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CHARACTERS OF THE IRREDUCIBLE HIGHEST WEIGHT MODULES
OVER THE VIRASORO AND SUPER-VIRASORO ALGEBRAS

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ABSTRACT - The characters of all irreducible highest weight modules over the Virasoro algebra and the $N = 1$ super-Virasoro algebras (i.e. Neveu-Schwarz and Ramond superalgebras) are given in an explicit and unified for all algebras form incorporating all previously known results. Extensions to the $N = 2$ super-Virasoro case and to the super-Kac-Moody algebra $osp(2,2)^{(1)}$ are discussed.

1. INTRODUCTION

This paper is a natural continuation of our previous papers [8],[10] on the structure of the reducible (and indecomposable) highest weight modules (HWM) over the Virasoro algebra ($\hat{W}$) and Neveu-Schwarz ($\hat{S}$) and Ramond ($\hat{R}$) superalgebras.

In [8] we gave an explicit parametrization of the Feigin-Fuchs [15] multiplet classification of the reducible HWM over $\hat{W}$. In [10] we presented the multiplet classification of the reducible HWM over $\hat{S}$ and $\hat{R}$. From the results of [15] (the details are given in [16]) and [10] it is clear that this is also a parametrization of the irreducible HWMs obtained by the factorization of the largest proper submodules of the corresponding Verma modules.

Thus the next step would be to present explicit formulae for the characters of the irreducible HWM. An additional motivation to do this were the papers on the Virasoro algebra by Rocha-Caridi [33] and Feigin and Fuchs [16] which became available to us after [8],[10]. Rocha-Caridi gives the general character formulae and explicit character formulae for the discrete series [17] of unitarizable HWM with central charge $c < 1$. (Some of the latter formulae were recast in simple multiplicative expressions in [29].) Feigin and Fuchs give the character formulae for a more general case (than the unitarizable) in a very unexplicit parametrization. We were further motivated by the applications of character formulae in two-dimensional conformal invariant theories [25],[20],[4],[38] and furthermore that character formulae for the nonunitary cases are also relevant for physics [36],[24].

Our aim in this paper is to give explicit and simple expressions for the characters of all irreducible HWM over $\hat{W}$, $\hat{S}$ and $\hat{R}$. For $\hat{W}$ this amounts almost

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This paper is in final form and no version of it will be submitted for publication elsewhe

to an application of our very explicit results from [8] to the general formulae [16],[33]. (We should mention of course that simple expressions were already available besides [29] for the cases $c = 1$ [26],[27], $c = 0$ [28],[34],[35], $c = 25,26$ [34],[35].) However, because of the explicit parametrization in [8] we can also simplify the general derivation in some cases. Further, to derive the general character formulae for the $\hat{S}$ and $\hat{R}$ cases we use the results of [10] and their striking similarity with the $\hat{W}$-case. Here most formulae are new. (Partial results which appeared in [27],[29],[23] are commented below.) This similarity enables us also to present all character formulae in a unified fashion which is an additional advantage of our results.

The organization of the paper is as follows. In Sec. 2.1, we present the necessary definitions and introduce our notation. In Sec. 2.2 we recall the results on the multiplet classification of [8],[10]. In Sec. 3 we present the character formulae. In Sec. 4 we formulate our results as a theorem. Then we comment on current and further research for the so-called $N = 2$ extensions of the Virasoro algebra and for the super-Kac-Moody algebra $osp(2,2)^{(1)}$.

2. PRELIMINARIES

2.1 Definitions and notation

The Virasoro algebra [21] $\hat{W}$ is a complex Lie algebra with basis z, L_i, $i \in \mathbb{Z}$ and nontrivial Lie bracket

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{z}{12}(i^3 - i)\delta_{i,-j} \quad , \tag{1}$$

while z belongs to the centre of $\hat{W}$.

The Neveu-Schwarz superalgebra [31] $\hat{S}$ is a complex Lie superalgebra with basis z, L_i, $i \in \mathbb{Z}$, J_α, $\alpha \in \mathbb{Z} + \frac{1}{2}$ and Lie brackets

$$[L_i, L_j] = (i-j)L_{i+j} + \frac{z}{8}(i^3 - i)\delta_{i,-j} \quad , \tag{2a}$$

$$[J_\alpha, J_\beta]_+ = 2L_{\alpha+\beta} + \frac{z}{2}(\alpha^2 - 1/4)\delta_{\alpha,-\beta} \quad , \tag{2b}$$

$$[L_i, J_\alpha] = \left(\frac{i}{2} - \alpha\right) J_{i+\alpha} \quad , \tag{2c}$$

while z belongs to the centre of $\hat{S}$.

The Ramond superalgebra [32] $\hat{R}$ is a complex Lie superalgebra with basis and Lie brackets as for $\hat{S}$, however, $\alpha \in \mathbb{Z}$. Further, $\hat{Q}$ will denote $\hat{W}$, $\hat{S}$ or $\hat{R}$ when a statement holds for all three algebras. The elements z, L_i are even and J_α are odd. The grading of $\hat{Q}$ is given by $\deg z = 0$, $\deg L_i = i$, $\deg J_\alpha = \alpha$. We have the obvious decomposition

$$\hat{Q} = \hat{Q}_+ \oplus \hat{Q}_0 \oplus \hat{Q}_- \tag{3}$$

where $\hat{W}_\pm$ are generated by $L_{\pm i}$, $i = 1,2.$, $\hat{S}_\pm$ are generated by $J_{\pm a}$, $\alpha = \frac{1}{2}, \frac{3}{2}$, $\hat{R}_\pm$ are generated by $L_{\pm 1}$, $J_{\pm 1}$; $\hat{W}_0$ and $\hat{S}_0$ are complexly spanned by L_0 and z ; $\hat{R}_0$ is complexely spanned by z, L_0 and J_0 which are related, however, by (2b): $L_0 = (J_0)^2 + z/16$.

A highest weight module (HWM) over $\hat{Q}$ is characterized by its highest weight $\lambda \in Q_0^*$ and highest weight vector v_0 such that

$$L_i v_0 = 0 , \quad i > 0 , \quad \text{for } \hat{Q} , \tag{4a}$$

$$J_\alpha v_0 = 0 , \quad \alpha > 0 , \quad \text{for } \hat{S}, \hat{R} , \tag{4b}$$

$$L_0 v_0 = \lambda(L_0) v_0 = h v_0 , \quad \text{for } \hat{Q} , \quad h \in \mathbb{C} , \tag{5}$$

$$z v_0 = \lambda(z) v_0 = c v_0 , \quad \text{for } \hat{Q} , \quad c \in \mathbb{C} . \tag{6}$$

Further we shall work with the largest HWM with highest weight λ - the so-called Verma module $V^\lambda = V^{h,c}$. (Any other HWM with hw λ may be obtained as a factor module of $V^{h,c}$.) We know (see [26],[27] for $\hat{W}$, $\hat{S}$ and [18] for $\hat{R}$) that the HWM $V^{h,c}$ is reducible if and only if h and c are related as follows:

$$h = h_{(n,m)} := h_0 + \frac{1}{4}(\alpha_+ n + \alpha_- m)^2 + \frac{1}{8}\mu \tag{7}$$

where $m,n \in \frac{1}{\nu}\mathbb{N}$,

$$\nu \equiv \begin{cases} 1 & \text{for } \hat{W} \\ 2 & \text{for } \hat{S}, \hat{R} , \end{cases}$$

$$m - n \in \mathbb{Z} + \mu \quad , \text{for } \hat{Q} ,$$

$$\mu \equiv \begin{cases} 0 & \text{for } \hat{W}, \hat{S} \\ 1/2 & \text{for } \hat{R} \end{cases} ,$$

$$h_0 = \begin{cases} (c-1)/24 & \text{for } \hat{W} , \\ (c-1)/16 & \text{for } \hat{S}, \hat{R} , \end{cases}$$

$$\alpha_\pm = \begin{cases} (\sqrt{1-c} \pm \sqrt{25-c})/\sqrt{24} & , \text{for } \hat{W} , \\ (\sqrt{1-c} \pm \sqrt{9-c})/2 & , \text{for } \hat{S}, \hat{R} . \end{cases} \tag{8}$$

We know that the reducible $V = V^{h,c}$, $h = h_{(n,m)}$, contains a proper submodule isomorphic to the HWM $V' = V^{k+\nu nm,c}$. The latter means that there exists a non-trivial invariant embedding map between V and V'. Equivalently we shall say that V and V' are partially equivalent. The reducible (and in some cases some irreducible) HWM are grouped into multiplets by the notion of partial equivalence. (The notion of a multiplet was introduced in [6],[9] in the context of real semi-simple Lie groups and algebras and was later generalized to infinite-dimensional Lie (super)-algebras [7],[8],[10] and to Lie superalgebras [14].) A set $\mathcal{M}$ of HWM over a complex (infinite-dimensional) (super) Lie algebra is said to form a <u>multiplet</u> if: (1) $V \in \mathcal{M} \to \mathcal{M} \supset \mathcal{M}_V \neq \emptyset$, where $\mathcal{M}_V$ is the set of all HWM $V' \neq V$ and partially equivalent to V; (2) $\mathcal{M}$ does not have proper subsets fulfilling (1).

It is useful to represent a multiplet $\mathcal{M}$ by a connected oriented graph. The vertices of the graph correspond to the HWM of $\mathcal{M}$ and the arrows connecting the vertices correspond one-to-one to those embedding maps which are not compositions of other embedding maps [7],[8],[10]. As earlier we use the convention that $V \to V'$ mean that V' can be invariantly embedded in V (arrows point <u>to</u> the embedded HWM.) The multiplets represented by the same graph are said to belong to one and the same <u>type of multiplets</u> [6] and then one needs parametrization to distinguish the multiplets belonging to a fixed type. In some cases, as below, it is convenient to introduce subtypes when there is no convenient parametrization for the whole type. (That is a reflection of some finer differences in the structure of the representations.)

2.2 <u>Summary of the multiplet classification of the reducible HWM</u>

Further we recall the results on the multiplet classification of the reducible HWM for $\hat{W}$ [15] in the form given in [8] and for $\hat{S}$ and $\hat{R}$ [10]. There are five types in each case. For $\hat{S}$ and $\hat{R}$ the (sub)types are denoted by N^0, $N^1_\pm$, $N^{21}_\pm$, $N^{22}_\pm$, $N^{23}_\pm$ ($N^{2K}_\pm$ are subtypes or $N^2_\pm$) and the corresponding (sub)types for $\hat{W}$ are II, $III_\pm$, $III^0_\pm$, $III^{00}_\pm \begin{pmatrix} c>25 \\ c<1 \end{pmatrix}$, $III^{00}_\pm \begin{pmatrix} c=25 \\ c=1 \end{pmatrix}$ is the notation of [15]. The types II, N^0 occur when α_-/α_+ (see (8)) is not a real rational number. In all other cases α_-/α_+ is a real rational number and <u>either</u> $c \leq 1$ <u>or</u> $c \geq 25$ for $\hat{W}$, $c \geq 9$ for $\hat{S}$, $\hat{R}$. In the case $c < 1$ we have the (sub) types III_-, III^0_-, III^{00}_-, N^1_-, N^{21}_-, N^{22}_- and if $-\alpha_-/\alpha_+ = p/q$, $p\ q \in \mathbb{N}$, then in all cases:

$$c = 13 - 6\left(\frac{p}{q} + \frac{q}{p}\right)\ ,\ \text{for}\ \hat{W}\ , \qquad c = 5 - 2\left(\frac{p}{q} + \frac{q}{p}\right)\ ,\ \text{for}\ \hat{S}, \hat{R}\ . \tag{9a}$$

In the case $c > 25$ for $\hat{W}$ we have the subtypes III_+, III^0_+, III^{00}_+ and for $c > 9$ for $\hat{S}$, $\hat{R}$ we have the subtypes N^1_+, N^{21}_+, N^{22}_+; if $\alpha_-/\alpha_+ = p/q$, $p,q \in \mathbb{N}$ then in all cases

$$c = 13 + 6\left(\frac{p}{q} + \frac{q}{p}\right)\ ,\ \text{for}\ \hat{W}\ ,\ c = 5 + 2\left(\frac{p}{q} + \frac{q}{p}\right)\ ,\ \text{for}\ \hat{S}, \hat{R}\ . \tag{9b}$$

In the case $c = 1$ we have the subtypes III^{00}_-, N^{23}_- and for $c = 25$ (for $\hat{W}$)-III^{00}_+, $c = 9$ (for $\hat{S}$,$\hat{R}$) - N^{23}_+.

Now we give the explicit parametrization of all types (for $\hat{W}$ see [15] and Propositions 1.1-1.5, formulae (6),(13),(16) of [8] and for $\hat{S}$ and $\hat{R}$, Propositions 1-4 and formulae (6),(11),(14) of [10]):

$$\mathrm{II}, N^0 : \{c, n, m\},\ c \in \mathbb{C},\quad \alpha_-/\alpha_+ \text{ not a real rational number, } n,m \text{ as in (8);} \tag{10}$$

$$\mathrm{III}_\pm, N^1_\pm : \{p, q, m, n\},\ p,q \in \mathbb{N},\quad m,n \in \tfrac{1}{\nu}\mathbb{N}, \quad {*1)} \tag{11a}$$

such that

$$p<q,\ pXq,\ m-n \in \mathbb{Z}+\mu,\ pm > qn,\ m<\tilde{q}, \tag{11b}$$

where

$$\tilde{q} \equiv \begin{cases} q/2 \text{ if both } p \text{ and } q \text{ are odd for } \hat{S} \text{ and } \hat{R} \\ q \text{ otherwise} \end{cases}, \tag{11c}$$

(pXq means that p and q have no common divisor);

$$\underline{\mathrm{III}}^0_\pm, N^{21}_\pm : \{p, q, n\},\quad p,q \in \mathbb{N},\quad n \in \tfrac{1}{\nu}\mathbb{Z}_+, \quad {*2)} \tag{12a}$$

such that

$$0<n<\tilde{p}+\tilde{q},\ n-\tilde{p} \in \mathbb{Z}+\mu;\ 0=n<p<q,\ \tilde{p}=p(1-\mu);\ pXq; \tag{12b}$$

where $\tilde{p}$ is defined as $\tilde{q}$ in (11c);

$$\underline{\mathrm{III}}^{00}_\pm\binom{c>25}{c<1}, N^{22}_\pm : \{p, q\}, p,q \in \mathbb{N}, pXq,\quad \text{for } \hat{R}, \text{ either } p \text{ or } q \text{ is even;} \tag{13}$$

$$\underline{\mathrm{III}}^{00}_\pm\binom{c=25}{c=1} : \{\varepsilon\},\quad \varepsilon = 0,1\ ; \tag{14a}$$

$N^{22}_\pm\binom{c=9}{c=1}$: one type for $\hat{S}$ and one for $\hat{R}$ for each sign. (14b)

Note that in each case the types with $c \leqslant 1$ have the same parametrization as those with $c \geqslant 25$ for $\hat{W}$, $c \geqslant 9$ for $\hat{S}$, $\hat{R}$.

3. CHARACTER FORMULAE

3.1 Generalities

First we recall the weight space (or level) decomposition of $V^{h,c}$

$$V^{h,c} = \bigoplus_j V^{h,c}_j,\quad j \in \mathbb{Z}_+, \text{ for } \hat{W}, \hat{R},\quad j \in \tfrac{1}{2}\mathbb{Z}_+, \text{ for } \hat{S}, \tag{15a}$$

where the weight spaces $V^{h,c}_j$ are defined as the eigenspaces of L_0

$$V^{h,c}_j \equiv \{ v \in V^{h,c} \mid L_0 v = (h+j)v \}\ . \tag{15b}$$

($V_0^{h,c}$ is spanned by the highest weight vector v_0.) The character of $V^{h,c}$ is defined (cf. [27]) by

$$\mathrm{ch}\, V^{h,c}(t) = \sum_j \left(\dim V_j^{h,c}\right) t^{h+j} = t^h \sum_j p(j)\, t^j = t^h \psi(t) , \tag{16a}$$

where $p(j)$ is the partition function and $\psi(t)$ is given by [27],[18]

$$\psi(t) \equiv \begin{cases} \prod_{k \in N} (1-t^k)^{-1} , & \text{for } \hat{W} , \\ \prod_{k \in N} (1+t^{k-1/2})/(1-t^k) , & \text{for } \hat{S} , \\ \prod_{k \in N} (1+t^k)/(1-t^k) , & \text{for } \hat{R} . \end{cases} \tag{16b}$$

Next let $L^{h,c}$ denote the unique irreducible quotient of $V^{h,c}$, $L^{h,c} = V^{h,c}/I^{h,c}$, where $I^{h,c}$ is the maximal proper submodule of $V^{h,c}$. If $V^{h,c}$ is irreducible $L^{h,c} = V^{h,c}$. If $V^{h,c}$ is not irreducible the character formula $L^{h,c}$ is more complicated than (16a).

Further we shall recall the explicit parametrization [8],[10] of all reducible $V^{h,c}$ and then we shall derive the corresponding character formulae using the results of [16],[33] for $\hat{W}$ and of [10] for $\hat{S}$ and $\hat{R}$. The multiplets of types II, N^0 can be depicted as follows [15],[10]

$$V^{h,c} \longrightarrow V^{h+\nu nm,c} , \tag{17}$$

with $h = h_{(n,m)}$ given by (7), the embedded HWM being irreducible. Then using the results of [33] we obtain:

$$\begin{aligned} \mathrm{ch}\, L^{h,c} &= \mathrm{ch}\, V^{h,c} - \mathrm{ch}\, L^{h+\nu nm,c} = t^h (1-t^{\nu nm}) \psi(t) = \\ &= (1-t^{\nu nm})\, \mathrm{ch}\, V^{h,c} . \end{aligned} \tag{18}$$

3.2 The character formulae for the discrete cases $c \leq 1$

Before we continue we recall [8],[10] that for $c < 1$ and $-\alpha_-/\alpha_+ = p/q$, h from (7) is expressed as follows

$$h_{(n,m)} \equiv \frac{1}{4\nu pq} \left[\nu^2 (pm-qn)^2 - (p-q)^2 \right] + \tfrac{1}{8}\mu = h_{(\tilde{p}-n,\, \tilde{q}-m)} . \tag{19}$$

The HWM in the multiplets of types III_-, N_-^1 form 4 groups given as follows (cf. [8] Eq. (10), [10] Eq. (10)) with $h = h_{(n,m)}$ and c as in (9a)):

$$V_{0k} \equiv V^{h+\nu k(\tilde{p}\tilde{q}k + \tilde{q}n - \tilde{p}m),\, c} , \tag{20a}$$

$$h + \nu K(\tilde{p}\tilde{q}K + \tilde{q}n - \tilde{p}m) = h_{(2\tilde{p}K+n,\, m)} = h_{(\tilde{p}-n,\, 2\tilde{q}K+\tilde{q}-m)} \quad ;$$

$$V'_{0K} \equiv V^{h+\nu(\tilde{q}K+m)(\tilde{p}K+n),\, c} \quad ,$$
$$h + \nu(\tilde{q}K+m)(\tilde{p}K+n) = h_{(2\tilde{p}K+\tilde{p}+n,\, \tilde{q}-m)} = h_{(\tilde{p}-n,\, 2\tilde{q}K+\tilde{q}+m)} \quad ; \tag{20b}$$

$$V_{1K} \equiv V^{h+\nu K(\tilde{p}\tilde{q}K - \tilde{q}n + \tilde{p}m),\, c} \quad ,$$
$$h + \nu K(\tilde{p}\tilde{q}K - \tilde{q}n + \tilde{p}m) = h_{(n,\, 2\tilde{q}K+m)} = h_{(2\tilde{p}K+\tilde{p}-n,\, \tilde{q}-m)} \quad ; \tag{20c}$$

$$V'_{1K} \equiv V^{h+\nu(\tilde{q}K+\tilde{q}-m)(\tilde{p}K+\tilde{p}-n),\, c} \quad ,$$
$$h + \nu(\tilde{q}K+\tilde{q}-m)(\tilde{p}K+\tilde{p}-n) = h_{(n,\, 2\tilde{q}K+2\tilde{q}-m)} = h_{(2\tilde{p}K+2\tilde{p}-n,\, m)} \quad . \tag{20d}$$

These four groups are arranged in the multiplet represented by the following commutative diagram (cf. [15] and formulae (8) of [8],[10])

$$\begin{tikzcd}
 & V'_{00} \arrow[r] \arrow[ddr] & \cdots \arrow[r] \arrow[ddr] & V_{0K} \arrow[r] \arrow[ddr] & V'_{0K} \arrow[r] \arrow[ddr] & \cdots \\
V \arrow[ur] \arrow[dr] & & & & & \\
 & V'_{10} \arrow[r] \arrow[uur] & \cdots \arrow[r] \arrow[uur] & V_{1K} \arrow[r] \arrow[uur] & V'_{1K} \arrow[r] \arrow[uur] & \cdots
\end{tikzcd} \tag{21}$$

where $V = V_{00} = V_{10}$. Using this diagram in the Virasoro case from [15] the following character formulae were derived in [33] (presented in our notation):

$$\mathrm{ch}\, L_{\ell K} = \mathrm{ch}\, V_{\ell K} + \sum_{j>K} (\mathrm{ch}\, V_{0j} + \mathrm{ch}\, V_{1j}) - \sum_{j\geq K} (\mathrm{ch}\, V'_{0j} + \mathrm{ch}\, V'_{1j}) \,, \quad \ell = 0,1 \,, \tag{22a}$$

$$\mathrm{ch}\, L'_{\ell K} = \mathrm{ch}\, V'_{\ell K} + \sum_{j>K} (\mathrm{ch}\, V'_{0j} + \mathrm{ch}\, V'_{1j} - \mathrm{ch}\, V_{0j} - \mathrm{ch}\, V_{1j}) \,, \quad \ell = 0,1 \,. \tag{22b}$$

The most important ingredient in this derivation is the fact that $V'_{0K} + V'_{1K}$ is the largest proper submodule of both V_{0K} and $V_{1K} = V_{0K} \cap V_{1K}$; analogously $V_{0K+1} + V_{1K+1} = V'_{0K} \cap V'_{1K}$. Since the character of a quotient module $L = V/I$ is $\mathrm{ch}\, L = \mathrm{ch}\, V - \mathrm{ch}\, I$ and since I is again a quotient module here, then it is clear how (22) follows. It is also clear from [10] that (22) holds also in the Neveu-Schwarz and Ramond cases.

Now making use of (20) we shall give explicit and simple formulae for the characters. We have (with $h = h_{(n,m)}$ from (19)):

$$\mathrm{ch}\, L_{0K}(t) = t^h \psi(t) \Big\{ t^{\nu K(\tilde{p}\tilde{q}K+\tilde{q}n-\tilde{p}m)} + \sum_{j>K} \big(t^{\nu j(\tilde{p}\tilde{q}j+\tilde{q}n-\tilde{p}m)} + t^{\nu j(\tilde{p}\tilde{q}j-\tilde{q}n+\tilde{p}m)} \big) -$$

$$-\sum_{j\geq K}\left(t^{\nu(\tilde q j+m)(\tilde p j+n)} + t^{\nu(\tilde q j+\tilde q-m)(\tilde p j+\tilde p-n)}\right)\Big\} = \tag{23a}$$

$$= t^h\psi(t)\Big\{\sum_{j\geq K} t^{\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)} + \sum_{j<-K} t^{\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)} -$$

$$-t^{\nu nm}\Big[\sum_{j\geq K} t^{\nu j(\tilde p\tilde q j+\tilde q n+\tilde p m)} + \sum_{j<-K} t^{\nu j(\tilde p\tilde q j+\tilde q n+\tilde p m)}\Big]\Big\} = \tag{23b}$$

$$= ch\, V_{00}(t)\Big\{\sum_{\substack{j\geq K\\ j<-K}} t^{\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)}\left(1-t^{\nu m(2\tilde p j+n)}\right)\Big\}\ , \quad (ch\, V_{00}(t) = t^h\psi(t)). \tag{23c}$$

To obtain (23b) we changed $j \to -j$ for the second term and $j+1 \to -j$ for the fourth. Analogously we obtain the formulae for the other three series (again $h = h_{(n,m)}$ from (19)):

$$ch\, L_{1K}(t) = ch\, V_{00}(t)\Big\{\sum_{\substack{j\geq K\\ j<-K}} t^{\nu j(\tilde p\tilde q j-\tilde q n+\tilde p m)}\left(1-t^{\nu n(2\tilde q j+m)}\right)\Big\}\ ; \tag{24}$$

$$ch\, L'_{0K}(t) = t^h\psi(t)\Big\{\sum_{|j|>K} t^{\nu j(\tilde p\tilde q j-\tilde q n+\tilde p m)}\left(t^{\nu(\tilde q-m)(2\tilde p j+\tilde p-n)}-1\right)\Big\} = \tag{25a}$$

$$= ch\, V_{00}(t)\Big\{\sum_{|j|>K} t^{\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)}\left(t^{\nu(\tilde p-n)(2\tilde q j+\tilde q-m)}-1\right)\Big\}\ ; \tag{25b}$$

$$ch\, L'_{1K}(t) = t^h\psi(t)\Big\{\sum_{|j|>K} t^{\nu j(\tilde p\tilde q j-\tilde q n+\tilde p m)}\left(t^{\nu n(2\tilde q j+m)}-1\right)\Big\} = \tag{26a}$$

$$= ch\, V_{00}(t)\Big\{\sum_{|j|>K} t^{\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)}\left(t^{\nu m(2\tilde p j+n)}-1\right)\Big\}\ ; \tag{26b}$$

($ch\, L_{10}(t)$ is brought to the expression for $ch\, L_{00}(t)$ by the change $j \to -j$ in the first term.)

The HWM in the multiplets of types III_-^0, N_-^{21} form two groups given as follows (cf. (15) from [8] and (13) from [10])

$$V^0_{0K} \equiv V_{0K}\Big|_{m=\tilde{q}} = V'_{0,K-1}\Big|_{\substack{m=\tilde{q}\\ K\geq 1}} = V^{h^0+\nu\tilde{q}K(\tilde{p}K+n-\tilde{p}),\,c}\,, \quad h^0 \equiv h_{(n,\tilde{q})}\,, \tag{27a}$$

$$V^0_{1K} \equiv V_{1K}\Big|_{m=\tilde{q}} = V'_{1K}\Big|_{m=\tilde{q}} = V^{h^0+\nu\tilde{q}K(\tilde{p}K-n+\tilde{p}),\,c}\,, \left(V^0_{10} = V^0_{00}\right), \tag{27b}$$

$$h^0+\nu\tilde{q}K(\tilde{p}K+n-\tilde{p}) = h_{(2\tilde{p}K+n,\tilde{q})} = h_{(\tilde{p}-n,2\tilde{q}K)} \left(= h_{(2\tilde{p}K+n-\tilde{p},0)} \text{ for } K\geq 1\right), \tag{27c}$$

$$h^0+\nu\tilde{q}K(\tilde{p}K-n+\tilde{p}) = h_{(n,(2K+1)\tilde{q})} = h_{(2\tilde{p}K+\tilde{p}-n,0)} = h_{(2\tilde{p}K+2\tilde{p}-n,\tilde{q})}\,. \tag{27d}$$

These HWM are arranged in a multiplet so that V^0_{1K} is embedded in V^0_{0K} which is then embedded in V^0_{1K-1}; thus all are embedded in $V^0_{00} = V^0_{10}$ (cf. [8](14),[10] (12)). The embedding diagram is obtained from (21) by gluing together the coinciding HWM (the arrows between them disappear in the process) and further deleting the arrows which turn out to depict composition maps in this notation (these are the maps between V^0_{1K} and V^0_{1K+1} and between V^0_{0K} and V^0_{0K+1}). (A further coincidence occurs in the case $n = 0$, then $V^0_{1K} = V^0_{0K+1}$.) We stress that this connection between different types of multiplets is available to us because of the very explicit parametrization we have for the multiplets and the HWM in them. Consequently it is much easier to derive the character formulae directly from (22). Thus from (23a) we have for $\ell = 0$ and $K > 0$:

$$\begin{aligned} \mathrm{ch}\, L^0_{0K}(t) &= \mathrm{ch}\, V^0_{0K} + \sum_{j>K}\left(\mathrm{ch}\, V^0_{0j} + \mathrm{ch}\, V^0_{1j}\right) - \sum_{j\geq K}\left(\mathrm{ch}\, V^0_{0j+1} + \mathrm{ch}\, V^0_{1j}\right) = \\ &= \mathrm{ch}\, V^0_{0K} - \mathrm{ch}\, V^0_{1K} = \psi(t)\, t^{h_{(2\tilde{p}K+n,\tilde{q})}}\left(1 - t^{2\nu\tilde{q}K(\tilde{p}-n)}\right). \end{aligned} \tag{28}$$

Analogously we have from (22b) with $\ell = 1$ (in the cases $n > 0$):

$$\begin{aligned} \mathrm{ch}\, L^0_{1K}(t) &= \mathrm{ch}\, V^0_{1K} + \sum_{j>K}\left(\mathrm{ch}\, V^0_{0j+1} - \mathrm{ch}\, V^0_{0j}\right) = \\ &= \mathrm{ch}\, V^0_{1K} - \mathrm{ch}\, V^0_{0K+1} = \psi(t)\, t^{h_{(n,(2K+1)\tilde{q})}}\left(1 - t^{\nu\tilde{q}n(2K+1)}\right). \end{aligned} \tag{29}$$

Note that we cannot use (22a) with $\ell = 1$ for $\mathrm{ch}\, L^0_{1K}$, because that formula utilizes the embedding of V'_{1K} in V_{1K}; since these coincide now we would have obtained that the character is zero. For the same reason we cannot use (22a) with $\ell = 0$, $K = 0$ and (22b) with $\ell = 0$.

The HWM in the multiplets of types $III_{-}^{00}(c < 1)$, N_{-}^{22} form just one group parametrized as follows (cf.[8](17), [10](16)):

$$V_K^0 = V_{0K}^0\Big|_{\substack{\tilde{p}\to p' \\ n=p'}} = V_{1K}^0\Big|_{\substack{\tilde{p}\to p' \\ n=p'}} = V^{h^{00}+\nu p'q'K^2,\, c} , \tag{30a}$$

$$p'(\text{resp. } q') \equiv \begin{cases} \tilde{p}\ (\text{resp. } \tilde{q}) & \text{for } \hat{W}, \hat{S} \\ p/2\ (\text{resp. } q/2) & \text{for } \hat{R} \end{cases} ;\quad h^{00} \equiv h_{(p,q)} = -\frac{(p-q)^2}{4\nu pq} + \frac{1}{8}\mu , \tag{30b}$$

$$h^{00} + \nu p'q'K^2 = h_{((2K+1)p',q')} = h_{(0,2q'K)} = h_{(2p'K,0)} = h_{(p',(2K+1)q')} . \tag{30c}$$

The HWM are arranged in a single line multiplet so that V_{K+1}^0 is embedded in V_K^0; thus all are embedded in V_0^0. The embedding diagram is obtained from the previous case by gluing together V_{0K}^0 and V_{1K}^0. Because of this it is clear that the character formula is obtained from (30) (and not from (29)):

$$ch\, L_K^0(t) = ch\, V_K^0 - ch\, V_{K+1}^0 = \psi(t)\, t^{h_{(p',(2K+1)q')}}\left(1 - t^{\nu p'q'(2K+1)}\right) . \tag{31}$$

Character formulae (23)-(26),(28),(29),(31) exhaust all cases when $V^{h,c}$ is reducible and $c < 1$. We comment now on what was known previously.

1) The Virasoro $p/q = 2/3$, (then by (9a) $c = 0$), was studied in great detail in [28],[34],[35]. (Note that since the centre is trivial the Virasoro algebra is reduced to the so-called Lie algebra of vector fields on the circle or the Witt algebra.) Formula (23)-(26), (28) and (29) (for $(m,n) = (3,1),(3,0)$) and (31) were conjectured in [28]; (23)-(26) were proved in [34], (28),(29),(31) in [35].

<u>Remark</u> (Not connected with character formulae.) In the Virasoro case $c = 0$ the series $h_{(1,2t+1)}$, $h_{(t+2,3)}$, $h_{(t+2,1)}$, $(t \geq 0)$, were considered in [36]. It is not difficult to notice that $h_{(1,2t+1)}$ equals the value of h for V_{0s}, V'_{0s}, $((m,n) = (2,1))$, V_{1s}^0, $(n = 1)$, with $t = 3s$, $3s + 2$, $3s + 1$, $(s \geq 0)$, respectively; $h_{(t+2,3)}$ equals h for V_{0s+1}^0, V_{1s}^0 $(n = 1)$, V_{0s}^0, $(n = 0)$, V_s^0, with $t = 4s+3$, $4s+1$, $4s+2$, $4s$, $(s \geq 0)$, respectively, $h_{(t+1,1)}$ equals for V_{1s}, $V'_{0s}((m,n) = (2,1))$, V_{0s+1}^0, $(p/q = 3/2,\ n = 1)$, V_{1s}^0, $(p/q = 3/2,\ n = 2)$, with $t = 4s$, $4s+2$, $4s+3$, $4s+1$, $(s \geq 0)$, respectively. It is obvious that each of these series contains HWM of different structure and it is not clear to us why they are combined in such series. (The coincidence between the series $h_{(1,6s+3)}$ and $h_{(4s+3,3)}$ and between $h_{(1,6s+5)}$ and $h_{(4s+3,1)}$ is not mentioned in [36].)

2) The Virasoro character formulae (23) (or (24)) with $K = 0$ for the unitarizable HWM $c < 1$ were derived by Rocha-Caridi [33]. These are the HWM $L = L_{00} = L_{10}$, $q = p + 1$, $c = 1 - 6/p(p+1)$, $p = 3, 4, \dots$ (For $p = 2$, treated

above, L is the trivial 1-dimensional representation of $\hat{W}$.) Their unitarity was conjectured in [17] and proved by explicit construction in [22]. (The proof of nonunitarizability of the other HWM $L^{h,c} \neq V^{h,c}$ with $c < 1$ was given recently in [19].) Our character formula (23) is simpler than that of [33] (except in the cases $p = 3, 4$ which are further worked out). Recently, Kac and Wakimoto [29] have rearranged some of these formulae to obtain multiplicative expressions for the characters. (For these comparisons our parameters $\{p,q,m,n\}$ should be replaced by $\{m, m+1, q, p\}$.)

3) The Virasoro character formulae (23)-(26) were given in [16],[25]. As we mentioned those in [16] are very unexplicit. As the emphasis of [25] is on modular invariance no derivation is given for the character formulae, but it is obvious that the authors follow [33] and do not use our results from [8]. Thus their character formulae are cast in the more complicated form used in [33].

4) For the Neveu-Schwarz and Ramond superalgebras the character formulae for the unitarizable HWM with $c < 1$ were given recently without derivation [29], [23]. These HWM are again $L = L_{00} = L_{10}$ for $\tilde{q} = \tilde{p} + 1$, $\tilde{p} = 1, 3/2, 2, \ldots$ (In the case $\tilde{p} = 1$ L is the trivial 1-dimensional representation of $\hat{S}$.) So these are again formulae (23) (or (24)) with $\nu = 2$ and $K = 0$. As for $\hat{W}$ in [29] many of these formulae are cast in a multiplicative form.

We turn now to the case $c = 1$. The two $\hat{W}$-multiplets $III_-^{00}(c = 1)$, the $\hat{S}$ multiplet N_-^{23} and the $\hat{R}$ multiplet N_-^{23} contain HWM parametrized as follows:

$$\check{V}_K^{\varkappa} \equiv V^{\check{h}_K^{\varkappa}, 1} \quad , \quad \check{h}_K^{\varkappa} \equiv \frac{1}{\nu}\left(K + \varkappa/2\right)^2 + \frac{1}{8}\mu \quad , \quad \left(K \in Z_+\right), \tag{32a}$$

where

$$\varkappa \equiv \begin{cases} \varepsilon = 0,1 & , \text{ for } \hat{W} \quad (\nu = 1, \mu = 0), \\ 2\mu = 0,1 & , \text{ for } \hat{S}, \hat{R} \text{ resp. }, \quad (\nu = 2) \end{cases} . \tag{32b}$$

Note that $\check{V}_K^{\varkappa=\varepsilon} = \tilde{V}_K^{\varepsilon}$, $\check{h}_K^{\varkappa=\varepsilon} = h_K^{\varepsilon}$ in (18) of [8], $\check{V}_K^{\varkappa=2\mu} = V_K$, $\check{h}_K^{\varkappa=2\mu} = h_K$ in (19) of [10]. The $\varepsilon = 0$ $\hat{W}$-multiplet corresponds to the $\hat{S}$-multiplet, while the $\varepsilon = 1$ $\hat{W}$-multiplet corresponds to the $\hat{R}$ multiplet in a sense which will become clearer below.

Using analogous considerations as above we can show that the character formulae are

$$ch\, \check{L}_K^{\varkappa}(t) = ch\, \check{V}_K^{\varkappa} - ch\, \check{V}_{K+1}^{\varkappa} = \psi(t)\, t^{\check{h}_K^{\varkappa}} \left(1 - t^{(2K+1+\varkappa)/\nu}\right) . \tag{33}$$

Formula (33) is known for a long time for $\hat{W}$ and $\hat{S}$ [26], [27].

Finally we shall make the more or less obvious remark that whenever embeddings between HWM can be depicted by a single line then in the expressions of the type $1 - t^x$ in the character formulae x is the difference between the weights, and $-x$ is the degree of homogeneity of the corresponding singular vector v_s in V (such that it has the characteristics of the highest weight vector v' of the module V' embedded in V). The degrees of all singular vectors realizing all noncomposition embedding maps were given explicitly in [8] and [10].

3.3 The character formulae for the discrete cases $c \geqslant 25(\hat{W})$, $c \geqslant 9(\hat{S},\hat{R})$

Consider the category with objects the HWM $V^{h,c}$ and with morphisms the nontrivial embedding maps (determined up to nonzero complex multiple) between the HWM. Then [15],[10] the correspondence $V^{h,c} \leftrightarrow V^{h',c'}$, where $h' = \frac{1}{\nu} + \frac{1}{4}\mu - h$, $c' = 26 - c$ for $\hat{W}$, $c' = 10 - c$ for $\hat{S}$, $\hat{R}$, is extended to an isomorphism between the corresponding categories *3). Because of this the parametrization of types III_+, N^1_+, III^0_+, N^{21}_+, III^{00}_+ $(c > 25)$, N^{22}_+ is obtained from formula (19),(20),(27),(30) by the change $p \to -p$, $n \to -n$ (or equivalently, $q \to -q$, $m \to -m$). The HWM in types III^{00}_+ $(c = 25)$, N^{22}_+ are parametrized as in (32) with $h^{\varkappa}_K$ replaced by

$$\check{h}^{\varkappa +}_k \equiv \frac{1}{\nu} + \frac{1}{4}\mu - \check{h}^{\varkappa}_k = \frac{1}{\nu} + \frac{1}{8}\mu - \frac{1}{\nu}\left(k + \varkappa/2\right)^2 \quad . \tag{34}$$

The HWM are arranged in the multiplets as before, however, all arrows are turned in the opposite direction. Thus if for $c \leqslant 1$ a module V' was embedded in V'', now $V''_{n\to -n,\ p\to -p}$ is embedded in $V'_{n\to -n,\ p\to -p}$. In particular, the HWM V_{00}, V^0_{00}, V^0_0, $\check{V}^{\varkappa}_0$ are now irreducible.

It is clear that the character formulae will involve finite sums in all cases. Thus for types III_+, N^1_+ we have instead of (22) (cf. [16] for $\hat{W}$):

$$ch\, L_{\ell k} = ch\, V_{\ell k} + \sum_{j=1}^{k-1}\left(ch\, V_{0j} + ch\, V_{1j}\right) + ch\, L_{00} - \sum_{j=0}^{k-1}\left(ch\, V'_{0j} + ch\, V'_{1j}\right), \ \ell = 0,1;\ k>0; \tag{35a}$$

$$ch\, L'_{\ell k} = ch\, V'_{\ell k} + \sum_{j=0}^{k-1}\left(ch\, V'_{0j} + ch\, V'_{1j}\right) - ch\, L_{00} - \sum_{j=1}^{k}\left(ch\, V_{0j} + ch\, V_{1j}\right), \ \ell = 0,1 \ . \tag{35b}$$

Using the knowledge of $ch\, V^{h,c}$, in particular $ch\, V_{0j} = ch\, V_{1,-j}$, $ch\, V'_{0,-j} = ch\, V'_{1,j-1}$, we obtain

$$ch\, L_{\ell k} = \sum_{j=-(k-1)}^{k}\left(ch\, V_{\ell j} - ch\, V'_{0,-j}\right), \quad \ell = 0,1;\ \ k>0; \tag{36a}$$

$$ch\, L'_{\ell k} = \sum_{j=-k}^{k}\left(ch\, V'_{\ell j} - ch\, V_{0j}\right), \quad \ell = 0,1 \quad ; \tag{36b}$$

and explicitly

$$\mathrm{ch}\,L_{0K}(t) = \mathrm{ch}\,L_{00}(t)\sum_{j=1-K}^{K} t^{-\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)}\left(1-t^{\nu n(2\tilde q j-m)}\right), \; K>0; \tag{37}$$

$$\mathrm{ch}\,L_{1K}(t) = \mathrm{ch}\,L_{00}(t)\sum_{j=1-K}^{K} t^{-\nu j(\tilde p\tilde q j-\tilde q n+\tilde p m)}\left(1-t^{\nu m(2\tilde p j-n)}\right), \; K>0; \tag{38}$$

$$\mathrm{ch}\,L'_{0K}(t) = t^h\psi(t)\sum_{j=-K}^{K} t^{-\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)}\left(t^{-\nu m(2\tilde p j+n)}-1\right) = \tag{39a}$$

$$= \mathrm{ch}\,L_{00}(t)\sum_{j=-K}^{K} t^{-\nu j(\tilde p\tilde q j-\tilde q n+\tilde p m)}\left(t^{-\nu n(2\tilde q j+m)}-1\right) \; ; \tag{39b}$$

$$\mathrm{ch}\,L'_{1K}(t) = t^h\psi(t)\sum_{j=-K}^{K} t^{-\nu j(\tilde p\tilde q j+\tilde q n-\tilde p m)}\left(t^{-\nu(\tilde p-n)(2\tilde q j+\tilde q-m)}-1\right) = \tag{40a}$$

$$= \mathrm{ch}\,L_{00}(t)\sum_{j=-K}^{K} t^{-\nu j(\tilde p\tilde q j-\tilde q n+\tilde p m)}\left(t^{-\nu(\tilde q-m)(2\tilde p j+\tilde p-n)}-1\right); \tag{40b}$$

$$\mathrm{ch}\,L_{00}(t) = \mathrm{ch}\,L_{10} = \mathrm{ch}\,V_{00} = \mathrm{ch}\,V_{10} = t^h\psi(t)\;;$$

where

$$h = h^+_{(n,m)} \equiv \frac{1}{4\nu pq}\left[(p+q)^2-\nu^2(pm-qn)^2\right]+\frac{1}{8}\mu \;\left(=\frac{1}{\nu}+\frac{1}{4}\mu-h_{(n,m)}\right) \tag{41}$$

Next for types III^0_+, N^{21}_+ we have

$$\mathrm{ch}\,L^0_{0K}(t) = \mathrm{ch}\,V^0_{0K}-\mathrm{ch}\,V^0_{1K-1} = \psi(t)\,t^{h^+_{(2\tilde p q+n,\tilde q)}}\left(1-t^{\nu\tilde q n(2K-1)}\right), \; K>0,\; n>0; \tag{42}$$

$$\mathrm{ch}\,L^0_{1K}(t) = \mathrm{ch}\,V^0_{1K}-\mathrm{ch}^0_{0K} = \psi(t)\,t^{h^+_{(n,(2K+1)\tilde q)}}\left(1-t^{2\nu\tilde q K(\tilde p-n)}\right), \; K>0. \tag{43}$$

Further for types III^{00}_+ $(c > 25)$, N^{22}_+ we have:

$$\mathrm{ch}\,L^0_K(t) = \mathrm{ch}\,V^0_K-\mathrm{ch}\,V^0_{K-1} = \psi(t)\,t^{h^+_{(p',(2K+1)q')}}\left(1-t^{\nu p'q'(2K-1)}\right), \; K>0. \tag{44}$$

Finally for the types III^{00}_+ $(c = 25)$, N^{23}_+ we have (using (32), (34)):

$$\mathrm{ch}\,\check L^{\varkappa}_K(t) = \mathrm{ch}\,\check V^{\varkappa}_K-\mathrm{ch}\,\check V^{\varkappa}_{K-1} = \psi(t)\,t^{\check h^{\varkappa+}_K}\left(1-t^{(2K-1+\varkappa)/\nu}\right), \; K>0. \tag{45}$$

Formulae (37)-(40), (42)-(44) were given in [35] for $\hat W$ in the case $p/q = 2/3$, (then by (9b) $c = 26$). Formula (45) for $\hat W$ was also given in [35].

4. FINAL STATEMENT AND RELATED RESEARCH

We have proven the following

Theorem. The characters of all irreducible highest weight modules $L^{h,c}$ over the Virasoro algebra and Neveu-Schwarz and Ramond superalgebras, such that $L^{h,c}$ is a proper quotient of the reducible Verma module $V^{h,c}$, are given by formulae (18),(23)-(26),(28),(29),(31),(33),(38)-(40),(42)-(45).

Finally we comment on related current and further research. Recently the so-called $N = 2$ extensions of the Virasoro algebra [1] received much attention [5],[30],[37],[3]. (In that terminology the Virasoro algebra is the $N = 0$ case, while Neveu-Schwarz and Ramond superalgebras are $N = 1$ extensions.) The determinant formulae for all three cases (Neveu-Schwarz-type, or A, Ramond-type, or P, twisted type, or T *4)) and a classification of the unitary highest weight representations was given in [3]. (For partial results obtained earlier or simultaneously see [5],[30],[37].) Using these determinant formulae we have determined the structure of the reducible Verma modules over these superalgebras [12],[13].

There are three mastertypes. Mastertype F includes the Virasoro $N = 0,1$ types (given in Sec. 2.2 here). It includes Verma modules for which the functions $f^X_{r,s}$ [3](X = A,P,T) vanish, but g^X_k [3] (X = A,P) do not vanish. Mastertype G includes Verma modules for which $g^X_k = 0$, but $f^X_{r,s} \neq 0$ (X = A,P). The G-types are common with the types of the osp(2,2) case [11]. (We recall that the osp(2,2) superalgebra is a finite-dimensional subalgebra of the A superalgebra.) Mastertype H includes Verma modules for which both $f^X_{rs} = 0 = g^X_k$ (X = A,P) *5). These types may be viewed as "combinations" of the G or H types. Alternatively they are represented by some of the types of the super-Kac-Moody algebra $osp(2,2)^{(1)}$ [11]. The character formulae for the F types are obtained from those given here by identification of parameters [12],[13]. The character formulae for the unitary, resp. non-unitary irreducible HWM are presented in [12], resp. [13]. We also would like to note that the $N = 2$ Neveu-Schwarz- $osp(2,2)^{(1)}$ correspondence is a continuation of the Virasoro - $A_1^{(1)}$ and of the $N = 1$ Neveu-Schwarz - $osp(1,2)^{(1)}$ = super - $A_1^{(1)}$ correspondences (cf. [8],[10]).

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FOOTNOTES

*1) Our parametrization notation is borrowed from the fundamental paper by Belavin, Polyakov and Zamolodchikov [2] which started the renewed interest in 2-dimension conformal theories.

*2) There is a misprint in the preprint version of [8], n = 0 being omitted.

*3) For $\hat{W}$ the similarity between the cases $c = 0$ and $c = 26$ and between $c = 1$ and $c = 25$ was called duality in [35] (it is attributed to Meurman, unpublished).

*4) In [37] P and T are called Ramond I and Ramond II algebras.

*5) These include the discrete unitarizable series with $\tilde{c} < 1$ denoted by A_0, $P_0^{\pm}$ [3] and some of the A_2, $P_2^{\pm}$ [3] cases ($\tilde{c} \geq 1$).

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