Matts Essén

Harmonic majorization and thinness

In: Zdeněk Frolík and Vladimír Souček and Marián J. Fabián (eds.): Proceedings of the 14th Winter School on Abstract Analysis. Circolo Matematico di Palermo, Palermo, 1987. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 14. pp. [295]--304.

Persistent URL: http://dml.cz/dmlcz/701902

HARMONIC MAJORIZATION AND THINNESS

Matts Essén

At the Srni conference on "Abstract Analysis" in January 1986, this survey was presented to a very general audience. In the talk, it was necessary to explain a number of basic concepts. These facts, including references to the literature, are here given in an appendix.

The theme of the survey is to describe how some results from the classical H^p-theory for functions analytic in the unit disc in the plane can be generalized to higher dimensions. At the end, we give analogues in $\underline{R}^d$, $d \geq 3$, of the Zygmund L log L-theorem and the M. Riesz theorem on conjugate functions.

In order to avoid technicalities, we have sometimes restricted ourselves to cases which are typical rather than general (as an example, let us mention that the main example in Problem I is $\Phi(t) = t^p$, $p > 0$, although many other choices are possible). A more complete treatment can be found in the references.

Let D be a domain in $\underline{R}^d$, $d \geq 2$, such that the complement of D has positive capacity.

Problem I. Let Φ: $[0,\infty) \to [0,\infty)$ be unbounded at infinity and such that $\Phi(|x|)$ is subharmonic in $\underline{R}^d$. When does $\Phi(|x|)$ have a harmonic majorant in D?

Here $|\cdot|$ denotes the Euclidean norm.

An important example is $\Phi(t) = t^p$, where $p > 0$ is given. If we take $d = 2$ and let D be the angle $D = \{z = re^{i\theta}: |\theta| < \alpha/2\}$, it is known that $|z|^p$ has a harmonic majorant in D if and only if $\alpha < \pi/p$. In the case when such majorants exist, the least one is $\operatorname{Re}(z^p/\cos(p\alpha/2)) = r^p \cos p\theta/\cos(p\alpha/2)$. If p is given and if the domain D is too big, there will not exist any harmonic majorant of $|z|^p$ in D. There are similar conclusions of this type for more general versions of Problem I.

This paper is in final form and no version of it will be submitted for publication elsewhere.

As a second example, let us take $U = \{z \in \underline{C}: |z| < 1\}$ and consider a domain D such that $|w|^p$ has a harmonic majorant $h(w)$ in D. Let $F: U \to D$ be an analytic function. Taking $w = F(z)$, we integrate the inequality

$$|F(re^{i\theta})|^p \leq h(F(re^{i\theta}))$$

and obtain

$$\|F\|^p_{H^p} = \sup_{r<1} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta/(2\pi) \leq h(F(0)) < \infty .$$

We use that $h \circ F$ is harmonic in U. This means that $F \in H^p(U)$, a class which has been extensively studied (cf. [4]). This well-known argument proves the first half of

Theorem 1. a) *If* $F: U \to D$ *is analytic and* $|w|^p$ *has a harmonic majorant in* D, *then* $F \in H^p(U)$.
b) *If every analytic function* $F: U \to D$ *is in* $H^p(U)$, *then* $|w|^p$ *has a harmonic majorant in* D.

The proof of the second half of Theorem 1 will be given later.

In higher dimension, we lose the analytic functions mapping the unit ball into D. However, it is still possible to study Problem I in D.

If D is a bounded domain, we can solve the Dirichlet problem in D with boundary values $\Phi(|x|)$: there exists a function h harmonic in D with values $\Phi(|x|)$ on ∂D (with the possible exception of a polar subset of ∂D). According to the maximum principle, we shall have $\Phi(|x|) \leq h(x)$, $x \in D$.

For unbounded domains, the situation is more complicated. A natural candidate for a harmonic majorant would be a harmonic function in D with boundary values $\Phi(|x|)$. Assume that there exists such a function H which is the limit of an increasing sequence of harmonic functions which are bounded in D: let us say that H is a formal majorant of $\Phi(|x|)$ (cf. Ch. 3 in [1]). The following example will show that a formal majorant is not necessarily a majorant of $\Phi(|x|)$ in D.

Let $D = \{x \in \underline{R}^d: |x| > 1\}$. The function $H(x) = 1$, $x \in D$, is a formal harmonic majorant. It is clear that we have $H(x) < |x|^p$ in D.

What is wrong in the example is that the complement CD is too small at infinity. We can prove the following result (cf. Theorems 1 and 3 in [9]).

Theorem 2. *The following statements are equivalent.*
a) $\Phi(|x|)$ *has a harmonic majorant in* D.
b) *There exists a formal majorant* H *and* CD *is not thin at infinity*. (*In the case* $d = 2$, *we have to assume that* $\lim \Phi(t)/(\log t) = \infty$, $t \to \infty$.)

An equivalent way of describing non-thinness of CD at infinity is to say that infinity must be a regular point for the Dirichlet problem.

To connect our problem with the geometry of D, we need the concept of harmonic measure. Let $R > 0$ be given and let ω_R be the harmonic function in that component of $D_R = \{x \in D: |x| < R\}$ which contains the origin (which we assume to be in D) which has boundary values 1 on $\{|x| = R\} \cap \overline{D}$ and 0 on $\{|x| < R\} \cap \partial D$: ω_R is called the harmonic measure of this part of ∂D_R with respect to D_R.

<u>Conjecture</u>. <u>Problem I has an affirmative answer if and only if</u>

$$\int^{\infty} \Phi'(t)\omega_t(0)dt < \infty . \tag{1}$$

The conjecture is known to be true for a large class of functions Φ which includes $\Phi(t) = t^p$ (cf. Theorems 2 and 4 in [9]; also cf. Theorems 2.2. 3.1, 3.5. and 3.6 in [3]). Also cases when the conjecture does not hold are well understood but it would take us too far to go into the details here. As an example, we mention that if $\Phi_0(t) = (\log t)(\log\log t)$ for t large, then it is known that $\Phi_0(|x|)$ has a harmonic majorant in $D \subset \underline{R}^2$ if and only if

$$\int^{\infty} \omega_t(0)dt/t < \infty , \tag{2}$$

(cf. Theorem 3 in [6]): we note that $\Phi_0'(t) \approx t^{-1}\log\log t$, t large.

The point of conditions of type (1) is that it is often possible to estimate $\omega_t(0)$ in terms of the geometry of D. We have a general criterion which can be used to solve Problem I. For an example of such an estimate, we refer to [8].

To make the picture complete, we have to define the formal harmonic majorant. Let v_t be the harmonic function in D which is 1 on $\{|x| > R\} \cap \partial D$ and 0 on $\{|x| \leq R\} \cap \partial D$. Also when D is unbounded, we can construct such a function. We consider

$$H(x) = \int_0^{\infty} \Phi(t)d(-v_t(x)) . \tag{3}$$

If $H(0)$ is defined, we can use Harnack's principle to deduce that $H(x)$ will be finite for all $x \in D$. Furthermore, H will be harmonic in D with boundary values $\Phi(|x|)$. H is the formal harmonic majorant in Theorem 2. If CD is not thin at infinity, H will be a harmonic majorant of $\Phi(|x|)$.

We can now give an outline of the proof of the second half of Theorem 1. Let $F: U \to D$ be a universal covering map such that $F(0) = 0$ (cf. the Appendix). The nontangential maximal function NF is defined by

$$NF(e^{i\theta}) = \sup |F(z)| , \quad z \in S(\theta) ,$$

where $S(\theta) = \{z: |z| < \sqrt{3}/2\} \cup \{z: |z| \geq \sqrt{3}/2, \ |\arg (e^{i\theta} - z)| < \pi/3\}$. In Section 6 in [9], we find the following estimate:

$$\omega_R(0) \leq (2/\pi)\, m\{\theta: NF(e^{i\theta}) \geq R\} .$$

Here $m(\cdot)$ denotes Lebesgue measure on the unit circle. It follows that

$$\int_0^\infty \omega_R(0)\, d R^p \le 4\|NF\|_p^p \le \text{Const.}\ \|F\|_p^p < \infty .$$

In the last step, we use a result of Hardy and Littlewood (cf. Theorem IV.40 p. 186 in [15]).

This means that (1) holds with $\Phi(t) = t^p$. We conclude that $|w|^p$ has a harmonic majorant in D. The second half of Theorem 1 is proved.

At this point, let us stop for a moment and consider the history of the subject. A beginning can be found in Chapter 3 of the thesis of A. Beurling [1]. In the plane, he defines harmonic measures $\alpha(\cdot,t)$ and $\beta(\cdot,t)$ which are our functions $1 - v_t$ and ω_t. Beurling uses Stieltje's integrals of the type given above in (3) to find "formal" solutions of problems on harmonic majorization. For details, the reader is referred to [1].

In [3], D. Burkholder gave results on harmonic majorization which were stated and proved in the language of probability (stopping-times, Brownian motion). Let X be a Brownian motion in $\underline{R}^d$ starting at $x \in D$ and let τ be the first time X leaves D:

$$\tau(\omega) = \inf\{t > 0: X_t(\omega) \notin D\} .$$

As an example of the results of [3], we mention that $\tau^{1/2} \in L^p$ (where $0 < p < \infty$) if and only if $|x|^p$ has a harmonic majorant in D: this is one of the basic problems in this survey.

If a problem on harmonic majorization can be solved using probability, there is also a solution which uses classical analysis. In [9], the questions which Burkholder asked in [3] are studied from the point of view of analysis. The reader is recommended to compare these two papers.

It is well known that $\omega_R(0)$ can be interpreted as the probability that a Brownian motion starting at $0 \in D$ will hit the sphere $\{|x| = R\}$ before it hits ∂D. There is a connection between the behaviour of $\omega_R(0)$ as $R \to \infty$ and the size of CD at infinity. We can give a precise formulation of this fact if we assume that CD is not thin at infinity. (Very often, different proofs are required in [9] in the cases $d = 2$ and $d \ge 3$. This dichotomy is present in the following lemma (cf. the proofs of Lemma 4 and Lemma 6 in [9]).)

<u>Lemma 1. A necessary and sufficient condition for</u> CD <u>not to be thin at infinity is that</u>

a) $\omega_R(0) \to 0$, $R \to \infty$, <u>when</u> $d \ge 3$,

b) $(\log R)\omega_R(0) \to 0$, $R \to \infty$, <u>when</u> $d = 2$.

We use this result to prove a simple Phragmén-Lindelöf theorem (cf. Lemma 1 in [6]). Theorem 3 should be compared to Theorem 5.16 in [11].

Theorem 3. a) Let u be subharmonic in $D \subset \underline{R}^d$, $d \geq 3$, and assume that CD is not thin at infinity. If u is bounded above in D and if $u \leq 0$ on ∂D, then we have $u \leq 0$ in D.

b) Let u be subharmonic in $D \subset \underline{R}^2$ and assume that CD is not thin at infinity. If

$$\sup u(z)/(1 + \log^+|z|) < \infty , \quad z \in D ,$$

and if $u \leq 0$ on ∂D, then we have $u \leq 0$ in D.

To prove Theorem 3 in the case $d \geq 3$, let us assume that M is an upper bound for u in D. Using the maximum principle in the bounded set $D \cap \{|x| < R\}$ (cf. the Appendix), we see that $u(x) \leq M\,\omega_R(x)$, $x \in D \cap \{|x| < R\}$. From Lemma 1 and Harnack's principle, we deduce that for all $x \in D$, we have $\omega_R(x) \to 0$, $R \to \infty$. It follows that $u(x)$ is non-positive in D, and the first part of Theorem 3 is proved. The second part of the proof is similar.

In the remaining part of the paper, we shall assume that the domain D is contained in the half-space $\{x \in \underline{R}^d : x_1 > 0\}$ where $d \geq 2$. E will be the complement of D with respect to the half-space, i.e., $E = \{x_1 > 0\} \setminus D$.

Problem II. Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing convex function with $\Phi(0) = 0$ such that $\Phi(t)/t \to \infty$, $t \to \infty$. When does $\Phi(x_1)$ have a harmonic majorant in D?

Minimal thinness at infinity (cf. the Appendix) plays the same role for this problem as the one that thinness played for Problem I. We have the following analogue of Theorem 2 (cf. Theorem 1 in [6]).

Theorem 4. The following statements are equivalent.

a) $\Phi(x_1)$ has a harmonic majorant in D.

b) There exists a formal harmonic majorant in D and E is not minimally thin at infinity.

In this context, the harmonic measures are defined as follows. Let $x_0 \in D$ be given. Let $t > 0$ be given and let ω_t be the harmonic function in that component of $D \cap \{0 < x_1 < t\}$ containing x_0 which has boundary values 1 on $\overline{D} \cap \{x_1 = t\}$ and 0 on $\partial D \cap \{0 < x_1 < t\}$. We let v_t be the harmonic function in D which has boundary values 1 on $\partial D \cap \{x_1 > t\}$ and 0 on $\partial D \cap \{x_1 \leq t\}$. The formal harmonic majorant of $\Phi(x_1)$ in D is given by formula (3). A connection between the behavior of $\omega_t(x_0)$ as $t \to \infty$ and the size of E at infinity is given by

Lemma 2. A necessary and sufficient condition for E not to be minimally thin at infinity is that $t\,\omega_t(x_0) \to 0$, $t \to \infty$.

(Cf. Lemma 2 in [6].)

Also here, there is a Phragmén-Lindelöf theorem.

Theorem 5. *Let $d \geq 2$. Let u be subharmonic in $D \subset \{x \in \underline{R}^d : x_1 > 0\}$ and assume that E is not minimally thin at infinity. If*

$$\sup u(x)/(1+x_1) < \infty , \quad x \in D ,$$

and if $u \leq 0$ on ∂D, then we have $u \leq 0$ in D.

In the proof, we apply Lemma 2 in the same way as when we used Lemma 1 to prove Theorem 3. We omit the details.

We would like to give an analogue for Problem II of the conjecture (which is almost a theorem) mentioned in the discussion of Problem I. If Φ is the convex function in Problem II, we introduce

$$\Lambda(t) = \int_0^t s\,d(\Phi(s)/s) , \quad t > 0 .$$

We say that Φ satisfies a doubling condition if there exists a constant C such that $\Phi(2t) \leq C\Phi(t)$ for all sufficiently large t.

Theorem 6. *Assume that Φ and Λ satisfy doubling conditions and that we have $\limsup_{t\to\infty} \Lambda(t)/t > 0$. Then $\Phi(x_1)$ has a harmonic majorant in D if and only if*

$$(4) \qquad \int_0^\infty \Lambda'(t)\omega_t(x_0)dt < \infty .$$

(Cf. Theorem 2a in [6].)

Remark 1. If $\Phi(t) = t \log^+ t$, we have $\Lambda(t) = (t-1)^+$ and Theorem 6 says that $x_1 \log^+ x_1$ has a harmonic majorant in D if and only if

$$(5) \qquad \int_0^\infty \omega_t(x_0)dt < \infty .$$

Remark 2. The proof that (2) is a necessary and sufficient condition for $(\log^+|z|)(\log^+\log^+|z|)$ to have a harmonic majorant in $D \subset \underline{R}^2$ has a structure which is similar to the proof of Theorem 6. This solved a question which was left open in [9]. For details, we refer to Theorem 3 in [6].

Remark 1 plays a crucial role in the proof of

Theorem 7. *Let $d \geq 2$. Then $|x|$ has a harmonic majorant in $D \subset \{x_1 > 0\}$ if and only if $x_1 \log^+ x_1$ has a harmonic majorant in D.*

(Cf. Theorem 4 in [6].)

In the case $d = 2$, there is a classical result of a similar type: The Zygmund L log L-theorem (cf. Section 4.3 in [4]). If F is analytic in the unit disc U, we say that $\mathrm{Re}\, F \in L \log L$ if

$$\sup_{0<r<1} \int_0^{2\pi} |\operatorname{Re} F(re^{i\theta})| \log^+ |\operatorname{Re} F(re^{i\theta})| d\theta < \infty .$$

Theorem A. *Let F be analytic in U.*

a) *If* $\operatorname{Re} F \in L \log L$, *then* $F \in H^1(U)$.

b) *If* $F \in H^1(U)$ *and* $\operatorname{Re} F > 0$ *in* U, *then* $\operatorname{Re} F \in L \log L$.

Remark. Another variation on the $L \log L$-theme in the plane can be found in [10].

To compare Theorem A and Theorem 7 in the case $d = 2$, we note that it follows from the discussion of Theorem 1 that

i) if $F: U \to D$ is analytic and if $|x|$ has a harmonic majorant in D, then $F \in H^1(U)$;

ii) if $F: U \to D$ is analytic and if $x_1 \log^+ x_1$ has a harmonic majorant in D, then $\operatorname{Re} F \in L \log L$;

iii) if every analytic function $F: U \to D$ is in $H^1(U)$, then $|x|$ has a harmonic majorant in D.

Theorem 7 tells us that if one of the two harmonic majorants exists and if $F(U) \subset D$, then we have both $F \in H^1(U)$ and $\operatorname{Re} F \in L \log L$. To go the other way, we have to assume that all analytic functions $F: U \to D$ are in $H^1(U)$: then it follows that $|x|$ has a harmonic majorant in D.

While Theorem A is a statement of properties of one analytic function F, Theorem 7 describes a property of the domain D.

In the case $1 < p < \infty$, we can prove (cf. [7])

Theorem 8. *Let $d \geq 2$ and let p be given, $1 < p < \infty$. Then $|x|^p$ has a harmonic majorant in $D \subset \{x_1 > 0\}$ if and only if x_1^p has a harmonic majorant in D.*

The associated classical result in the case $d = 2$ is due to M. Riesz.

Theorem B. *Let F be analytic in U and let p be given, $1 < p < \infty$. Then we have*

$$\|\operatorname{Im} F\|_p^p \leq C_p' \|\operatorname{Re} F\|_p^p , \tag{6}$$

$$\|F\|_{H^p}^p \leq C_p \|\operatorname{Re} F\|_p^p , \tag{7}$$

where

$$\|h\|_p^p = \sup_{r<1} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta/(2\pi) .$$

The classical form of the inequality is (6) (cf. Section 4.1 in [4]). The best constant C_p' was determined by Pichorides in 1972. The best constant C_p is also known (cf. [5]). The proof of Theorem 8 combines ideas from [5] and [6].

The relation between Theorems B and 8 is similar to that between Theorems A and 7. We omit the details.

As a final remark, we note that Problem II is a special case of a more general problem treated in [6]. If $\Omega \subset \underline{R}^d$ is a given domain, we let Ψ be a minimal harmonic function associated with a Martin boundary point of Ω (which in Problem II is ∞). Let $D \subset \Omega$ be another given domain. Let Φ be as in Problem II. The general problem is to find criteria for $\Phi \circ \Psi$ to have a harmonic majorant in D.

Appendix

For the convenience of the reader, we give here some basic facts on subharmonic functions and thinness. A more complete treatment of the area can e.g. be found in the two books [11] and [12].

A function u is harmonic in a domain $D \subset \underline{R}^d$ if $\Delta u = 0$ in D, where Δ is the classical Laplace operator. A function u is subharmonic in D if u is semicontinuous from above and we have $\Delta u \geq 0$ in D in the distributional sense. Other definitions are possible (cf. [11], [12]). u is superharmonic if $-u$ is subharmonic.

If $\Phi: [0,\infty) \to [0,\infty)$ is a given increasing function, $\Phi(|x|)$ will be subharmonic in $\underline{R}^d$ if

$$\Phi''(t) + (d-1)t^{-1}\Phi'(t) \geq 0, \quad t \geq 0 .$$

If we choose $\Phi(t) = t^p$, $p > 0$, we have the example $|x|^p$.

If u is subharmonic in D, we say that h is a harmonic majorant of u in D if h is harmonic in D and $u \leq h$ in D.

If D is a bounded domain, we can solve the Dirichlet problem in D with boundary values f on the boundary ∂D: there exists a function $u = H_f$ harmonic in D such that $u = f$ on ∂D.

I. If ∂D is smooth and f is continuous, this means that

$$u(y) \to f(x), \quad y \to x \in \partial D, \quad y \in D .$$

II. For more general domains and functions f, we can solve the Dirichlet problem by applying the Perron-Wiener-Brelot method (cf. Ch. 8 in [12] or 5.6 in [11].

III. A set E is thin at $x \in \underline{R}^d$ provided that the Wiener criterion holds (cf. Theorem 10.21 in [12]). This is a convenient way of characterizing sets which are small near a given point. The characterization is in terms of capacity.

IV. A point $x \in \partial D$ is a regular boundary point if for every $f \in C(\partial D)$, we have $\lim H_f(y) = f(x)$, $y \to x$, $y \in D$ (cf. Ch. 8, Section 3 in [12]). A point $x \in \partial D$ is regular if and only if the complement CD of D is not thin at x. The set of irregular boundary points is a set of capacity zero (cf. Theorems 10.9 and 10.12 in [12] or Ch. 5 in [11]).

V. For a bounded domain D, we have the maximum principle (cf. Theorem 5.16 in [11]). Let $F \subset \partial D$ have capacity zero. Let u be subharmonic and bounded above in D and assume that for all $x \in \partial D \setminus F$, we have $\limsup u(y) \leq 0$, $y \to x$, $y \in D$. Then it follows that $u(y) \leq 0$, $y \in D$.

Let us now assume that $D = \{x \in \underline{R}^d : x_1 > 0\}$. If E is a closed subset of D and if the function f is defined on E, the reduced function of f with respect to E in D is defined by

$$R_f^E(x) = \inf \{u(x) : u \geq f \text{ on } E,\ u \text{ is superharmonic in } D,\ u \geq 0 \text{ in } D\} .$$

According to the Perron-Wiener-Brelot method, R_f^E will be harmonic in $D \setminus E$ with boundary values f on ∂E and 0 on ∂D (let us assume that $\partial E \cap \partial D = \emptyset$). R_f^E is not necessarily superharmonic in D. However, the lower regularization

$$\hat{R}_f^E(x) = \liminf R_f^E(y) , \quad y \to x , \quad y \in D ,$$

will be superharmonic in D (cf. Ch. 7, Section 3 in [12]).

The set $E \subset D$ is said to be minimally thin at infinity if there exist points in $D \setminus E$ where $\hat{R}_{x_1}^E(x) < x_1$ (cf. Definition XII.4 in [2]).

If D is a domain in $\underline{R}^2$, there exists a universal covering surface $\hat{D}$ over D: if CD has positive capacity, we can choose $\hat{D}$ to be the unit disc U (cf. [14]). This means that there exists an analytic function F from U onto D (which is not necessarily simply connected) such that for each point $w \in D$, there is an open neighborhood $\underline{0}$ of w such that each connected component of $F^{-1}(\underline{0})$ is mapped homeomorphically by F onto $\underline{0}$.

Remark. For a general discussion of covering spaces, we refer to [13].

References

[1] BEURLING,A. "Êtudes sur un problème de majoration". Thèse, Uppsala 1933.

[2] BRELOT M. "On topologies and boundaries in potential theory", Springer Lecture Notes in Math. 175 (1971).

[3] BURKHOLDER D.L. "Exit times of Brownian motion, harmonic majorization, and Hardy spaces", Adv. in Math. 26 (1977), 182-205.

[4] DUREN P.L. "Theory of H^p spaces", Academic Press, London (1970).

[5] ESSÉN M. " A superharmonic proof of the M. Riesz conjugate function theorem", Ark. för mat. 22 (1984), 241-249.

[6] ESSÉN M. "Harmonic majorization, harmonic measure and minimal thinness", U.U.D.M. Report 1985:13.

[7] ESSÉN M. " A generalization of the M. Riesz theorem on conjugate functions to $\underline{R}^d$, $d \geq 2$", manuscript 1986.

[8] ESSÉN M. and HALISTE K. "A problem of Burkholder and the existence of harmonic majorants of $|x|^p$ in certain domains in $\underline{R}^d$", Ann. Acad. Sci. Fenn. 9 (1984), 107-116.

[9] ESSÉN M., HALISTE K., LEWIS J.L. and SHEA D.F. "Harmonic majorization and classical analysis", J. London Math. Soc. (1986), to appear.

[10] ESSÉN M., SHEA D.F. and STANTON C.S. "A value-distribution criterion for the class $L \log L$, and some related questions", Ann. de l'Inst. de Fourier 35, 4(1985), 127-150.

[11] HAYMAN W.K. and KENNEDY P.B. "Subharmonic functions", Academic Press (1976).

[12] HELMS L.L. "Introduction to potential theory", Wiley-Interscience (1969).

[13] SINGER I.M. and THORPE J.A. "Lecture Notes on elementary topology and geometry", Scott, Foresman and Comp. (1967).

[14] SPRINGER G. "Introduction to Riemann surfaces" Addison-Wesley (1957).

[15] TSUJI M. "Potential theory in modern function theory", Maruzen, Tokyo (1959).

Department of Mathematics
University of Uppsala
Thunbergsvägen 3
S-752 38 UPPSALA, Sweden