WSAA 14

Henryk Hudzik
Musielak-Orlicz algebras

In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [335]--338.

Persistent URL: http://dml.cz/dmlcz/701905

Terms of use:

© Circolo Matematico di Palermo, 1987

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

MUSIELAK - ORLICZ ALGEBRAS

Henryk Hudzik

There are characterized Musielak-Orlicz spaces which are Banach algebras under pointwise multiplication of functions. It is an extension of the results of [1]. Let (T,Σ,μ) be a space of positive σ-finite measure and let $\Phi: T\times\mathbb{R} \longrightarrow [0,+\infty]$ be a Musielak-Orlicz function, i.e. $\Phi(t,\cdot)$ is convex, even, vanishing and continuous at 0 and not identically equal to 0 for μ-a.a. $t\in T$ and $\Phi(\cdot,u)$ is a Σ-measurable function for any $u\geq 0$. Let L^Φ be the corresponding Musielak-Orlicz space, i.e. L^Φ consists of all equivalence classes of Σ-measurable functions $f: T\longrightarrow\mathbb{R}$ for which there exists $\lambda>0$ such that $M_\Phi(\lambda f)=\int_T \Phi(t,\lambda f(t))d\mu<+\infty$. With respect to the Luxemburg norm $\|\ \|_\Phi$, defined by

$$\|f\|_\Phi=\inf\{\lambda>0:\ M_\Phi(\lambda^{-1}f)\leq 1\},$$

L^Φ is a Banach function space with the Fatou property (see [3-6]).

Henceforth, T_a and $\mathbb{N}$ denote the non-atomic and purely atomic part of T, respectively, i.e. the atoms will be identified with positive integers. For $n\in\mathbb{N}$ we write $\Phi_n(\cdot)$ instead of $\Phi(n,\cdot)$. L^∞ denotes the space of μ-essentially bounded functions on T with the norm defined by $\|f\|_\infty=\operatorname{ess\,sup}_{t\in T}|f(t)|$ for any $f\in L^\infty$.

LEMMA. $L^\Phi\subset L^\infty$ if and only if there exists $\alpha\in(0,+\infty)$ such that

(i) $\Phi(t,\alpha)=+\infty$ for μ-a.a. $t\in T_a$, <u>and</u>

(ii) $\Phi_n(\alpha)\mu(\{n\})\geq 1$ for all $n\in\mathbb{N}$.

Moreover, the inequality $\|f\|_\infty\leq\alpha\|f\|_\Phi$ holds for any $f\in L^\Phi$ when conditions (i) and (ii) are fulfilled.

Proof. Sufficiency. Assume that conditions (i) and (ii) hold. If $f\in L^\Phi$, then $M_\Phi(f/r\|f\|_\Phi)\leq 1$ and so $|f(t)|/r\|f\|_\Phi\leq\alpha$ for any $r>1$ and for μ-a.a. $t\in T$. Hence it follows that $\|f\|_\infty\leq\alpha\|f\|_\Phi$.

Necessity. Assume that $L^\Phi\subset L^\infty$. Then $L^\Phi(T_a)\subset L^\infty(T_a)$, i.e. $L^\Phi(T_a)\subset L^{\Phi_0}(T_a)$, where Φ_0 is the Orlicz function defined by $\Phi_0(u)$

This paper is in final form and no version of it will be submitted for publication elsewhere

$=0$ for $0 \le u \le 1$ and $\Phi_0(u) = +\infty$ for $u > 1$. However, this inclusion is possible if there exist $k > 0$, a set T_0 of measure 0 and a non-negative μ-summable over T_a function h such that

$$\Phi_0(u) \le \Phi(t, ku) + h(t)$$

for all $t \in T_a \setminus T_0$ and $u \ge 0$ (see [5]). Hence it follows that $\Phi(t, 2k) = +\infty$ for μ-a.a. $t \in T_a$. So, condition (i) holds with $\alpha = 2k$.

Now, we shall prove the necessity of condition (ii). Assume that this condition does not hold. There is a sequence (n_k) of positive integers such that $\Phi_{n_k}(2^k)\,\mu(\{n_k\}) \le 2^{-k}$ for $k = 1, 2, \ldots$. Defining $f = \sum_{k=1}^{\infty} 2^k e_{n_k}$, where e_{n_k} is the n_kth basic sequence in l^1, we have

$$M_\Phi(f) = \sum_{k=1}^{\infty} \Phi_{n_k}(2^k)\mu(\{n_k\}) \le \sum_{k=1}^{\infty} 2^{-k} = 1,$$

i.e. $f \in L^\Phi \setminus L^\infty$. This ends the proof.

DEFINITION. A Banach function space $(X, \|\cdot\|)$ is called a Banach quasi-algebra (algebra) if it is an algebra, i.e. $f \cdot g \in X$ whenever $f, g \in X$ and if there is $K > 0$ such that $\|f \cdot g\| \le K \|f\| \|g\|$ ($\|f \cdot g\| \le \|f\| \|g\|$) for all $f, g \in X$.

THEOREM 1. The following conditions are equivalent:

(i) L^Φ is an algebra :

(ii) $L^\Phi \subset L^\infty$:

(iii) L^Φ is a Banach quasi-algebra:

(iv) There is $\alpha \in (0, +\infty)$ such that:

(a) $\Phi(t, \alpha) = +\infty$ for μ-a.a. $t \in T_a$, <u>and</u>

(b) $\Phi_n(\alpha)\,\mu(\{n\}) \ge 1$ for all $n \in \mathbb{N}$.

Proof. (i)$\Longrightarrow$(ii). If L^Φ is an algebra, then for any $f \in L^\Phi$ we may define on L^Φ the operator π_f by $\pi_f g = f \cdot g$ for any $g \in L^\Phi$. It is obvious that π_f is an orthomorphism (see [7] and [8]), i.e. $\inf(|\pi_f g|, |h|) = 0$ whenever $\inf(|g|, |h|) = 0$ in L^Φ (obviously L^Φ is a lattice under the natural order relation $f \le g$ if and only if $f(t) \le g(t)$ for μ-a.a. $t \in T$). However, it follows by [7], Th.8 that $f \in L^\infty$. So, $L^\Phi \subset L^\infty$. The implication (ii)$\Longrightarrow$(iii) follows by LEMMA . Indeed, if $f, g \in L^\Phi$, then for any $r > 1$,

$$M_\Phi\left(\frac{f \cdot g}{r\alpha \|f\|_\Phi \|g\|_\Phi}\right) \le M_\Phi\left(\frac{f \cdot g}{r \|f\|_\Phi \|g\|_\infty}\right) \le M_\Phi\left(\frac{f}{r \|f\|_\Phi}\right) \le 1,$$

i.e. $\|f \cdot g\|_\Phi \le \alpha \|f\|_\Phi \|g\|_\Phi$. The implication (iii)$\Longrightarrow$(i) is obvious and the equivalence (ii)$\Longleftrightarrow$(iv) follows by LEMMA . The proof is finished.

THEOREM 2. L^{Φ} is a Banach algebra if and only if there exists $\alpha \in (0,1]$ such that:

(i) $\Phi(t,\alpha) = +\infty$ for μ-a.a. $t \in T_a$, <u>and</u>

(ii) $\Phi_n(\alpha)\mu(\{n\}) \geq 1$ for any $n \in \mathbb{N}$.

Proof. The sufficiency follows by LEMMA .. Now, we shall prove the necessity. Denote $\alpha = \sup[u \geq 0: \Phi(t,u) < +\infty$ for μ-a.a. $t \in T]$. Assuming that $\alpha > 1$ and defining $\beta = \alpha^{2/3}$, we have $\Phi(t,\beta) < +\infty$ for μ-a.a. $t \in T$. Assume that $\mu(T_a) > 0$ and C is a subset of T_a of positive and finite measure. Define

$$C_n = \{t \in C: \Phi(t,\beta) \leq n\},\ n = 1,2,\ldots$$

This sequence is ascending and $\mu(\bigcup_n C_n) = \mu(C)$. So, there is an index k such that $0 < \mu(C_k) < +\infty$. Defining $f = \beta\chi_{C_k}$, we have

$$M_{\Phi}(f) = \int_{C_k} \Phi(t,\beta)dt \leq k\,\mu(C_k) < +\infty.$$

There exists a set $D \subset C_k$, $D \in \Sigma$, such that $M_{\Phi}(f\chi_D) \leq 1$. However,

$$M_{\Phi}((f\chi_D)^2) = \int_D \Phi(t,\alpha^{4/3})d\mu = +\infty.$$

Hence it follows that $\|f\chi_D\|_{\Phi} \leq 1$ and $\|(f\chi_D)^2\|_{\Phi} > 1$, i.e. L^{Φ} is not a Banach algebra.

For the proof of necessity of condition (ii), assume that L^{Φ} is a Banach algebra. Every element $f = \chi_{\{n\}}$ belongs to L^{Φ} $(n=1,2,\ldots)$. We have $\|f\|_{\Phi} \leq \|f\|_{\Phi}^2$, i.e. $\|f\|_{\Phi} \geq 1$ and so $M_{\Phi}(f) = \Phi_n(1)\mu(\{n\}) \geq 1$ for $n = 1,2,\ldots$. The proof is finished.

COROLLARIES: (i). Let μ be a purely atomic measure and $\Phi = (\Phi_n)$, where $\Phi_n(u) = u^{p_n}$, where $1 \leq p_n < +\infty$ for any $|u| \geq 0$ and $n \in \mathbb{N}$. Then L^{Φ} is a Banach quasi-algebra if $\inf_n \mu(\{n\}) > 0$. L^{Φ} is a Banach algebra if and only if $\inf_n \mu(\{n\}) \geq 1$.

(ii). We may define for any Musielak-Orlicz function a subspace E^{Φ} of L^{Φ} by

$$E^{\Phi} = \{f \in L^{\Phi}: M_{\Phi}(\lambda f) < +\infty \text{ for any } \lambda > 0\}.$$

It is clear that the condition $\Phi(t,\alpha) = +\infty$ for μ-a.a. $t \in T$, where $0 < \alpha < +\infty$, implies that $E^{\Phi} = \{0\}$. So, in the case of a non-atomic measure, no non-trivial space E is an algebra under pointwise multiplication of functions.

(iii). It is well known that any Banach quasi-algebra can be renormed to be a Banach algebra (see [9]). For Musielak-Orlicz spaces the following is true: if a Musielak-Orlicz space L^{Φ} is a Banach quasi-algebra, then there is a Musielak-Orlicz function Φ_1 equivalent to Φ (i.e. $L^{\Phi} = L^{\Phi_1}$) such that L^{Φ} equipped with the norm $\|\ \|_{\Phi_1}$ is a Banach algebra. For this purpose it suffices to put $\Phi_1(t,u) = \Phi(t,\alpha u)$, where α is a positive constant satisfying conditions (i) and in THEOREM 1. It is evident that $\|\ \|_{\Phi_1} = \alpha \|\ \|_{\Phi}$. So, $\|f\cdot g\|_{\Phi_1} = \alpha\|f\cdot g\|_{\Phi} \leq \alpha^2\|f\|_{\Phi}\|g\|_{\Phi} = \|f\|_{\Phi_1}\|g\|_{\Phi_1}$ for all $f,g \in L^{\Phi_1}$.

REMARKS. It is obvious that L^{Φ} is an algebra if and only if $f^2 \in L^{\Phi}$ whenever $f \in L^{\Phi}$. It is equivalent to $L^{\Phi} \subset L^{\Psi}$ which it is equivalent to $\Psi \dashv \Phi$, where $\Psi(t,u) = \Phi(t,u^2)$ for all $u \geq 0$ and μ-a.a. $t \in T$. The relation $\Psi \dashv \Phi$ is characterized for example in [5]. In the case of a non-decreasing but non-convex Musielak-Orlicz function Φ it is possible that $\Psi \dashv \Phi$ also for a non-atomic measure. For example, the function $\Phi(u) = \log(1+|u|)$ satisfies the inequality $\Phi(u^2) \leq 3\,\Phi(u)$ for all $u \geq 0$. Orlicz algebras generated by non-convex Orlicz functions has considered N.J. Kalton in [2].

REFERENCES

[1] HUDZIK H. "Orlicz spaces of essentially bounded functions and Banach-Orlicz algebras", Arch. Math. 44 (1985),535-538.

[2] KALTON N.J. "Subalgebras of Orlicz spaces and related algebras of analytic functions",Ark. Math. 18 (1980), 223-254.

[3] KRASNOSELSKII M.A. and RUTICKII Ya.B.,"Convex functions and Orlicz spaces", Gröningen 1961.

[4] LUXEMBURG W.A.J. "Banach function spaces", Thesis, Delft 1955.

[5] MUSIELAK J. "Orlicz spaces and modular spaces", Lecture Notes in Math. 1034 (1983).

[6] MUSIELAK J. and ORLICZ W. "On modular spaces", Studia Math. 18 (1959), 49-65.

[7] ZAANEN A.C. "Examples of orthomorphism", J. Appr. Theory 13 (1975), 192-204.

[8] ZAANEN A.C. "Riesz spaces II", Amsterdam-New York-Oxford (1983).

[9] ŻELAZKO W. "Banach algebras", Warszawa 1968 (in Polish).

INSTITUTE OF MATHEMATICS
A. MICKIEWICZ UNIVERSITY
POZNAŃ, POLAND