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AN APPLICATION OF QUATERNIONIC ANALYSIS TO THE SOLUTION OF TIME-INDEPENDENT MAXWELL EQUATIONS AND OF STOKES' EQUATION [1)]

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1. Indroduction

This paper presents an approximative solution of special boundary value problems of equations of mathematical physics in 3-dimensional, in general multiple connected domains with smooth boundaries. Several numerical methods are at present successful applied for solving of these problems. However, the numerical effort of difference methods, finite element methods or the Galerkin method is very expensive. Therefore it is necessary to construct techniques for use in practice, which give approximative solutions by small computing-time and a good estimate of the error. The boundary collocation method possesses these advantages, because it is a synthesis of analytical and numerical techniques. The classical functiontheory is an important practical tool for calculation of plane problems. Basing on the results of R. FUETER (Switzerland), A.W.BIZADSE (USSR) in the seventies several mathematicians from various countries founded a functiontheory of quaternions and the Clifford analysis. Important papers are written by R.DELANGHE, F.F. BRACKX, F.SOMMEN (Belgium), J.BURES, V. SOUCEK (Czechoslovakia), P.LOUNESTO (Finland) , J.RYAN, A.SUDBERY (England), B.GOLDSCHMIDT (GDR). The authors of this paper also have written some publications about applications of quaternionic analysis for construction of approximative solutions of boundary value problems. They descibed in previous papers [5], [13] [14] an effective representation of solutions for some important equations of mathematical physics by the aid of a general operater calculus. Some of the numerical methods recently developed make use of the quaternionic calculus (see [3] , [7]). In our paper

1) This paper is in final form and no version of it will be submitted for publication elsewhere.

we shall apply the boundary collocation method for approximative solution of Stokes' problem and for time-independent Maxwell equations. A special method of decomposition gives favourable approximative solutions for the user.

2. An operator calculus

Let u and v be fourdimensional vectors written in the form $u = (u_0, \hat{u})$ and $v = (v_0, \hat{v})$ with $\hat{u} = (u_1, u_2, u_3)$ and $\hat{v} = (v_1, v_2, v_3)$ By introduction of the non-commutative product

$$u \circ v = (u_0 v_0 - (\hat{u}, \hat{v}),\ u \times v + u_0 \hat{v} + v_0 \hat{u}) \qquad (2.1)$$

we obtain the structure of a skew field. If $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0)$, $e_2 = (0,0,1,0)$ and $e_3 = (0,0,0,1)$, so the quaternion u allowed the representation

$$u = \sum_{i=0}^{3} u_i e_i$$

Furthermore let be $\operatorname{Re} u = u_0$, $\operatorname{Im} u = \sum_{i=1}^{3} u_i e_i$ and $\overline{u} = u_0 e_0 - \sum_{i=1}^{3} u_i e_i$. It is evident, that hold true the relations

$$\begin{aligned} e_0 \circ e_i &= e_i \circ e_0 & & i = 0,1,2,3 \\ e_i \circ e_j &= e_j \circ e_i & & i = j,\ i = 1,2,3,\ j = 1,2,3 \\ e_i^2 &= -1 e_0 & & i = 1,2,3 \\ e_0^2 &= 1 e_0 & & . \end{aligned}$$

Let $G \subset R^3$ be a bounded domain with sufficient smooth boundary $\Gamma = \partial G$. The Banach spaces C_Q^k, L_Q^p, H_Q^s, $W_{2,Q}^1$ of quaternionic functions are defined by their components, which belong to the spaces C^k, L^p, H^s, W_2^1. In the spaces $H_Q^s(G)$ will be introduced a scalar product $(u,v)_Q$ by the formula

$$(u,v)_Q = \int_G u(x) \circ v(x)\, dG_x \quad ,$$

where $u = (u_0, -\hat{u})$ is the conjugate quaternion.

For $u \in C_Q^1(G)$ we can define the following operators. Let be $\lambda \geqslant 0$, then by

$$\nabla_\lambda \circ u = (-\operatorname{div} \hat{u},\ \operatorname{grad} u_0 + \operatorname{rot} \hat{u}) + \lambda u \qquad (2.2)$$

is designed the 3-dimensional analogue to the Cauchy-Riemann operator. The operator

$$(F_\lambda u)(x) = \frac{1}{4\pi} \int_\Gamma r_\lambda(x-y) \circ n(y) \circ u(y) \, d\Gamma_y \qquad , x \notin \Gamma \quad , \qquad (2.3)$$

where

$$r_\lambda(x) = \left(\frac{\lambda \cos \lambda |x|}{x}, \frac{x_1}{|x|^2}\left(\lambda \sin \lambda |x| + \frac{\cos \lambda |x|}{|x|}\right), \dots, \frac{x_3}{|x|^2}\left(\lambda \sin \lambda |x| + \frac{\cos \lambda |x|}{|x|}\right)\right)$$

and $n = (0, \hat{n})$ with $\hat{n} = (n_1, n_2, n_3)$ the unit vector of the outer normal on the surface Γ in the point y, is the 3-dimensional analogue to the classical Cauchy integral operator, [5]. Weakly singular integral operator

$$(T_\lambda u)(x) = \frac{1}{4\pi} \int_G r(x-y) \circ u(y) \, dG_y \qquad (2.4)$$

represented the 3-dimensional analogue to the T-operator of the classical functiontheory.
For $u \in C_Q^1(\Gamma)$ the operator

$$(S_\lambda u)(x) = \frac{1}{2\pi} \int_\Gamma r(x-y) \circ n(y) \circ u(y) \, d\Gamma_y \qquad , x \in \Gamma \quad , \qquad (2.5)$$

exists in the sense of Cauchy's principal value. We introduce the projectors

$$(\tilde{Q}_\lambda u)(x) = 2^{-1}(u(x) - (S_\lambda u)(x)) \qquad (2.6)$$

$$(\tilde{P}_\lambda u)(x) = 2^{-1}(u(x) + (S_\lambda u)(x)) \quad . \qquad (2.7)$$

Between all these operators exist numerable relations, which included in a general operator theory (see [3],[11],[14]) . Finally we need the multiplication operator

$$(Mu)(x) = m(x) \left((1-2)^{-1}(21-2) u_0, \hat{u} \right) , \qquad (2.8)$$

where $1 \in R \setminus \{2\}$, $m \in C_R^1(G)$.
These operators enable us to describe a lot of systemes of partial differential equations in a favourable manner. We consider the following system of equations :

$$\nabla_\lambda \circ M \nabla_\lambda \circ u = 0 \qquad \text{in} \quad G \qquad (2.9)$$

$$\gamma_0 u = g \qquad \text{on} \quad \Gamma \quad , \qquad (2.10)$$

where by $\gamma_0 u$ is denoted the trace of the quaternionic function on the boundary Γ. For particular choice of the numbers 1 and and the function m(x) the following Dirichlet problems are obtained :

a) $\lambda = 0$, $1 = 0$; $m \equiv 1$ — Laplace equation

b) $\lambda = 0$, $1 = 0$; $m = \frac{\varepsilon}{\mu \varkappa}(x)$ — equation of the magnetic field (μ permeability ε dielectric constant, $\varkappa$ conductivity)

c) $\lambda = 0$ 1 0,2 ; $m \equiv 1$ — equations of linear theory of elasticity (1 Poisson number)

d) $\lambda^2 > 0$, $1 = 0$; $m \equiv 1$ — Helmholtz equation

Remark 2.1
For $\lambda = 0$ the index λ shall be omitted.

3. A boundary collocation method

Let $G \subset R^3$ be a bounded domain with sufficient smooth boundary . An elliptic differential operator with constant coefficients is denoted by A . We look for the solution of the boundary value problem

$$\begin{aligned} A\,u &= 0 \quad \text{in } G \\ R\,u &= g \quad \text{on} \end{aligned} \tag{3.1}$$

in suitable chosen spaces. The domain is denoted by $D(A) \subseteq X$, where X is a normed space and $Y = X \cap \ker A$. By K is designed the set of coefficients furnished by an algebraic structure of a ring. In Y is defined an addition " + " and a multiplication "$\circ$" by elements of K in a suitable manner, so that Y has the structure of a right vector space. We look for an approximative solution for the problem (3.1) in the form

$$u_n(x) = \sum_{j=1}^{n} \varphi_j(x) \circ a_j \quad , \tag{3.2}$$

where the coefficients $a_j \in K$, $j = 1,\ldots,n$, shall be defined by the equations

$$(Ru_n)(x_j) = g(x_j) \quad , x_j \in \Gamma \quad , j=1,\ldots,n \tag{3.3}$$

Let us consider the boundary value problem

$$\begin{aligned} \nabla_\lambda \circ M \nabla_{-\lambda} u &= 0 \quad \text{in } G \\ \gamma_0 u &= g \quad \text{on } \Gamma \end{aligned} \tag{3.4}$$

where l, and m fulfil the above-mentioned premises. Then the following theorem holds true

Theorem 3.1

Let be $g \in H_Q^s(\Gamma)$ $s > 2^{-1}$; λ, l and $m(x)$ is chosen as in a) c) or d). Then it follows

$$u = v + T_\lambda M^{-1} w \qquad (3.5),$$

where v and w are the unique solutions of the boundary value problems

$$\begin{aligned} \nabla_\lambda \circ v &= 0 \quad \text{in } G \\ \gamma_0 v &= \tilde{P}_\lambda g \quad \text{on } \Gamma \end{aligned} \qquad (3.6)$$

and

$$\begin{aligned} w &= 0 \quad \text{in } G \\ \gamma_0 T_\lambda M^{-1} w &= \tilde{Q}_\lambda g \quad \text{on } \Gamma \end{aligned} \qquad (3.7)$$

Remark 3.1

By realization of the assumption in the case b) we obtain the same result, if $m(x) = \text{const.}$. In the general case we also conjecture the correctness of Theorem 3.1 .

Theorem 3.2 [4]

Let be G_I, G, G_A bounded domains with sufficient smooth boundaries $\Gamma_I = \partial G_I$, $\Gamma = \partial G$ and $\Gamma_A = \partial G_A$, so that $\bar{G}_I \subset G$ and $\bar{G} \subset G_A$. Furthermore the sets of points $\{x_i\}_{i=1}^{\infty} \subset \Gamma_A$ and $\{y_j\}_{j=1}^{\infty} \subset \Gamma_I$ are dense subsets of Γ_A respectively Γ_I . Then it holds for $s > 0$:

$\{r_\lambda(x-x_i)\}$ is Q - complete in $H_Q^s(G) \cap \ker \nabla_\lambda \circ$,
$\{\gamma_0 r_\lambda(x-x_i)\}$ is Q - complete in $H_Q^s(\Gamma) \cap \operatorname{Im} \tilde{P}_\lambda$
$\{\gamma_0 r_\lambda(x-y_i)\}$ is Q - complete in $H_Q^s(\Gamma) \cap \operatorname{Im} \tilde{Q}_\lambda$
$\{\gamma_0 r_\lambda(x-x_i)\} \cup \{\gamma_0 r_\lambda(x-y_i)\}$ is Q - complete in $H_Q^s(\Gamma)$.

Remark 3.2

The proofs in the cases a), c) and d) can be found in [4] ,[5]. Applications to the case c) are also included in [14]. The paper [6]is dedicated to the investigation of this method in the case of parabolic equations. A transfer of these results to the biharmonic equation is made in the paper [10].

Remark 3.3

On the base of the system $\{r_\lambda\}$we can construct systems, which have too, and a=b= (0,0,0,0). System of equations (4.4) we can rewrite in the simple form

$$\begin{aligned} \nabla\circ E &= \varrho \\ \nabla\circ H &= \alpha\circ E \end{aligned} \qquad (4.5)$$

By help of the multidimensional generalized Vekua's theory (see [8]) we receive the representations for the solution

$$\begin{aligned} E &= T\varrho + \phi_E \\ H &= T\alpha E + \phi_H \end{aligned} \qquad (4.6)$$

where ϕ_E and ϕ_H belong to $\ker \nabla\circ$. By substitution of E in in the second equation it follows

$$\begin{aligned} E &= T\varrho + \phi_E \\ H &= T\alpha T\varrho + T\alpha\phi_E + \phi_H \end{aligned} \qquad (4.7)$$

On the assumption $\gamma_0 H = h$, it follows immediately $\phi_H = F h$. The multidimensional boundary formulas of Plemelj-Sochotzki (see [1]) leads to the formula

$$\tilde{Q} h = \gamma_0 T\alpha T\varrho + \gamma_0 T\alpha F\phi_E \qquad (4.8)$$

on the surface Γ.
Equation (4.8) can be multiplied by α^{-1}, because α is a scalar quantity greater than zero. The operator $\gamma_0 TF$ is continuously invertible in the pair of spaces $[H^s_Q \cap \operatorname{Im}\tilde{P}, H^s_Q \cap \operatorname{Im}\tilde{Q}]$ as shown in a general case later. Hence, it follows

$$\phi_E = \alpha^{-1} F(\gamma_0 TF)^{-1}\tilde{Q}h - F(\gamma_0 TF)^{-1}\gamma_0 T^2\varrho \quad . \qquad (4.9)$$

By putting in (4.7), we obtain

$$E = \alpha^{-1} F(\gamma_0 TF)^{-1}\tilde{Q}h + (I - F(\gamma_0 TF)^{-1}\gamma_0 T)\, T\varrho \qquad (4.10)$$

$$H = TF(\gamma_0 TF)^{-1}\tilde{Q}h + Fh + \alpha T(I - F(\gamma_0 TF)^{-1}\gamma_0 T)\, T\varrho \quad . \qquad (4.11)$$

The operator $(I - F(TF)^{-1}\gamma_0 T) \equiv \mathbb{P}_1$ is the L^2_Q- orthoprojector onto the subspace $\nabla\circ \overset{\circ}{W}{}^1_{2,Q}(G)$. We obtain

$$E = \alpha^{-1} F(\gamma_0 TF)^{-1}\tilde{Q}h + \mathbb{P}_1 T\varrho$$

better numerical properties. In the paper [7] are described a method for construction of such a system and numerical comparisons with the Galerkin method.

4. An approximative solution of the time-independent Maxwell equations

In the domain G is an electric charge distributed with the density $\varrho_0(x) = \varrho_0$. We shall compute the stationary electric field $\hat{E} = (E_1, E_2, E_3)$ and the stationary magnetic field $\hat{H} = (H_1, H_2, H_3)$, if there are given the dielectric constant $\varepsilon = \varepsilon(x)$, the permeability $\mu = \mu(x)$, the electric conductivity $\varkappa = \varkappa(x)$ and the magnetic field on the boundary Γ.
The Maxwell equations read as follows in the time-independent case

$$\begin{aligned} \operatorname{div} \varepsilon \hat{E} &= \varrho_0 & \qquad \operatorname{div} \mu \hat{H} &= 0 \\ \operatorname{rot} \hat{E} &= 0 & \qquad \operatorname{rot} \hat{H} &= \varkappa \hat{E} \end{aligned} \tag{4.1}$$

By using of a well-known multiplication rule it holds

$$\begin{aligned} \operatorname{div} \varepsilon \hat{E} &= \varrho_0 \\ \operatorname{rot} \varepsilon \hat{E} &= \operatorname{grad} \varepsilon \times \hat{E} \end{aligned} \tag{4.2}$$

$$\begin{aligned} \operatorname{div} \mu \hat{H} &= 0 \\ \operatorname{rot} \mu \hat{H} &= \mu \varkappa \hat{E} + \operatorname{grad} \mu \times \hat{H} \end{aligned} \tag{4.3}$$

By setting $E = (0, \varepsilon \hat{E})$ and $H = (0, \mu \hat{H})$ we obtain fourdimensional vectors, which are needed for our method. We denote by $a = (0, \varepsilon^{-1} \operatorname{grad} \varepsilon)$, $b = (0, \mu^{-1} \operatorname{grad} \mu)$ and $\varrho = (-\varrho_0, 0, 0, 0)$. It follows by using the Nabla operator in the sense of the quaternionic multiplication from (4.2) and (4.3)

$$\begin{aligned} \nabla \circ E &= \varrho + \operatorname{Im}(a \circ E) \\ \nabla \circ H &= \frac{\mu \varkappa}{\varepsilon} E + \operatorname{Im}(b \circ H) \end{aligned} .$$

The abbreviation $\alpha = \frac{\mu \varkappa}{\varepsilon}$ leads to the system of differential equations

$$\begin{aligned} \nabla \circ E &= \varrho + \operatorname{Im}(a \circ E) \\ \nabla \circ H &= \alpha E + \operatorname{Im}(b \circ H) \end{aligned} \tag{4.4}$$

First let ε, μ and $\varkappa$ be constant quantities. Therefore $\alpha = \text{const.}$

$$H = TF(\gamma_o TF)^{-1}\tilde{Q}h + Fh + \alpha T \Pi_1 T \varrho \quad .$$

From the paper[13] we know, that the equation

$$\gamma_o TF\phi_H = \tilde{Q}h$$

is in a certain sense equivalent to the Dirichlet problem follows

$$\tilde{\phi}_H = F(TF)^{-1}\tilde{Q}h \in \ker \nabla\circ \quad .$$

By introducing the orthoprojector

$$\Pi_2 \equiv F(TF)^{-1}T$$

onto the subspace $\ker \nabla\circ \cap L^2_Q(G)$ finally we obtain

$$E = \alpha^{-1}\tilde{\phi}_H + \Pi_1 T\varrho \tag{4.12}$$

$$H = T\tilde{\phi}_H + Fh + \alpha T \Pi_1 T\varrho \tag{4.13}$$

Now we shall approximate each of the terms in the formulas (4.12) and (4.13). Let $h \in H^s_Q(\Gamma)$ then holds

$$h = \lim_{n\to\infty} \sum_{j=1}^{n} (\gamma_o r(x-x_i)\circ a_i^{(n)} + \gamma_o r(x-y_i)\circ b_i^{(n)}) \tag{4.14}$$

in $H^s_Q(\Gamma)$, where $a_i^{(n)}$ and $b_i^{(n)}$ are unknown quaternionic constands. F is a continuous operator in $[H^s_Q(\Gamma), H^{s+0.5}(G)]$, $\ker F = H^s_Q(\Gamma) \cap \operatorname{Im} \tilde{Q}$ and $r(x-x_i)$ is the generalized analytical continuation of $\gamma_o r(x-x_i)$. Therefore from (4.14) follow the representations

$$Fh = \lim_{n\to\infty} \sum_{i=1}^{n} r(x-x_i)\circ a_i^{(n)} \quad \text{in } H^{s+0.5}_Q(G) \tag{4.15}$$

and

$$\tilde{Q}h = \lim_{n\to\infty} \sum_{i=1}^{n} \gamma_o r(x-y_i)\circ b_i^{(n)} \quad \text{in } H^s_Q(\Gamma) \tag{4.16}$$

We mention, that $\operatorname{Im} \tilde{P} = \ker \tilde{Q}$. The quaternionic function $T\tilde{\phi}_H \in H^{s+0.5}_Q(G)$ is a solution of the Dirichlet problem

$$\begin{aligned} \Delta v &= 0 \quad \text{in } G \\ \gamma_o v &= \tilde{Q}h \quad \text{on } \Gamma \end{aligned} \tag{4.17}$$

The boundary value problem (4.17) is correctly given. Hence, $\tilde{Q}h$

can be approximated by (4.16) and the method (3.2)-(3.3) is applicable. This leads to

$$T\widetilde{\Phi}_H = \lim_{n\to\infty} |x-x_i|^{-1} c_i^{(n)} \quad \text{in } H_Q^{s+0.5}(G) \qquad (4.18)$$

and after derivation by $\nabla\circ$

$$\widetilde{\Phi}_H = \lim_{n\to\infty}\sum_{i=1}^{n}(x-x_i)|x-x_i|^{-3} c_i^{(n)} \quad \text{in } H_Q^{s-0.5}(G) \ .$$

The item $T\Pi_1 T\varrho$ is the solution of the boundary value problem

$$\begin{aligned} \Delta w &= \varrho && \text{in } G \\ \gamma_o w &= 0 && \text{on } \Gamma \ . \end{aligned} \qquad (4.19)$$

Problem (4.19) can be transformed by subtraction of a special solution $K\varrho$ (for instance, $K\varrho = \frac{1}{4\pi}\int|x-y|^{-1}\varrho\, dG_y$) into a problem of the form (4.17), which can be solved again by (3.2)-(3.3). Now, we have

$$T\Pi_1 T\varrho = \lim_{n\to\infty}\sum_{i=1}^{n}|x-x_i|^{-1} d_i^{(n)} + K\varrho \qquad (4.20)$$

and

$$\Pi_1 T\varrho = \lim_{n\to\infty}\sum_{i=1}^{n}(x-x_i)|x-x_i|^{-3} d_i^{(n)} \qquad (4.21)$$

Finally, we get to the formulas

$$E = \alpha^{-1}\lim_{n\to\infty}\sum_{i=1}^{n}(x-x_i)|x-x_i|^{-3}c_i^{(n)} + \lim_{n\to\infty}\sum_{i=1}^{n}(x-x_i)|x-x_i|^{-3}d_i^{(n)} + \nabla\cdot K\varrho \qquad (4.22)$$

$$H = \lim_{n\to\infty}\sum_{i=1}^{n}|x-x_i|^{-1} c_i^{(n)} + \lim_{n\to\infty}\sum_{i=1}^{n} r(x-x_i)\, a_i^{(n)} + \alpha\lim_{n\to\infty}\sum_{i=1}^{n}|x-x_i|^{-1}d_i^{(n)} + \alpha K\varrho \qquad (4.23)$$

Remark 4.1

The advantage of the boundary collocation method to get differentiable solutions is consequently used in the construction of the single terms of the representation of solutions (4.22) and (4.23).

Remark 4.2

The presented method allowed the above-constructed system of differential equations to be split and the numerical solution to be attributed to the research of two simple boundary value problems (for h and $K\varrho$). Now these boundary value problems can be easily computed by the help of boundary collocation methods or other approximative methods in dependence of the existing software. It is also possible to use the obtained expressions of the relations (4.22) and (4.23) for an approximative solution of system (4.5).

Now let us consider the case in which $\varkappa=\varkappa(x)$ is a smooth function of the class $C^1(G)$. The coefficients ε and μ are for the present constants. Therefore α is not a constant, it is also a smooth function of the class $C^1(G)$.
By assumption of an existing inverse operator to the boundary operator $\gamma_0 T\alpha F$ in the pair of Banach spaces $[\mathrm{Im}\tilde{Q}\cap L^2_Q(\Gamma), \mathrm{Im}\tilde{P}\cap L^2_Q(\Gamma)]$ it follows for the analytic quaternionic functions Φ_E and Φ_H

$$\begin{aligned}\Phi_H &= Fh\\ \Phi_E &= F(\gamma_0 T\alpha F)^{-1}(\tilde{Q}h - \gamma_0(T\alpha T\varrho))\end{aligned}$$

and hence for E and H

$$E = F(\gamma_0 T\alpha F)^{-1}\tilde{Q}h + (I - F(\gamma_0 T\alpha F)^{-1}\gamma_0 T\alpha)T\varrho \tag{4.24}$$

$$H = T\alpha F(\gamma_0 T\alpha F)^{-1}\tilde{Q}h + Fh + T\alpha(I - F(\gamma_0 T\alpha F)^{-1}\gamma_0 T\alpha)T\varrho \tag{4.25}$$

It remains to show the existence of the operator $(\gamma_0 T\alpha F)^{-1}$. For this purpose, we construct by help of the scalar product in the space $L^2_Q(G)$

$$\langle \nabla\circ\alpha\nabla\circ u, u\rangle_{L^2_Q} = \int_G \overline{\nabla\circ\alpha\nabla\circ u}\circ u\, dG \qquad , u\in \overset{\circ}{W}{}^1_{2,Q}(G).$$

By using the unit quaternions e_i $i = 0,1,2,3$ holds

$$\nabla\circ\alpha\nabla\circ u = \sum_{i=1}^{3} e_i\frac{\partial}{\partial x_i}\Big(\alpha\sum_{j=1}^{3} e_j\frac{\partial}{\partial x_j}\Big)\sum_{k=0}^{3} u_k e_k \tag{4.26}$$

It follows

$$\begin{aligned}\int_G \overline{\nabla\circ\alpha\nabla\circ u}\circ u\, dG &= \int_G \overline{\Big(\sum_{i,j=1}^{3}\sum_{k=0}^{3} e_i e_j e_k\frac{\partial}{\partial x_i}\alpha\frac{\partial}{\partial x_j}u_k}\sum_{l=0}^{3} u_l e_l\, dG =\\ &= -\int_G \Big(\alpha\sum_{i,j=1}^{3}\sum_{k=0}^{3}\bar{e}_k\bar{e}_j\bar{e}_i\frac{\partial}{\partial x_j}u_k\sum_{l=0}^{3}\frac{\partial}{\partial x_i}u_l e_l\Big)\, dG =\\ &= -\int_G \alpha\sum_{\substack{j=1\\k=0}}^{3}\bar{e}_k e_j\frac{\partial}{\partial x_j}u_k\circ\nabla\circ u\, dG = \int_G \alpha\overline{\nabla u}\circ\nabla u\, dG = \int_G \alpha|\nabla u|^2 dG \geq 0 .\end{aligned}$$

It is easy to obtain that from $\int_G \overline{\nabla\circ\alpha\nabla\circ u}\circ u\, dG = 0$ follows $\nabla\circ u = 0$ and finally

$$\langle \nabla\circ\alpha\nabla\circ u, u\rangle \geq \min_x \alpha\int_G |\nabla u|^2\, dG \geq \min_x \alpha\,\lambda_1(\Delta)\|u\|^2_{L_Q(G)} . \tag{4.27}$$

Inequality (4.27) shows positive definiteness of the operator $(\nabla\circ\alpha\nabla, \gamma_0)$ in $\overset{\circ}{W}{}^1_{2,Q}(G)$. Hence, we have proved the existence of a solution u of the boundary value problem

$$\begin{aligned} \nabla\circ\alpha\nabla\circ u &= 0 \quad \text{in } G \\ \gamma_0 u &= g \quad \text{on } \Gamma \quad . \end{aligned} \tag{4.28}$$

By applying the multidimensional Vekua's theory [8] [12] it follows from (4.28)

$$u = Fg + T\alpha^{-1}\phi ,$$

where $\phi \in \ker \nabla\circ$. This representation leads to the formula

$$\tilde{Q}g = T\alpha^{-1}F\phi .$$

Now the existence of the operator $(T\alpha^{-1}F)^{-1} \in L(L^2_Q \cap \mathrm{Im}\tilde{Q}, L^2_Q \cap \mathrm{Im}\tilde{P})$ is obvious, because for all $g \in L^2_Q$ a quaternionic function $\phi \in \ker \nabla\circ$ can be found. By setting $g \equiv 0$ on Γ we obtain

$$0 = \gamma_0 u = \gamma_0 T\alpha^{-1}F\gamma_0\phi \quad , \quad \gamma_0\phi \in \mathrm{Im}\tilde{P}$$

and because of the uniquness of the solution of (4.28) $T\alpha^{-1}F\phi = 0$ and so $\phi = 0$.

Let us now consider the case, if $\varepsilon = \varepsilon(x)$ and $\mu = \mu(x)$ are scalar functions which depend on x . First we shall obtain special solutions of the system of differential equations by using the following iteration method

$$E_n = T\varrho + T\mathrm{Im}(a\circ E_{n-1}) \tag{4.29}$$

$$H_m = T\alpha E_* + T\mathrm{Im}(b\circ H_{m-1}) \quad , \tag{4.30}$$

where $E_* = \lim_{n\to\infty} E_n$. For proving the convergence of this method let us make the following calculations. It is true

$$E_n - E_{n-1} = T\mathrm{Im}\, a\circ(E_{n-1} - E_{n-2}) = (T\mathrm{Im}\, a\circ)^{n-1}T \quad \text{if} \quad E_0 = 0,$$

therefore

$$E_n = \sum_{k=1}^{n} (E_k - E_{k-1}) = I + T\mathrm{Im}\, a\circ + \ldots + (T\mathrm{Im}\, a\circ)^{n-1}\, T\varrho$$

Assuming $\|T\mathrm{Im}\, a\circ\|_{L^2_Q(G)} \le q < 1$ existed the infinite sum

$$E_* = \sum_{k=0}^{\infty} (T\mathrm{Im}\, a\circ)^k T\varrho \quad . \tag{4.31}$$

Accordingly, holds by setting $H_0 = 0$

$$H_m - H_{m-1} = (T\mathrm{Im}\, b\circ)^{m-1}(H_1 - H_0) = (T\mathrm{Im}\, b\circ)^{m-1} T\alpha E_*$$

and for $\|T\mathrm{Im}\, b\circ\|_{L^2_Q(G)} \le q < 1$ it follows

$$H_* = \lim_{m\to\infty} H_m = \sum_{k=0}^{\infty} (T\mathrm{Im} b\circ)^k T\alpha E_* \quad . \tag{4.32}$$

The pair (E_*, H_*) fulfils the systems of differential equations (4.4). By applying the multidimensional generalized Vekua's theory we immediatly obtain the general solution

$$E_{**} = T\,(\varrho + \mathrm{Im}(a\circ E_*)\,) + \phi_{E_*}$$

$$H_{**} = T\alpha T\varrho + T(\alpha T\mathrm{Im}(a\circ E_*) + \alpha\phi_{E_*} + \mathrm{Im}\,(b\circ H_*)\,) + \phi_{H_*}^{\times} .$$

With the abbreviations

$$\phi_h = (T\alpha T)^{-1}\tilde{Q}h \quad , \quad \Pi_{1,\alpha} = F(\gamma_0 T\alpha F)^{-1}\gamma_0\, T \qquad \Pi_{1,\alpha} = I - \Pi_{2,\alpha}\alpha$$

we get to the formulas (4.33) - (4.34)

$$E_{**} = \Pi_{1,\alpha} T\varrho + \phi_h - \Pi_{2,\alpha}(\alpha T\mathrm{Im}(a\circ E_*) + \mathrm{Im}(b\circ H_*)) \tag{4.33}$$

$$H_{**} = Fh + T\alpha(\,\Pi_{1,\alpha}T(\varrho + \mathrm{Im}(a\circ E_*)\,) + \Pi_{2,\alpha}\mathrm{Im}(b\circ H_*) + \phi_h \tag{4.34}$$

5. The solution of Stokes' equation

By the help of the above-introduced operator calculus, we want to prepare the solution of Stokes' system by an integral representation in such a manner, that a computation by a boundary collocation method is easy to do. Stokes' system means [15]

$$\Delta u - \nabla p = f \quad \text{in } G \tag{5.1}$$

$$\mathrm{div}\, u = g \tag{5.2}$$

$$\gamma_0 u = h \quad \text{on } \Gamma \tag{5.3}$$

The necessary condition

$$\int_G g \; dG = \int_\Gamma (n,h) \; d\Gamma \tag{5.4}$$

is true. This system describes the stationary motion of a homogeneous viscose fluid for small Reynold's numbers. Here $u = (0, \hat{u})$, $\hat{u} = (u_1, u_2, u_3)$ mean the velocity of the fluid and p the hydrostatic pressure. Function g is a measure for the compressibility of the fluid. In the case $g = 0$ the fluid is not compressible. The boundary condition (5.3) means an adhesion at the boundary of the domain for $h = 0$. A detailed dicussion of the references is given by A. VALLI [15] and in the book of O.A. LADYZENSKAJA [9]

The generalized Vekua's theory gives us the possibility to write equation (5.1) in the form

$$u = T p + \phi_2 + T\phi_1 + T^2 f \quad , \quad \phi_1 \quad \ker \nabla\circ \tag{5.5}$$

By putting (5.5) in (5.2) we receive

$$\begin{aligned} g &= \operatorname{div} u = \operatorname{div} T p + \operatorname{div} \operatorname{Im}\phi_2 + \operatorname{div} \operatorname{Im} T\phi_1 + \operatorname{div} \operatorname{Im} T^2 f = \\ &= - \operatorname{Re} \nabla\circ T p - \operatorname{Re} \nabla\circ\phi_2 - \operatorname{Re} \nabla\circ T\phi_1 - \operatorname{Re} \nabla\circ T^2 f = \\ &= - p - \operatorname{Re}(\phi_1 + Tf) \; . \end{aligned}$$

We obtain for p

$$p = -(g + \operatorname{Re}(\phi_1 + Tf)) \tag{5.6}$$

By setting the expression (5.6) in (5.5) we get to a representation for u

$$u = - T g + T \operatorname{Im}\phi_1 + T \operatorname{Im} Tf + \phi_2 \tag{5.7}$$

It is easy to see, that a spezial solution is given by

$$(u_s, p_s) = (-Tg + T\operatorname{Im} Tf, -g - \operatorname{Re} Tf) \; . \tag{5.8}$$

With $v = u - u_s$ and $q = p - p_s$ it follows

$$\Delta v - \nabla q = 0 \quad \text{in } G \tag{5.9}$$

$$\operatorname{div} v = 0 \tag{5.10}$$

$$\gamma_0 v = H \quad \text{on } \Gamma \tag{5.11}$$

where $H = h + \gamma_0 T(\mathrm{Im}\, Tf - g)$. It is well-known that for solvability the condition

$$\int_\Gamma (H,n)\, d\Gamma = 0 \tag{5.12}$$

is necessary, where $(H,n) = H_1 n_1 + H_2 n_2 + H_3 n_3$.

This condition can be formulated in terms of h and g in the following way :

$$0 = \int_\Gamma (H,n)\, d\Gamma = \int_\Gamma (h,n)\, d\Gamma + \int_\Gamma \gamma_0 T\mathrm{Im}\, Tf\, d\Gamma - \int_\Gamma \gamma_0 Tg\, d\Gamma .$$

By using the theorem of Gauss-Ostrogradski we get

$$\int_\Gamma \gamma_0 T\mathrm{Im}\, Tf\, d = \int_G \mathrm{div}\, T\mathrm{Im}\, Tf\, dG = 0$$

and

$$\int_\Gamma \gamma_0\, Tg\, d = \int_G g\, dG$$

and therefore

$$\int_G g\, dG = \int_\Gamma (h,n)\, d\Gamma . \tag{5.13}$$

For that reason, the condition (5.13) is necessary for solution (see [15]). Let us now solve the new problem (5.9)-(5.10)-(5.11) . By using (5.6) and (5.7) we obtain

$$q = -g - \mathrm{Re}\,\Phi_1 \tag{5.14}$$

$$v = T\mathrm{Im}\,\Phi_1 + \Phi_2 \tag{5.15}$$

We shall approximate the function Φ_1 by

$$\Phi_1^{(n)} = \sum_{i=1}^{n} r(x-x_i) \cdot a_i \quad , \tag{5.16}$$

where $r(x-x_i) = r_0(x-x_i)$. With the notations

$$r^{(j)} = \frac{x^{(j)} - x_i^{(j)}}{|x - x_i|^3} \qquad j = 1,2,3$$

finally follows

$$T\mathrm{Im}\,\Phi_1^{(n)} = \sum_{i=1}^{n} \Big\{ (Tr^{(1)} e_1)\, a_i^{(0)} + (Tr^{(2)} e_1)\, a_i^{(3)} - (Tr^{(3)} e_1)\, a_i^{(2)} +$$

$$(Tr^{(2)}e_2)\, a_i^{(0)} + (Tr^{(3)}e_2)\, a_i^{(1)} - (Tr^{(1)}e_2)\, a_i^{(3)} +$$

$$+(Tr^{(3)}e_3)\, a_i^{(0)} + (Tr^{(1)}e_3)\, a_i^{(2)} - (Tr^{(2)}e_3)\, a_i^{(1)} \Big\} \qquad (5.17)$$

The functions $Tr^{(i)}e_j$ can be calculated analogously to the paper [14 , p. 281]. The calculation is made with accuracy of a function, which belongs to ker $\nabla\circ$, This analytic function shall be added to Φ_2 and the resulting function is wanted by the ansatz

$$\Phi_2^{(m)} = \sum_{i=1}^{m} r(x-x_i)\circ c_i \qquad (5.18)$$

where c_i are quaternionic constants. These shall be defined by equations, which arise by putting some collocation points for x in the formula

$$\sum_{i=1}^{m} r(x-x_i)\circ c_i = (Fh)(x) \qquad (5.19)$$

For instance, $(Fh)(x)$ can be calculated by (4.15). We obtain the constants $a_i^{(j)}$ by using the boundary value condition (5.10) from the system of equations

$$(TIm\, \Phi_1^{(n)})(z_j) = (\widehat{Q}H)(z_j) \qquad j = 1,\ldots,4n \quad ,$$

where $(QH)(x)$ is represented by (4.16) . Consequently the approximative solution of Stokes' system (5.9)-(5.10)-(5.11) is given by

$$v^{n,m} = TIm\, \Phi_1^{(n)} + \sum_{i=1}^{m} (0,(x-x_i)|x-x_i|^{-3})\circ c_i \qquad (5.20)$$

and

$$q^{(n)} = -g + \sum_{i=1}^{n}\sum_{j=1}^{3} r^{(j)}\, a_i^{(j)} \qquad (5.21)$$

REFERENCES

[1] BIZADSE A.W. "Grundlagen der Theorie der analytischen Funktionen, DVW, Berlin ,(1973)

[2] CATTABRIGA L. " Su un probleme al contorno relativo al sistema di equazioni di Stokes", Rend.Sem.Mat. Univ. Padova 31 (1961) pp. 308 - 340

[3] GÜRLEBECK K. " Über die optimale Interpolation verallgemeinert analytischer Funktionen und ihre Anwendung zur näherungsweisen Lösung wichtiger Randwertaufgaben der mathematischen Physik, Dissertation (A) Techn. Hochsch. Karl-Marx-Stadt ,(1984)

[4] GÜRLEBECK K. " Über Interpolation und Approximation verallgemeinert analytischer Funktionen", Techn. Hochsch. Karl-Marx-Stadt, Wiss. Informationen 34, (1982)
[5] GÜRLEBECK K. " Hypercomplex Factorization of the Helmholtz equation, ZAA ,(in print)
[6] GÜRLEBECK K./SCHÜRER A. / SPRÖSSIG W. "Application of Boundary Collocation Methods in Physics and Engineering" ,WZ der Techn. Hochsch. Karl-Marx-Stadt, 28 (1986) H.2 , 168-174
[7] GÜRLEBECK K./SPRÖSSIG W./TASCHE M. "Numerical realization of boundary collocation methods" , Int. Ser. of Numerical Math., Vol. 75 (1985), Birkhäuser Verlag Basel , 206-217
[8] IFTIMIE V. Operateurs du type de Moisil-Teodorescu, Bull. Math. de la Royal Soc. de Roumanie 10 (58), (1966), 271-305
[9] LADYZENSKAJA, O.A. " The mathematical theory of viscous incompressible flow" , 2nd english ed., Gordon and Breach, New York-London-Paris (1969)
[10] LÖSCH U. , Diplomarbeit , Techn. Hochsch. Karl-Marx-Stadt,(1986)
[11] SPRÖSSIG W. "Über eine mehrdimensionale Operatorrechnung mit Operatoren über beschränkten Gebieten des Euklidischen Raumes und deren Anwendung zur Lösung partieller Differentialgleichungen, Dissertation (B), Techn. Hochsch. Karl-Marx-Stadt, (1979)
[12] SPRÖSSIG W. "Räumliches Analogon zum komplexen T-Operator, Beiträge zur Analysis 12 (1978) 127-137
[13] SPRÖSSIG W. "Anwendung der analytischen Theorie der Quaternionen zur Lösung räumlicher Probleme der linearen Elastizität in beschränkten Gebieten, ZAMM 59 , 741-743 (1979)
[14] SPRÖSSIG W. /GÜRLEBECK K. " A hypercomplex method of calculating stresses in three-dimensional bodies, Rend. Circ. Mat. Palermo (2) 6 (1984) 271-284
[15] VALLI A. "On the Integral Representation of the Solution to the Stokes System, Rend. Sem. Mat. Univ. Padova, Vol. 74 (1985)

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