r) = 0\}| > 0$, for otherwise τ_1 would be constant, contradicting $\tau_1(0) = 0$, $\tau_1(2) = -1$.

We next verify that $\tilde{w}$ is C^∞. Clearly, $\tilde{w}$ is smooth on the complement of $\tilde{w}^{-1}(w_+(C_0))$. Inasmuch as all derivatives of w_+ vanish on C_0, it will suffice to show that all derivatives of $\tilde{w}$ vanish on $\tilde{w}^{-1}(w_+(C_0))$. Thus fix $r_0 \in \tilde{w}^{-1}(w_+(C_0))$. Clearly, $\tilde{w}'(r_0) = 0$ because $(\tilde{w}^{-1})'(t_0) \leqq (w_+^{-1})'(t_0) = -\infty$, $t_0 = \tilde{w}(r_0)$. Assume inductively that $\tilde{w}^{(k)}(r_0) = 0$, and set $s_0 = w_+^{-1}(t_0)$. Since for $t \notin w_+(C_0)$, $\tilde{w}^{(k)}(\tilde{w}^{-1}(t)) = w_+^{(1)}(w_+^{-1}(t))$, we have by (26)

$$0 = \lim_{s \to s_0} \left| \frac{w_+^{(k)}(s)}{s - s_0} \right| = \lim_{t \to t_0} \left| \frac{w_+^{(k)}(w_+^{-1}(t))}{w_+^{-1}(t) - w_+^{-1}(t_0)} \right| \geqq$$

$$\geqq \lim_{t \to t_0} \left| \frac{w_+^{(k)}(w_+^{-1}(t))}{[w_+^{-1}(t) - c(t)] - [w_+^{-1}(t_0) - c(t_0)]} \right| = \lim_{r \to r_0} \left| \frac{\tilde{w}^{(k)}(r)}{r - r_0} \right| = \left| \tilde{w}^{(k+1)}(r_0) \right|,$$

because

$$\left| [w_+^{-1}(t) - w_+^{-1}(t_0)] - [c(t) - c(t_0)] \right| \geqq \left| w_+^{-1}(t) - w_+^{-1}(t_0) \right|.$$

Now define for $x \in \mathbf{R}^n$,

$$w(x) = \begin{cases} \tilde{w}(|x|), & |x| \leqq 2, \\ 0, & \text{otherwise}. \end{cases}$$

Since c is constant on neighbourhoods of 0 and of 1 because $C_0 \subset (0, 1)$, and since $\operatorname{spt} w_+ \subset \{r : r \leqq 1\}$, it follows that all derivatives of $\tilde{w}$ vanish at 0 and at 2, hence $w \in C^\infty(\mathbf{R}^n)$. Defining

$$\tau(r) = \tfrac{1}{2} \begin{cases} 0, & r < 0, \\ \tau_1(r)\, e_1, & 0 \leqq r \leqq 2, \\ -e_1, & r > 2, \end{cases}$$

we define $T(x) = x + \tau(|x|)$ and $u = w \circ T^{-1}$ as above. In view of (28), Remark 4.2 implies that u is extremal; $u \neq u^* = w$ because τ is not constant.

Finally, it is proved in [BZ] that $u \in C^\infty(\mathbf{R}^n)$.

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