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SYSTEMS OF COVERS OF FRAMES AND RESULTING SUBFRAMES

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A system of covers $\mathcal{A}$ on a frame L generates in a natural way certain (in general, non-reflexive) ordering $\overset{\mathcal{A}}{\triangleleft}$ on L and this in turn gives rise to a subset $L_{\mathcal{A}}$ of L (see 1.7 and 1.10 below). This has been used e.g. for describing the uniformizability (and complete regularity), regularity and metrizability of frames (see [6],[8]). In these notes we present some more facts concerning the mentioned construction.

The paper is divided into three sections. The first one is devoted to a detailed introduction into working with systems of covers and with the resulting orders. In section 2 we, first, summarize a few known facts, some of them in a slightly modified light. Then, when having realized that the existence of special $\mathcal{A}$ such that $L = L_{\mathcal{A}}$ can characterize some properties (general $\mathcal{A}$ characterize the regularity, uniformity bases $\mathcal{A}$ characterize the complete regularity, countable $\mathcal{A}$ the metrizability) the question naturally arises as to when L equals $L_{\mathcal{A}}$ with finite resp. one-element $\mathcal{A}$. As an answer, a characterization of atomic Boolean algebras among frames is obtained. Section 3 concerns injectively semireflective subcategories $\mathcal{C}$ of the category of frames. In particular we show that if $\mathcal{C}$ is contained in the category of regular frames, there are always systems of covers $\mathcal{A}(\mathcal{C},L)$ such that the coreflection is given as the correspondence $L \longmapsto L_{\mathcal{A}(\mathcal{C},L)}$.

1. Preliminaries: Systems of covers

1.1. In this paper, USL_1 is the category of all complete upper semilattices with unit, and their $(\bigvee,1)$-preserving homomor-

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This paper is in final form and no version of it will be submitted for publication elsewhere.

phisms, FRM (see,e.g. [4]) is the category of frames (complete lattices satisfying the complete distributivity law $(\bigvee a_i)\wedge b = \bigvee(a_i\wedge b)$) and the $(\bigvee,\wedge,0,1)$-preserving homomorphisms. Thus, FRM can be viewed as a (not full) subcategory of USL_1.

In the first two sections we will work in FRM only, the upper semilattices and their homomorphisms will play a role as a suitable more general framework for some results in section 3. The reader should note that all the definitions and some of the facts in section 1 remain valid in the USL_1-context as well, since an upper semilattice with 1 is automatically a lattice and hence formulas like $a\wedge b \neq 0$ make sense. We repeat, however, that in the first part of the article we are interested in frames only.

<u>1.2.</u> Recall that a <u>cover</u> of a frame L is a subset $A\subseteq L$ such that $\bigvee A = 1$. We say that a cover A <u>refines</u> a cover B and write

$$A \prec B$$

if $\forall a\in A\ \exists b\in B$ such that $a\leq b$. Thus, in particular,

$$A\subseteq B \Rightarrow A\prec B \ .$$

If $\mathcal{A}$, $\mathcal{B}$ are systems of covers we say that $\mathcal{A}$ <u>majorizes</u> $\mathcal{B}$ and write

$$\mathcal{A} \text{ maj } \mathcal{B}$$

if $\forall A\in\mathcal{A}\ \exists B\in\mathcal{B}$ such that $B\prec A$. In particular,

$$\mathcal{A}\subseteq\mathcal{B} \Rightarrow \mathcal{A} \text{ maj } \mathcal{B} .$$

<u>1.3.</u> If A_1, A_2 are covers of L we write

$$A_1\wedge A_2 \quad \text{for} \quad \{a_1\wedge a_2 \mid a_i\in A_i\} .$$

Obviously, $A_1\wedge A_2$ is a cover and we have

(1.3.1) $A_1\wedge A_2 \prec A_i \quad (i = 1,2)$,

(1.3.2) if $A_i\prec B_i$ (i=1,2) then $A_1\wedge A_2 \prec B_1\wedge B_2$.

<u>1.4.</u> For a cover A put

$$A^{(2)} = \{a\vee b \mid a,b\in A,\ a\wedge b\neq 0\},$$
$$A^{*} = \{\bigvee X \mid X\subseteq A \text{ such that } a,b\in X \Rightarrow a\wedge b\neq 0\}.$$

We easily see that

(1.4.1) $(A_1\wedge A_2)^{(2)} \prec A_1^{(2)}\wedge A_2^{(2)}$,

(1.4.2) $(A_1\wedge A_2)^{*} \prec A_1^{*}\wedge A_2^{*}$.

(Let us prove, say (1.4.2), just as an exercise of the work with the notions. Let $X\subseteq A_1\wedge A_2$ be such that $x,y\in X \Rightarrow x\wedge y\neq 0$. Put $X_i = \{a_i\in A_i \mid \exists\ a_{3-i}\in A_{3-i}$ such that $a_1\wedge a_2\in X\}$. Obviously $a_i,b_i\in X_i \Rightarrow a_i\wedge b_i\neq 0$, hence $\bigvee X_i\in A_i^{*}$ and we have $\bigvee X \leq \bigvee X_1\wedge\bigvee X_2$.)

Recall (see [6]) that a system of covers $\mathcal{A}$ is said to be a uniformity basis (resp. weak uniformity basis; we will write briefly u-basis and wu-basis) if for each $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ such that

$$B^* \prec A \quad (\text{resp. } B^{(2)} \prec A)$$

1.5. Some special systems of covers: Let L_1 be a subset of L. Denote by

$$\mathcal{D}(L,L_1), \text{ resp. } \mathcal{F}in(L,L_1)$$

the system of all two-element, resp. all finite, covers A of L such that $A \subseteq L_1$.

Obviously, unions of (weak) uniformity bases are (weak) uniformity bases. Consequently, on each L there is the largest (weak) uniformity basis. It will be denoted by

$$\mathcal{U}(L) \quad (\text{resp. } w\mathcal{U}(L)).$$

1.6. If $A \subseteq L$ is a cover and $x \in L$ we write

$$Ax = \bigvee\{a \in A \mid a \wedge x \neq 0\}.$$

Obviously,

$$(1.6.1) \qquad A \prec B \Rightarrow Ax \leq Bx,$$

$$(1.6.2) \qquad A(Ax) \leq A^{(2)}x.$$

1.7. Let $\mathcal{A}$ be a system of covers of a frame L. We write

$$x \overset{\mathcal{A}}{\lhd} y$$

if there is an $A \in \mathcal{A}$ such that $Ax \leq y$. Obviously we have the formulas

$$(1.7.1) \qquad \mathcal{A} \text{ maj } \mathcal{B} \Rightarrow (x \overset{\mathcal{A}}{\lhd} y \Rightarrow x \overset{\mathcal{B}}{\lhd} y),$$

$$(1.7.2) \qquad x \leq x' \overset{\mathcal{A}}{\lhd} y' \leq y \Rightarrow x \overset{\mathcal{A}}{\lhd} y.$$

1.8. We say that $\mathcal{A}$ has the property (M) if

$$(\mathrm{M}) \qquad x_i \overset{\mathcal{A}}{\lhd} y_i \ (i=1,2) \Rightarrow x_1 \wedge x_2 \overset{\mathcal{A}}{\lhd} y_1 \wedge y_2 .$$

The following is obvious:

Proposition: Let $\mathcal{A}$ be such that for any two $A_1, A_2 \in \mathcal{A}$ there is an $A \in \mathcal{A}$ with $A \prec A_i$ (i=1,2). Then $\mathcal{A}$ has the property (M). □

Thus, by (1.3.1) and (1.4.1), $\mathcal{U}(L)$ and $w\mathcal{U}(L)$ have (M). Obviously so has $\mathcal{F}in(L,L_1)$ whenever L_1 is a subframe of L.

1.9. Proposition: Let L_1 be a subframe of L. Then

$$\overset{\mathcal{D}(L,L_1)}{\lhd} \equiv \overset{\mathcal{F}in(L,L_1)}{\lhd}$$

Proof: Trivially, $x \overset{\mathcal{D}}{\lhd} y \Rightarrow x \overset{\mathcal{F}in}{\lhd} y$. Now, let $x \overset{\mathcal{F}in}{\lhd} y$. Thus, there are $a_1,\dots,a_n \in L$ such that $\bigvee_{i=1}^{n} a_i = 1$ and $a_i \wedge x \neq 0 \Rightarrow a_i \leq y$. Put $a = \bigvee\{a_i \mid a_i \wedge x \neq 0\}$, $b = \bigvee\{a_i \mid a_i \wedge x = 0\}$. Obviously, $\{a,b\}x = a \leq y$ and hence $x \overset{\mathcal{D}}{\lhd} y$. □

1.9.1. Corollary: If L_1 is a subframe of L, then $\mathcal{D}(L,L_1)$ has the property (M). □

(Note that common refinements in $\mathcal{D}$ are extremely rare. Thus, the premise in Proposition 1.8 is far from being a necessary condition for (M).)

1.10. We put

$$[L:\mathcal{A}] = \{x \in L \mid x = \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x\}\}.$$

(In [6], [7], [8] $[L:\mathcal{A}]$ was written as $L_{\mathcal{A}}$. We are changing the notation to avoid too complex indices.)

1.11. Proposition: (1) $[L:\mathcal{A}]$ is always an upper sub-semilattice of L. If $\mathcal{A}$ has the property (M) then $[L:\mathcal{A}]$ is a subframe of L.

(2) If $\mathcal{A}$ maj $\mathcal{B}$ then $[L:\mathcal{A}] \subseteq [L:\mathcal{B}]$.

Proof: (1) Obviously, $1_L \in [L:\mathcal{A}]$. Let x_j $(j \in J)$ be in $[L:\mathcal{A}]$. Thus $x_j = \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x_j\}$ and hence $\bigvee_J x_j = \bigvee_{j \in J} \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x_j\} \leq \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} \bigvee_J x_j\} \leq \bigvee_J x_j$ (recall 1.7.2). If $\mathcal{A}$ has (M) and $x_1, x_2 \in [L:\mathcal{A}]$ we have $x_1 \wedge x_2 = \bigvee\{y_1 \mid y_1 \overset{\mathcal{A}}{\lhd} x_1\} \wedge \bigvee\{y_2 \mid y_2 \overset{\mathcal{A}}{\lhd} x_2\} = \bigvee\{y_1 \wedge y_2 \mid y_1 \overset{\mathcal{A}}{\lhd} x_1$ & $y_2 \overset{\mathcal{A}}{\lhd} x_2\} \leq \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x_1 \wedge x_2\} \leq x_1 \wedge x_2$.

(2) immediately follows from (1.7.1). □

1.12. Notation: Let

$$f: L \longrightarrow K$$

be a frame morphism. For covers A of L and systems of covers $\mathcal{A}$ we will use the notation

$$f(A) = \{f(a) \mid a \in A\},$$
$$f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\}.$$

1.13. Proposition: (1) $A \prec B \Rightarrow f(A) \prec f(B)$,

(2) $\mathcal{A}$ maj $\mathcal{B} \Rightarrow f(\mathcal{A})$ maj $f(\mathcal{B})$,

(3) $f(A)f(x) \leq f(Ax)$

(4) $f(A)^{(2)} \subseteq f(A^{(2)})$,

(5) $f(A)^{*} \subseteq f(A^{*})$.

Proof is straightforward. Let us show, e.g., (5): If $u \in f(A)^{*}$ we have $u = \bigvee X$ for some $X \subseteq f(A)$ such that $x, y \in X \Rightarrow x \wedge y \neq 0$. Let $X = \{f(a) \mid a \in Y\}$; for $a, b \in Y$ we have $f(a \wedge b) = f(a) \wedge f(b) \neq 0$ and hence $a \wedge b \neq 0$ so that $\bigvee Y \in A^{*}$. Thus, $u = \bigvee X = f(\bigvee Y) \in f(A^{*})$. □

1.14. Corollary: If $f(\mathcal{A})$ maj $\mathcal{B}$ then

$$x \overset{\mathcal{A}}{\lhd} y \Rightarrow f(x) \overset{\mathcal{B}}{\lhd} f(y).$$

(By 1.13.(3), $x \overset{\mathcal{A}}{\triangleleft} y \Rightarrow f(x) \overset{f(\mathcal{A})}{\triangleleft} f(y)$. Use (1.7.1).) □

<u>1.15.</u> <u>Corollary</u>: Let $f: L \longrightarrow K$ be a frame morphism. Then

$$f(w\mathcal{U}(L)) \subseteq w\mathcal{U}(K),$$
$$f(\mathcal{U}(L) \subseteq \mathcal{U}(K).$$

Consequently,

$$x \overset{w\mathcal{U}(L)}{\triangleleft} y \Rightarrow f(x) \overset{w\mathcal{U}(K)}{\triangleleft} f(y),$$
$$x \overset{\mathcal{U}(L)}{\triangleleft} y \Rightarrow f(x) \overset{\mathcal{U}(K)}{\triangleleft} f(y).$$

(The inclusions follow from 1.13.(4),(5). Then we use 1.14.) □

<u>1.16.</u> <u>Corollary</u>: Let $f: L \rightarrow K$ be a frame morphism, let $f(L_1) \subseteq$ $\subseteq K_1$. Then

$$x \overset{\mathfrak{D}(L,L_1)}{\triangleleft} y \Rightarrow f(x) \overset{\mathfrak{D}(K,K_1)}{\triangleleft} f(y). \quad \square$$

<u>2. Some properties of frames represented in the form $L=[L:\mathcal{A}]$</u>

<u>2.1.</u> Recall the following standard definitions (see, e.g. [4], cf[1]). In a frame one writes

$$x \triangleleft y$$

if there is a z such that $z \wedge x = 0$ and $y \vee z = 1$. One writes

$$x \triangleleft\!\triangleleft y$$

if there exist x_d indexed by some D dense subset of the unit interval (e.g., the set of all dyadic rationals) such that

$$x = x_0 \;, \; y = y_0 \quad \text{and } d < e \Rightarrow x_d \triangleleft x_e.$$

(More exactly, one should indicate the frame in question, writing, say, $\triangleleft_L$, $\triangleleft\!\triangleleft_L$. We will do it in section 3, here there is no danger of confusion.)

A frame is said to be regular (resp.completely regular) if

$$\forall x \in L \qquad x = \bigvee \{y \mid y \triangleleft x\}$$
$$(\text{resp. } \forall x \in L \qquad x = \bigvee \{y \mid y \triangleleft\!\triangleleft x\}).$$

<u>2.2.</u> <u>Lemma</u>: Let $\mathcal{A}$ be a system of covers. Then

$$x \overset{\mathcal{A}}{\triangleleft} y \Rightarrow x \triangleleft y.$$

If $\mathfrak{D}(L,L)$ maj $\mathcal{A}$ then

$$\overset{\mathcal{A}}{\triangleleft} \equiv \triangleleft .$$

<u>Proof</u>: Let $Ax \leq y$. Put $z = \bigvee \{a \mid a \wedge x = 0\}$. Since A is a cover, $Ax \vee z = 1$ and hence $y \vee z = 1$. By the distributivity $x \wedge z = 0$. Now, let $\mathfrak{D}(L,L)$ maj $\mathcal{A}$. By the first implication and by (1.7.1) it suffices to prove that $x \triangleleft y \Rightarrow x \overset{\mathfrak{D}}{\triangleleft} y$. Let $z \wedge x = 0$, $z \vee y = 1$. Then $\{y,z\}$ is a cover and $\{y,z\}x \leq y$. □

2.3. Lemma: Let $\mathcal{A}$ ba a wu-basis. Let $x \overset{\mathcal{A}}{\lhd} y$. Then there is a z such that $x \overset{\mathcal{A}}{\lhd} z \overset{\mathcal{A}}{\lhd} y$.

Proof: Let $Ax \leq y$. We have a $B \in \mathcal{A}$ such that $B^{(\omega)} \prec A$. Hence $B(Bx) \leq B^{(\omega)}x \leq Ax \leq y$ by (1.6.2) and (1.6.1). Put $z = Bx$. □

2.4. From 2.2. and 2.3. we immediately obtain

Corollary: Let $\mathcal{A}$ be a wu-basis. Then

$$x \overset{\mathcal{A}}{\lhd} y \Rightarrow x \lhd\!\lhd y. \ \square$$

2.5. Lemma: We have

$$\overset{\mathcal{U}(L)}{\lhd} \equiv \overset{w\mathcal{U}(L)}{\lhd} \equiv \lhd\!\lhd .$$

Proof: Since by (1.7.1) and 2.4,

$$x \overset{\mathcal{U}(L)}{\lhd} y \Rightarrow x \overset{w\mathcal{U}(L)}{\lhd} y \Rightarrow x \lhd\!\lhd y$$

it suffices to show that $x \lhd\!\lhd y \Rightarrow x \overset{\mathcal{A}}{\lhd} y$. By [6,Prop.5.2] , there is a u-basis $\mathcal{A}$ such that $x \lhd\!\lhd y \Rightarrow x \overset{\mathcal{A}}{\lhd} y$. Use (1.7.1). □

2.6. Theorem: The following statements are equivalent:

(1) L is regular,

(2) $L = [L:\mathcal{A}]$ for a system of covers $\mathcal{A}$,

(3) $L = [L:\mathfrak{D}(L,L)]$,

(4) for each $\mathcal{A}$ such that $\mathfrak{D}(L,L)$ maj $\mathcal{A}$, $L= [L:\mathcal{A}]$.

Proof: (2) $\Rightarrow$ (1) $\Rightarrow$ (4) by 2.2, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) is trivial. □

2.7. Theorem: The following statements are equivalent:

(1) L is completely regular,

(2) $L = [L:\mathcal{A}]$ for a u-basis $\mathcal{A}$,

(3) $L = [L:\mathcal{A}]$ for a wu-basis $\mathcal{A}$,

(4) $L = [L:\mathcal{U}(L)]$,

(5) $L = [L:w\mathcal{U}(L)]$.

Proof: By [6;Theorem 5.3], (1) $\Rightarrow$(2). Further, we have the implications

$$\begin{tikzcd}
(2) \arrow[r, Rightarrow, "1.11"] \arrow[d, Rightarrow, "\text{trivial}"] & (4) \arrow[d, Rightarrow, "1.11"] & \\
(3) \arrow[r, Rightarrow, "1.11"'] & (5) \arrow[r, Rightarrow, "2.4"'] & (1).
\end{tikzcd} \ \square$$

2.8. Theorem: The following statements are equivalent:

(1) L is metrizable,

(2) $L = [L:\mathcal{A}]$ for a countable system of covers $\mathcal{A}$,

(3) $L = [L:\mathcal{A}]$ for a countable wu-basis $\mathcal{A}$,

(4) $L = [L:\mathcal{A}]$ for a countable u-basis $\mathcal{A}$.

Proof: The equivalence (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) has been estabilished in [7;Theo.4.6]. In fact, the existence of a countable u-basis - in a slightly different form - was, in essence originally taken for the definition of a metrizable frame (locale) (see [3]). What was done in [7] was connecting this with the existence of certain type of a real function on L (the diameter). The equivalence (2) $\Leftrightarrow$ (3) is a generalization of Bing's metrizability theorem (see, e.g., [2]) to locales, and has been proved in [8]. $\square$

2.9. There are two reasons why we have presented the facts of 2.6 - 2.8 this explicitly although the essence is already published elsewhere. First, we want to stress the relation of 2.6.(2) and 2.8.(2) on the one side and the relation of 2.7.(2) and 2.8.(4) on the other side: We see that, in a sense, the regularity is an equally natural generalization of the metrizability as the complete regularity.

Second, the facts naturally introduce the question which we wish to deal with in the remainder of this section, namely:

What does $L = [L:\mathcal{A}]$ with a finite resp. one-element $\mathcal{A}$ say about L?

2.10. From now on untill the end of section 2,

$$\mathcal{A} = \{A_1, \ldots, A_n\}$$

is a finite system of covers of a frame L. Given a system B of elements of L, consider on it the equivalence relation generated by the relation

$$x\,R\,y \quad \text{iff} \quad x \wedge y \neq 0.$$

The equivalence classes will be called chain-components of B. We will denote

$$A = A_1 \wedge \ldots \wedge A_n\,, \text{ and}$$
$$\tilde{A} = \{\bigvee X \mid X \text{ is a chain component of } A\}.$$

Thus, $\tilde{A}$ is a disjoint cover of L.

2.11. Lemma: Let x be in $[L:\mathcal{A}]$. Then

(1) for each $a \in A$ we have either $a \leqslant x$ or $a \wedge x = 0$,

(2) for each $a \in \tilde{A}$ we have either $a \leqslant x$ or $a \wedge x = 0$.

Proof: (1) We have $x = \bigvee\{y \mid \exists i \mid A_i y \leqslant x\}$. Let $a = a_1 \wedge \ldots \wedge a_n$, $a_i \in A_i$, and let $a \wedge x \neq 0$. Hence

$$\bigvee\{a \wedge y \mid \exists i,\ A_i y \leqslant x\} \neq 0.$$

Thus there is an i and a y such that $a \wedge y \neq 0$ and $A_i y \leqslant x$. Then, however, $a_i \wedge y \neq 0$ and hence $a \leqslant a_i \leqslant x$.

(2) Let $a \wedge x \neq 0$ for some $a \in \tilde{A}$. Choose $u \in A$, $u \leqslant a$, such that $u \wedge x \neq 0$. Let v be an arbitrary non-zero element of A such that $v \leqslant a$.

We have a sequence

$$u = u_0, u_1, \dots, u_k = v \ ; \ u_i \in A, \ u_i \wedge u_{i+1} \neq 0.$$

By (1), $u_0 \leq x$. If $u_i \leq x$ we have $u_{i+1} \wedge x \neq 0$ and hence $u_{i+1} \leq x$ by (1) again. Thus, $v \leq x$ and hence finally $a \leq x$. $\square$

2.12. Corollary: For each $x \in [L:\mathcal{A}]$,

$$x = \bigvee \{a \mid a \in \tilde{A}, \ a \leq x\}$$

Consequently,

$$[L:\mathcal{A}] \subseteq [L:\{\tilde{A}\}]. \ \square$$

2.13. Lemma: Let B be a disjoint cover of a frame L. Then

$$[L:\{B\}] = \{\bigvee X \mid X \subseteq B\}$$

and consequently it is isomorphic to $\exp(B \setminus \{0\})$.

Proof: By 2.12 each $x \in [L:\{B\}]$ is of the form $\bigvee X$, $X \subseteq B$. On the other hand, let $X \subseteq B$ be arbitrary. For any $b \in B$ we have $Bb = b$ so that $\bigvee X$ is in $[L:\{B\}]$. We have the mutually inverse isomorphisms

$$\exp(B \setminus \{0\}) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\bigvee X \mid X \subseteq B\}$$

given by $\varphi(X) = \bigvee X$, $\psi(u) = \{b \in B \mid 0 \neq b \leq u\}$. $\square$

2.14. Proposition: Let A be an arbitrary cover of L. Then $[L:\{A\}] = [L:\{\tilde{A}\}]$ and hence $[L:\{A\}]$ is an atomic Boolean algebra.

Proof: By 2.12 (applied for n = 1) we have

$$[L:\{A\}] \subseteq [L:\{\tilde{A}\}].$$

On the other hand, $A \dashv \tilde{A}$, hence $\{\tilde{A}\}$ maj$\{A\}$ and hence $[L:\{\tilde{A}\}] \subseteq [L:\{A\}]$ by 1.11. Finally, $[L:\{\tilde{A}\}]$ is atomic boolean by 2.13. $\square$

2.15. Theorem: The following statements are equivalent:

(1) L is an atomic Boolean algebra,

(2) $L = [L:\mathcal{A}]$ for a finite $\mathcal{A}$,

(3) $L = [L:\{A\}]$ for a cover A,

(4) $L = [L:\{A\}]$ for a disjoint cover A.

Proof: (1) $\Rightarrow$ (4): It suffices to take the cover consisting of the atoms.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1): By 2.12, $L = [L:\mathcal{A}] \subseteq [L:\{\tilde{A}\}] \subseteq L$, hence $L = [L:\{\tilde{A}\}]$ and this is an atomic Boolean algebra by 2.14. $\square$

2.16. Note: We are here concerned with the condition $L = [L:\mathcal{A}]$ rather then with the form of the general $[L:\mathcal{A}]$. About that we have got just the statements 1.11.(1) and 2.14. In fact, these are, in essence, the only general characteristics one can present. As we will prove elsewhere, any complete lattice C can be represented as $[L:\mathcal{A}]$; moreover, one can choose a representation with atomic boolean L, and in the case of finite C even with a two-element $\mathcal{A}$.

3. The subframes $[L:\mathcal{A}]$ and injectively coreflective subcategories of RegFRM

3.1. We will say that a subcategory $\mathcal{C}_1$ of a concrete category $\mathcal{C}$ is injectively coreflective if the coreflection transformation $\gamma_C: F(C) \longrightarrow C$ (where F is the coreflection functor $\mathcal{C} \rightarrow \mathcal{C}_1$) consist of injective morphisms γ_C.

3.2. A system $(R_L)_{L \in USL_1}$ of binary relations on upper semilattices L is said to be admissible if

(i) $xR_Ly \Rightarrow x \leq y$ in L,

(ii) for every morphism $f:L \rightarrow K$

$$xR_Ly \Rightarrow f(x)R_Kf(y).$$

A system $(R_L)_{L \in FRM}$ of binary relations on frames L is said to be strongly admissible if (i),(ii) and, moreover,

(iii) $x_iR_Ly_i \ (i=1,2) \Rightarrow x_1 \wedge x_2 R_L y_1 \wedge y_2$.

3.3. Examples: (1) By 2.2., 1.8 and 1.16,

$$(\lhd_L)_L \quad \text{(recall 2.1)}$$

is a strongly admissible system.

(2) More generally, let us have a correspondence $L \mapsto L'$ associating with frames L subframes L' in such a way that for each morphism $f:L \longrightarrow K$ we have $f(L') \subseteq K'$. Then by 1.8, 1.9 and 1.16

$$(\overset{\mathcal{D}(L,L')}{\lhd})_L$$

is a strongly admissible system.

(3) By 2.5, 1.8 and 1.15,

$$(\lhd\!\lhd_L)_L$$

is a strongly admissible system.

3.4. Construction: Let $(R_L)_L$ be an admissible (resp. strongly admissible) system. Define a functor

$$F_1: USL_1 \longrightarrow USL_1 \quad (\text{resp. } F_1: FRM \longrightarrow FRM)$$

by putting

$$F_1(L) = \{x \mid x = \bigvee\{y \mid yR_Lx\}\}$$

$$\text{and} \quad F_1(f)(x) = f(x) \quad \text{for morphisms } f:L \rightarrow K$$

(it is easy to check that $F_1(L)$ is a sub-upper semilattice of L - resp. a subframe of L - and that $f(F_1(L)) \subseteq F_1(K)$ for morphisms

$f:L\to K$).

Further, define functors F_α for ordinals α as follows:

$F_0 = \mathrm{Id}$, $F_{\alpha+1} = F_1 \circ F_\alpha$, $F_\alpha(L) = \bigcap_{\beta<\alpha} F_\beta(L)$ for limit α ;

(of course) $F_\alpha(f)(x) = f(x)$ for all α .

Finally put

$$F(L) = \bigcap_{\alpha\in \mathrm{Ord}} F_\alpha(L) \ , \quad F(f)(x) = f(x).$$

3.5. Theorem: For each admissible (resp. strongly admissible) system $(R_L)_L$ on USL_1 (resp. on FRM) the full subcategory generated by all the L such that

$$(*)\quad \forall x\in L\ ,\ x = \bigvee\{y \mid yR_Lx\}$$

is an injectively coreflective subcategory.

On the other hand, for each injectively cpreflective subcategory $\mathcal{C}$ of USL_1 (resp.FRM) there is an admissible (resp. strongly admissible) system $(R_L)_L$ such that the objects of $\mathcal{C}$ are characterized by the formula $(*)$.

Proof: I. Take a (strongly) admissible system $(R_L)_L$ and consider the functors from 3.4. We immediately see that

L satisfies $(*)$ iff $L = F_1(L)$ iff $L = F(L)$.

Thus, writing γ_L for the inclusion $F(L) \subseteq L$ we see that for L general and K satisfying $(*)$ and$f:K\to L$ a morphism, $\gamma_L \circ F(f) = f$.

II. Now let $\mathcal{C}$ be an injectively coreflective subcategory, $F: USL_1 \to \mathcal{C}$ (resp. $F:FRM \to \mathcal{C}$) the coreflection functor. Without loss of generality we can assume that $F(L) \subseteq L$ for all L and that always $F(f)(x) = f(x)$. Define

$$xR_Ly \quad \text{iff} \quad \exists z\in F(L),\ x\leq z\leq y.$$

If $f:L\to K$ is a morphism and xR_Ly we have $f(x)\leq f(z)\leq f(y)$ with $z\in F(L)$ and hence $f(z)\in F(K)$. In the case of frames, if $x_iR_Ly_i$ ($i=1,2$) we have $z_i\in F(L)$ with $x_i\leq z_i\leq y_i$, hence $x_1\wedge x_2 \leq z_1\wedge z_2 \leq y_1\wedge y_2$ and since $F(L)$ is a subframe, $x_1\wedge x_2 R\, y_1\wedge y_2$. Now if $L=F(L)$ we have, for each $x\in L$, xR_Lx and hence $x = \bigvee\{y \mid yR_Lx\}$. On the other hand if $\forall x\in L\ x = \bigvee\{y\mid yR_Lx\}$ consider for yR_Lx a $z(y)\in F(L)$ such that $y\leq z(y)\leq x$. Thus, for each $x\in L$, $x = \bigvee\{z(y)\mid yR_Lx\}\in F(L)$. □

3.6. Remark: Thus (recall 3.3 and 2.1), the category

RegFRM

of regular frames is an injectively coreflective subcategory of FRM. Similarly for the subcategory of completely regular frames.

3.7. Lemma: Let K be a subframe of L and let $\mathcal{A}$ be a system of covers of K. Then $[L:\mathcal{A}] \subseteq K$.

Proof: Let x be in $[L:\mathcal{A}]$. We have $x = \bigvee\{y \mid y \overset{\mathcal{A}}{\lhd} x\} = \bigvee\{y \mid \exists A \in \mathcal{A},\ Ay \leq x\}$. Since, however, $y \leq Ay$ and $Ay \in K$, we obtain $x = \bigvee\{Ay \mid A \in \mathcal{A},\ y \in L \text{ and } Ay \leq x\} \in K$. □

3.8. Proposition: Let K be a regular subframe of L. Then

$$[L:\mathfrak{D}(L,K)] = K.$$

Proof: By 3.7, $[L:\mathfrak{D}(L,K)] \subseteq K$. On the other hand, let $x \in K$. Since K is regular, we have

$$x = \bigvee\{y \mid y \lhd_K x\}.$$

But if $y \lhd_K x$, we have $y \overset{\mathfrak{D}(L,K)}{\lhd} x$ and hence $x \in [L:\mathfrak{D}(L,K)]$. □

3.9. Corollary: Let $\mathcal{C}$ be an injectively coreflective subcategory of FRM such that $\mathcal{C} \subseteq$ RegFRM. Then there are $\mathcal{A}_L \subseteq \mathfrak{D}(L,L)$ such that we have $f(\mathcal{A}_L) \subseteq \mathcal{A}_L$ for each morphism $f: L \to K$ and that the coreflection is given by the correspondence

$$L \longmapsto [L:\mathcal{A}].$$

Proof: By 3.8 it suffices to take $\mathcal{A}_L = \mathfrak{D}(L,F(L))$, where F is the coreflection functor. □

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