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TOTALLY REAL SUBMANIFOLDS OF $S^6(1)$ WITH PARALLEL SECOND FUNDAMENTAL FORM

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Introduction. Let $S^6(1)$ be the unit six-dimensional Euclidean sphere. The aim of this note is to prove the following result.

Theorem. Let M be a totally real submanifold of $S^6(1)$ with parallel second fundamental form. If dim M = 3, then M is totally geodesic. If dim M = 2 and M is minimal, then M is totally geodesic or locally flat.

Minimal submanifolds of spheres with parallel second fundamental form were studied, for instance, in [3] and [4] .

Preliminaries. By using the cross-product in R^7 obtained as a restriction of the Cayley multiplication to the imaginary part of the Cayley algebra, we obtain an almost complex structure on $S^6(1)$ (see, for instance, [1],[2]). This almost complex structure will be denoted by J. If we denote by (,) the standard metric tensor field on $S^6(1)$, then ($S^6(1)$, J, (,)) is nearly Kählerian, i.e. $(\nabla' J)(X,X) = 0$, where ∇' is the Riemannian connection generated by (,). The skew-symmetric (1,2)-tensor field J will be denoted by G. The following formulas are known [1] , [2] :

(1.1) $(G(X,Y),Z) = -(G(X,Z),Y)$,

(1.2) $G(X,JY) = G(JX,Y) = -JG(X,Y)$,

(1.3) $(G(X,Y), G(Z,W)) = (X,Z)(Y,W) - (X,W)(Y,Z) + (JX,Z)(JW,Y) + (JX,W)(JY,Z)$,

(1.4) $(\nabla' G)(X,Y,Z) = (Y,JZ)X + (X,Z)JY - (X,Y)JZ$

for any vector fields X,Y,Z on $S^6(1)$.

The tangent bundle of a manifold N will be denoted by TN, the bundle of all unit tangent vectors by UN and the set of all vector fields on N, by $\mathfrak{X}(N)$. Let M be a submanifold of $S^6(1)$. $\mathcal{N}$ will denote the normal bundle of M in $S^6(1)$. The induced connections in the bundles TM and $\mathcal{N}$ will be denoted by ∇ and D respectively. R' , R and $R^\perp$ will denote the curvature tensors of the connections ∇', ∇

and D respectively. We have the formulas of Gauss and Weingarten:

(1.5) $\nabla'_X Y = \nabla_X Y + \alpha(X,Y)$,

(1.6) $\nabla'_X \xi = D_X \xi - A_\xi X$,

where α is the second fundamental form of M in $S^6(1)$, A is the Weingarten endomorphism and $X,Y \in \mathfrak{X}(M)$, ξ is a normal vector field on M. In the sequal we shall use the equations of Gauss, Codazzi and Ricci which are given by

(1.7) $(R(X,Y)Z,W) = (X,W)(Y,Z) - (X,Z)(Y,W) + (\alpha(X,W),\alpha(Y,Z)) - (\alpha(X,Z),\alpha(Y,W))$,

(1.8) $\nabla\alpha(X,Y,Z) = \nabla\alpha(Y,X,Z)$,

(1.9) $(R^\perp(X,Y)\xi, \eta) = ([A_\xi, A_\eta]X, Y)$,

for X,Y,Z,W tangent to M ; ξ, η normal to M.

Recall also that

(1.10) $\nabla^2\alpha(X,Y,Z,W) - \nabla^2\alpha(Y,X,Z,W) = R^\perp(X,Y)\alpha(Z,W) - \alpha(R(X,Y)Z,W) - \alpha(Z,R(X,Y)W)$

for $X,Y,Z,W \in \mathfrak{X}(M)$. A submanifold M in $S^6(1)$ is totally real if $JTM \in \mathcal{N}$. Of course, such a submanifold is 2 or 3-dimensional. A 3-dimensional totally real submanifold of $S^6(1)$ is minimal [1] . In contrast with this case there are non-minimal 2-dimensional totally real submanifolds of $S^6(1)$. For instance, we know [1], that $S^3(1/16)$ can be imbedded in $S^6(1)$ as a totally real submanifold. Of course, it is not totally geodesic, so there is a vector X tangent to $S^3(1/16)$ such that $\alpha'(X,X) \neq 0$, where α' is the second fundamental form of $S^3(1/16)$ in $S^6(1)$. Let $M = S^3(1/16) \cap X^\perp$, where $X^\perp$ is the orthogonal complement to X in R^4 . Then M is totally geodesic in $S^3(1/16)$. Hence M can be imbedded in $S^6(1)$ as a totally real submanifold and such that its second form α in $S^6(1)$ is equal to $\alpha'|_M$. Since $S^3(1/16)$ is minimal in $S^6(1)$ and $\alpha'(X,X) \neq 0$, M is not minimal in $S^6(1)$.

Proof of Theorem. Assume M is 3-dimensional. It is known, [1] , that

(2.1) $\{G(X,Y) : X,Y \in \mathfrak{X}(M)\} = \mathcal{N}$,

(2.2) $(\alpha(X,Y),JZ) = (\alpha(X,Z),JY)$

and

(2.3) $\alpha(X,JG(Y,Z)) = JG(\alpha(X,Y),Z) + JG(Y,\alpha(Z,X))$

for any $X,Y,Z \in \mathfrak{X}(M)$. The eqality (2.2) implies

(2.4) $(\nabla\alpha(W,X,Z),JY) - (\nabla\alpha(W,X,Y),JZ) = (\alpha(X,Y),G(W,Z)) - (\alpha(X,Z),G(W,Y))$

for $X,Y,Z,W \in \mathfrak{X}(M)$. Taking account of (1.1) and (2.3), we obtain

$$(\alpha(X,Y),G(W,Z)) = -(\alpha(X,JG(W,Z)),JY) = -(G(\alpha(X,W),Z),Y) - (G(W,\alpha(X,Z)),Y) = -(\alpha(X,W),G(Z,Y)) + (\alpha(X,Z),G(W,Y)).$$

Combining this with (2.4) we get

(2.5) $(\nabla\alpha(W,X,Z),JY) - (\nabla\alpha(W,X,Y),JZ) = (\alpha(X,W),G(Y,Z))$.

Since $\nabla\alpha = 0$ and (2.1) holds, $\alpha = 0$.

Assume now, that M is 2-dimensional. We set

K - the Gaussian curvature of M,

$\mathcal{N}\mathcal{H}$ - the orthogonal complement to $TM + JTM$ in $TS^6{}_{|M}$,

n - the projection onto $\mathcal{N}$ in $TS^6{}_{|M} = TM \oplus \mathcal{N}$,

t - the projection onto TM in $TS^6{}_{|M} = TM \oplus \mathcal{N}$,

p - the projection onto $TM + JTM$ in $TS^6{}_{|M} = (TM + JTM) \oplus \mathcal{N}\mathcal{H}$,

h - the projection onto $\mathcal{N}\mathcal{H}$ in $TS^6{}_{|M} = (TM + JTM) \oplus \mathcal{N}\mathcal{H}$,

Let V and U be an orthonormal basis in T_xM. By virtue of (1.1) and (1.2), $G(V,U) \in \mathcal{N}\mathcal{H}$. By formula (1.3) $G(V,U)$ is unit. If $\bar{V},\bar{U}$ is another orthonormal frame at x, then $\bar{V} = \beta_1 V + \beta_2 U$, $\bar{U} = \pm(-\beta_2 V + \beta_1 U)$, where $\beta_1^2 + \beta_2^2 = 1$ and consequently $G(\bar{V},\bar{U}) = \pm G(V,U)$. This means that im G defines a 1-dimensional vector subbundle of $\mathcal{N}\mathcal{H}$ and M is orientable iff this bundle is trivial. Taking account of (1.5) and (1.6), we obtain

(2.6) $D_X JY = G(X,Y) + nJ\alpha(X,Y) + J\nabla_X Y$,

(2.7) $A_{JY}X = -tJ\alpha(X,Y)$

for $X,Y \in \mathfrak{X}(M)$. The last equation implies

(2.8) $(\alpha(X,Y),JZ) = (\alpha(X,Z),JY)$

for $X,Y,Z \in \mathfrak{X}(M)$.

Let $x \in M$ and let M' be an oriented open neighbourhood of x. If $V \in UM'$, then U will denote the vector from UM' such that the pair (V,U) is positively oriented. We denote by ξ the vector G(V,U) which, of course, does not depend of the choice of V. If $V \in UM_x$, then $V,U,JV,JU,\xi,J\xi$ is an orthonormal basis in $\mathcal{N}_x$. Since moreover M is minimal, we have

$$\alpha(V,V) = a_1(V)JV + a_2(V)JU + a_3(V)\xi + a_4(V)J\xi$$
$$\alpha(V,U) = a_2(V)JV - a_1(V)JU + c(V)\xi + d(V)J\xi$$

for some real numbers $a_1(V)$, $a_2(V)$, $a_3(V)$, $a_4(V)$, $c(V)$, $d(V)$. Moreover $\|p\alpha\|^2 = 4(a_1(V)^2 + a_2(V)^2)$ and $\|h\alpha\|^2 = 2(a_3(V)^2 + a_4(V)^2 + c(V)^2 + d(V)^2)$. The following equalities are obvious

(2.9) $G(\xi,U) = -V$,
$G(\xi,V) = U$,
$G(J\xi,U) = JV$,
$G(J\xi,V) = -JU$.

Let $V \in UM_x$ and let γ_1, γ_2 be geodesics in M' determined by

(V,x) and (U,x) respectively. V,U will denote also vector fields defined along γ_1 and γ_2 and parallel with respect to ∇. Then $a_1(V)$, $a_2(V)$, $a_3(V)$, $a_4(V)$, $c(V)$, $d(V)$ are functions defined along γ_1 and γ_2, and they will be denoted by a_1, a_2, a_3, a_4, c, d respectively. By a straightforward computation and by using (1.4), (1.5), (1.6), (2.9), we obtain

$$-JU = \nabla' G(V,V,U) = -A_\xi V + a_3 V + cU + D_V\xi - a_4 JV - dJU,$$

i.e.

(2.10) $D_V\xi = a_4 JV + (d-1)JU.$

Of course $(D_V J\xi, J\xi) = 0$, and by (2.10) $(D_V\xi, J\xi) = 0$, i.e. $(D_V J\xi, \xi) = 0$. Consequently $D_V J\xi \in JTM$ and, by (2.6),

(2.11) $D_V J\xi = -a_3 JV - cJU.$

Similarily we get

$$JV = \nabla' G(U,V,U) = -A_\xi U + cV - a_3 U + D_U\xi - dJV + a_4 JU,$$

i.e.

(2.12) $D_U\xi = (1+d)JV - a_4 JU.$

Like in the previous case, we have

(2.13) $D_U J\xi = -cJV + a_3 JU.$

By virtue of (2.10) – (2.13), we obtain the following formulas

$$\begin{aligned}(2.14)\quad \nabla\alpha(V,V,V) = {} & (Va_1)JV + (Va_2)JU + a_3(d-1)JU - ca_4 JU \\ & + (Va_3)\xi - a_1 a_4\xi + (1-d)a_2\xi + (Va_4)J\xi \\ & + a_1 a_3 J\xi + a_2 cJ\xi ,\end{aligned}$$

$$\begin{aligned}(2.15)\quad \nabla\alpha(V,V,U) = {} & (Va_2)JV + ca_4 JV - da_3 JV - (Va_1)JU - cJU \\ & + (Vc)\xi + a_1(d-1)\xi - a_2 a_4\xi + (Vd)J\xi \\ & + a_2 a_3 J\xi - ca_1 J\xi ,\end{aligned}$$

$$\begin{aligned}(2.16)\quad \nabla\alpha(U,V,V) = {} & (Ua_1)JV + a_3(1+d)JV - a_4 cJV + (Ua_2)JU \\ & + (Ua_3)\xi + a_2 a_4\xi - a_1(d+1)\xi + (Ua_4)J\xi \\ & + ca_1 J\xi - a_2 a_3 J\xi\end{aligned}$$

and

$$\begin{aligned}(2.17)\quad \nabla\alpha(U,V,U) = {} & (Ua_2)JV + cJV - (Ua_1)JU - ca_4 JU + da_3 JU \\ & + (Uc)\xi - a_2(d+1)\xi - a_1 a_4\xi + (Ud)J\xi \\ & + a_2 cJ\xi + a_1 a_3 J\xi .\end{aligned}$$

By comparing (2.14) and (2.17), and using (1.8) we obtain at x

$$Va_2 - Ua_1 = 2ca_4 - 2da_3 + a_3 ,$$

and by (2.15), (2.16), (1.8)

$$Va_2 - Ua_1 = 2da_3 - 2ca_4 + a_3 .$$

Therefore $ca_4 = da_3$ at x. Of course this formula is valied on the whole UM. Now, formulas (2.14) – (2.17) can be rewritten in the following form

(2.18) $$\nabla\alpha(V,V,V) = (Va_1)JV + (Va_2)JU - a_3JU + (Va_3)\xi - a_1a_4\xi + (1-d)a_2\xi + (Va_4)J\xi + a_1a_3J\xi + a_2cJ\xi ,$$

(2.19) $$\nabla\alpha(V,V,U) = (Va_2)JV - (Va_1)JU - cJU + (Vc)\xi + a_1(d-1)\xi - a_2a_4\xi + (Vd)J\xi + a_2a_3J\xi - ca_1J\xi ,$$

(2.20) $$\nabla\alpha(U,V,V) = (Ua_1)JV + a_3JV + (Ua_2)JU + (Ua_3)\xi + a_2a_4\xi - a_1(d+1)\xi + (Ua_4)J\xi + ca_1J\xi - a_2a_3J\xi ,$$

(2.21) $$\nabla\alpha(U,V,U) = (Ua_2)JV + cJV - (Ua_1)JU + (Uc)\xi - a_2(d+1)\xi - a_1a_4\xi + (Ud)J\xi + a_2cJ\xi + a_1a_3J\xi .$$

Since $ca_4 = da_3$, we have $(h\alpha(V,V), Jh\alpha(V,U)) = 0$. It follows that the vectors $h\alpha(V,V)$ and $h\alpha(V,U)$ are proportional and consequently $\dim \operatorname{im} h\alpha = 1$. Consider the function

$$\chi : UM_x \ni X \longrightarrow \|h\alpha(X,X)\|^2 .$$

If V is a vector in which this function attains its maximum, then $(h\alpha(V,V), h\alpha(V,U)) = 0$. For this vector $h\alpha(V,U) = 0$ and consequently $(\alpha(V,V), \alpha(V,U)) = 0$. Moreover

$$\|\alpha(V,V)\|^2 = a_1(V)^2 + a_2(V)^2 + a_3(V)^2 + a_4(V)^2 = \frac{\|p\alpha\|^2}{4} + \frac{\|h\alpha\|^2}{2} ,$$

$$\|\alpha(V,U)\|^2 = a_1(V)^2 + a_2(V)^2 = \frac{\|p\alpha\|^2}{4}$$

The above formulas and the equation of Ricci give

$$(R^\perp(V,U)\alpha(V,V), \alpha(V,U)) = 2\{(\alpha(V,V), \alpha(V,U))^2 - \alpha(V,V)^2\alpha(V,U)^2\} = -\frac{\|p\alpha\|^2}{2}\left(\frac{\|p\alpha\|^2}{4} + \frac{\|h\alpha\|^2}{2}\right)$$

By the equation of Gauss $K = 1 - \frac{\|p\alpha\|^2}{2} - \frac{\|h\alpha\|^2}{2}$.

Consequently

(2.22) $$(R^\perp(V,U)\alpha(V,V), \alpha(V,U)) = -\frac{\|p\alpha\|^2}{2}\left(1-K-\frac{\|p\alpha\|^2}{4}\right) .$$

It is easy to check that $(R^\perp(V,U)\alpha(V,V), \alpha(V,U))$ does not depend of the choice of V and hence (2.22) holds for any $V \in UM$.

Now we shall use the assumption $\nabla\alpha = 0$. By virtue of (2.18) - (2.21) we see that $Va_1 = 0$, $Va_2 = 0$, $Ua_1 = 0$, $Ua_2 = 0$ and $a_3 = c = 0$ at x. Since $x \in M$ and $V \in UM_x$ are arbitrary, $a_3 = c = 0$ on the whole UM. Using once again formulas (2.18) and (2.19), we obtain

(2.23) $$a_1(V)\,a_4(V) = (1-d(V))\,a_2(V),$$
$$a_2(V)\,a_4(V) = (d(V)-1)\,a_1(V)$$

for every $V \in UM$. If for every $V \in UM_x$ $a_4(V) = 0$, then

(2.24) $(1-d(V))\, a_2(V) = (d(V)-1)\, a_1(V) = 0$ for every $V \in UM_x$.
But there is a vector $V \in UM_x$ (in which χ attains a maximum) such that $d(V) = 0$. For such a vector V, by (2.24), $a_1(V) = a_2(V) = 0$, i.e. $\|p\alpha\|_x = 0$. Assume now that there exists a vector $V \in UM_x$ such that $a_4(V) \neq 0$. The formulas (2.23) imply the equality

$a_1(V)^2\, a_4(V) + a_2(V)^2\, a_4(V) = 0$. Hence $\|p\alpha\|^2_x = 0$. Consequently $\|p\alpha\| = 0$ on M. Since $\nabla\alpha = 0$, (1.10) gives

$$0 = R^{\perp}(V,U)\,\alpha(V,V) - 2\,\alpha(R(V,U)V,V),$$

for any $V \in UM$. By virtue of (2.22), the obvious equality $A_{J\xi}V = a_4(V)V + d(V)U$ and the fact that $\alpha(V,U) = d(V)J\xi$, we have

$$0 = (\alpha(R(V,U)V,V), \alpha(V,U)) = (A_{\alpha(V,U)}V, R(V,U)V)$$
$$= d(V)\,(A_{J\xi}V, R(V,U)V) = -d(V)^2K_x \text{ for } V \in UM_x ,$$

$x \in M$. Hence $d(V) = 0$ for every $V \in UM_x$ or $K_x = 0$. In the first case

$a_4(V) = 0$ for every $V \in UM_x$. (If we put $\bar{V} = \frac{V+U}{2}$, then $d(\bar{V}) = a_4(V)$) Then $\alpha_x = 0$. The assumption $\nabla\alpha = 0$ and the Gauss equation imply that M has constant Gaussian curvature. Hence K = 0 on the whole M or M is totally geodesic. The proof is completed.

Examples. It is easy to find 2 and 3-dimensional great spheres in $S^6(1)$ which are totally real. Now let M be the pythagorean

product $S^1(1/2) \times S^1(1/2)$ (see [4] . Example 5.3). Then M is a minimal submanifold of $S^3(1)$ with parallel second fundamental form ([4] , Ex. 5.3, Lemma 5.2). Since $S^3(1)$ can be imbedded in $S^6(1)$ as a totally real totally geodesic submanifold, M can be imbedded in $S^6(1)$ as a totally real minimal submanifold with parallel second fundamental form. Of course M is locally flat.

Remark. If M is an almost complex submanifold of $S^6(1)$ with parallel second fundamental form, then M is totally geodesic. It follows from the formula (4.13) in [2] .

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