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PRODUCTS OF LOCALLY CONNECTED LOCALES

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Products of connected topological spaces are connected for a very simple reason: in $X\times Y$ one has the connected copies $X\times\{y\}$ of X and it suffices, e.g., to cross them by a connected $\{x\}\times Y$. More generally, if X is connected and Y general, and if $X\times Y$ is decomposed into two disjoint open sets U_1, U_2 we can consider again the connected $X\times\{y\}$ and realize that each of them is contained either in U_1 or in U_2 ; this gives rise to the obvious decomposition $Y = V_1 \cup V_2$ such that $U_i = X\times V_i$.

Now when dealing with general locales one cannot imitate the mentioned reasoning. We do not have the points which have been so important. The question naturally arises as to whether the facts hold true at all, i.e.:

> Are products of connected locales connected?
>
> More generally if A is a connected locale and if $1(A\oplus B) = a_1 \vee a_2$ with $a_1 \wedge a_2 = 0$, is there a decomposition $1(B) = b_1 \vee b_2$ such that $a_i = 1(A)\oplus b_i$?

The first of the mentioned problems seems to be open, the second one is answered in the negative (a counterexample, which is rather complex, will be presented elsewhere). The purpose of this article is to deal with the simplest positive case, namely that of locally connected locales. Namely, we prove that the answer of the second question is affirmative if A is a product of connected locally connected locales (see Theorem 4.7 and also 4.8). We prove, too, that the answer is affirmative in the case of general connected A and locally connected B (Proposition 4.10). Besides, the almost trivial case of A a product of spatial locales and B spatial is dealt with (Theorem 3.12).

The usual notation and terminology of the theory of locales is used (as, e.g., in[2],[3]). In the definitions of connectedness and local connectedness we keep the classical form, not the modified one from [4]. In expressing facts, the locale point of view is preferred (to keep parallel with the topological spaces), on the other side, for simplicity reasons, we count and work with symbols in frames (see 1.1 and, in particular, section 2).

1. Preliminaries

1.1. A frame (locale) is a complete lattice A satisfying the distributivity law

$$a \wedge \bigvee_J b_i = \bigvee_J (a \wedge b_i).$$

The bottom resp. top of A will be denoted by

$$O(A) \quad \text{resp.} \quad 1(A)$$

or simply by 0 resp. 1 if there is no danger of confusion. A locale A is said to be nontrivial if $O(A) \neq 1(A)$.

Frame morphisms are mappings $f:A \longrightarrow B$ such that $f(0) = 0$, $f(1) = 1$, $f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$ and $f(\bigvee_J a_j) = \bigvee_J f(a_j)$. The resulting category will be denoted by

Frm,

its opposite, the category of locales, by

Loc.

Throughout the paper we will often use the locale point of view while the notation will be kept as in Frm. Thus, we may speak about a sublocale B of A, but represent it as a surjective morphism $f:A \longrightarrow B$. Or speaking about products of locales, the diagrams will be written as coproducts of frames.

1.2. For a topological space X denote by

$$\mathcal{O}(X)$$

the locale of its open sets. If $f:X \longrightarrow Y$ is a continuous map then $\mathcal{O}(f): \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ defined by $\mathcal{O}(f)(u) = f^{-1}(u)$ is obviously a frame morphism. Thus, a (covariant) functor

$$\mathcal{O}: \mathrm{Top} \longrightarrow \mathrm{Loc}$$

is obtained. A locale isomorphic to an $\mathcal{O}(X)$ is said to be spatial.

1.3. A subset $U \subseteq A$ of a locale is called cover if $\bigvee U = 1$, it is said to be a basis of A if

$$\forall a \in A \ \exists\, U(a) \subseteq U \text{ s.t. } a = \bigvee U(a).$$

Obviously, each basis of A is a cover of A.

1.4. For an element a of a locale A denote by

$$[a]$$

the interval $\{x \mid x \leq a\}$. It will be viewed as a locale endowed by the 0, $\wedge$ and $\bigvee$ from A and by $1([a]) = a$.

The frame morphism

$$p = p_a : A \longrightarrow [a]$$

given by $p(x) = a \wedge x$ represents the embedding of $[a]$ in A as a sublocale.

1.5. The complement of an $x \in A$, i.e. the largest $y \in A$ such that $x \wedge y = 0$, will be denoted by

$$\bar{x}.$$

An element is said to be complemented if

$$x \vee \bar{x} = 1.$$

1.6. Let U be a subset of a locale A. A U-chain between $a, b \in A$ is a sequence $u_1, \ldots, u_n$ in U such that

$$a \wedge u_1 \neq 0 \ , \ u_i \wedge u_{i+1} \neq 0 \text{ for } i = 1, \ldots n-1,$$
$$\text{and } u_n \wedge b \neq 0.$$

A subset $U \subseteq A$ is said to be chained if there is a U-chain between any two of its elements.

1.7. We say that sublocales $f : A \longrightarrow B$ and $g : A \longrightarrow C$ meet if there is a commutative diagram in Frm

$$\begin{array}{ccccc} & & B & & \\ & \overset{f}{\nearrow} & & \searrow & \\ A & & & & D \\ & \underset{g}{\searrow} & & \nearrow & \\ & & C & & \end{array}$$

with a non-trivial D.

A system $\mathcal{F}$ of sublocales of A is said to be chained if for any f, g in $\mathcal{F}$ there is a sequence

$$f = f_0, f_1, \ldots, f_n = g$$

in $\mathcal{F}$ such that f_i meets f_{i+1} for any $i = 0, \ldots, n-1$.

1.8. Let U be a cover of A. For an $x \in A$ put

$$\mathcal{C}(x, U) = \{u \mid u \in U \text{ and there is a U-chain between } x \text{ and } u\},$$
$$c(x, U) = \bigvee \mathcal{C}(x, U).$$

Obviously,

$$x \leq c(x,U), \text{ and}$$
$$c(x,U) \text{ is complemented.}$$

(Indeed, $1 = \bigvee U = c(x,U) \vee d$ where $d = \bigvee(U \setminus \mathscr{C}(x,U))$. Obviously, $c(x,U) \wedge d = 0$.)

1.9. A morphism $f:A \longrightarrow B$ is said to be dense if

$$f(a) = 0 \Rightarrow a = 0.$$

More generally, a system of morphisms $f_i : A \to B_i$ $(i \in J)$ is said to be collectionwise dense if

$$(\forall i\ f_i(a) = 0) \Rightarrow a = 0.$$

1.10. Lemma: Let $f_i : A \longrightarrow B_i (i \in J)$ be collectionwise dense, let a,b be complemented in A and let $f_i(a) = f_i(b)$ for all $i \in J$. Then $a = b$.
Proof: We have $f_i(a \wedge \bar{b}) = f_i(a) \wedge f_i(\bar{b}) = f_i(b) \wedge f_i(\bar{b}) = 0$, hence $a \wedge \bar{b} = 0$ and similarly $\bar{a} \wedge b = 0$. Thus, $a = a \wedge (b \vee \bar{b}) = a \wedge b = (\bar{a} \wedge b) \vee (a \wedge b) = (\bar{a} \vee a) \wedge b = b$. □

2. What we will need on products

2.1. Products of locales A_i will be dealt with as coproducts of frames

$$(A_j \longrightarrow \bigoplus_{i \in J} A_i)_{j \in J} \quad .$$

If $a_{i_k} \in A_{i_k}$, the symbol

$$(+) \qquad a_{i_1} \oplus \ldots \oplus a_{i_n}$$

stands for

$$q_{i_1}(a_{i_1}) \wedge \ldots \ q_{i_n}(a_{i_n}).$$

To simplify the notation, the elements of the form (+) will be often written as

$$\bigoplus_J a_i \quad (= \bigwedge_J q_i(a_i)).$$

Then, of course, we must not forget that all but finitely many a are equal to the respective $1(A_i)$.

If $f_i : A_i \longrightarrow B_i$ is a collection of morphisms then $\bigoplus_J f_i : \bigoplus_J A_i \longrightarrow \bigoplus B_i$ designates the naturally resulting morphism between the products (defined by $\bigoplus f_i \circ q_j = q_j' \circ f_j$). In the case of small collection we write $f \oplus g$, $f_1 \oplus \ldots \oplus f_n$ etc. We see easily that $\bigoplus_J f_i(\bigoplus_J a_i) = \bigoplus_J f_i(a_i)$.

2.2. We will need the following properties of the products (see,e.g. [1]).

(α) The elements of the form (+) constitute a basis of $\bigoplus_J A_i$.

(β) Let us call an element (x_i) of the cartesian product $\bigtimes_J A_i$ acceptable if $x_i = 1(A_i)$ for all but finitely many i. Let M be a set of acceptable elements such that

(1) $(x_i)_J \in M$ & $(\forall i, y_i \leq x_i)$ & $(y_i)_J$ acceptable $\Rightarrow$ $\Rightarrow (y_i)_J \in M$.

(2) Let $(x_{i r})_{i\in J} \in M$ be such that for $i \neq i_0$ $x_{ir} = x_i$ independently on $r \in R$. Put $x_{i_0} = \bigvee_{r\in R} x_{i_0 r}$. Then $(x_i)_J \in M$.

Then if

$$\bigoplus_{i\in J} a_i \leq \bigvee\{\bigoplus_J x_i \mid (x_i)_J \in M\},$$

we necessarily have $(a_i)_J \in M$.

(γ) $\bigoplus_J a_i = 0$ iff $\exists k, a_k = 0$.

(δ) If $a_k \neq 0$ for $k \neq j$ and $\bigoplus a_i \leq \bigoplus b_i$ then $a_j \leq b_j$.

<u>2.3.</u> <u>Proposition</u>: Let $a_i = 1(A_i)$ for all but finitely many $i \in J$. Then $\bigoplus_J [a_i]$ is isomorphic to $[\bigoplus_J a_i]$.

<u>Proof</u>: Consider the subobjects

$$p: \bigoplus_J A_i \longrightarrow [\bigoplus_J a_i],$$
$$p_k: A_k \longrightarrow [a_k]$$

(recall 1.4) and the coproduct of frames

$$(q_k : A_k \longrightarrow \bigoplus_J A_i)_{k\in J}.$$

Define

$$q'_k : [a_k] \longrightarrow [\bigoplus_J a_i]$$

by putting $q'_k(x) = \bigoplus_J x_i$ where $x_k = x$ and $x_i = a_i$ otherwise. It is easy to check that

q'_k are frame morphisms,

for $x_i \leq a_i$ (and all but finitely many $= a_i$), $\bigwedge_J q_i(x_i) = \bigwedge_J q'_i(x_i)$, and $q'_k \circ p_k = p \circ q_k$.

Let $f_k : [a_k] \longrightarrow B$ be frame morphisms. Then there is a $\varphi : \bigoplus A_i \longrightarrow B$ such that $\varphi \circ q_k = f_k \circ p_k$. For $u \in [\bigoplus a_i]$ put $f(u) = \varphi(u)$. Thus, f obviously preserves 0, $\wedge$ and $\bigvee$. Moreover, $f(1([\bigoplus a_i])) = \varphi(\bigwedge q_k(a_k)) = \bigwedge \varphi\, q_k(a_k) = \bigwedge f_k(1([a_k])) = 1(B)$ so that f is a morphism and we see immediately that $f \circ q'_k = f_k$. Finally, if $f \circ q'_k = f_k$, we have $f(\bigoplus x_i) = f(\bigwedge q_i(x_i)) = f(\bigwedge q'_i(x_i)) = \bigwedge f q'_i(x_i) = \bigwedge f_i(x_i)$ so that f is uniquely determined. □

<u>2.4.</u> <u>Lemma</u>: Let $(f_i : A \longrightarrow A_i)_J$ be collectionwise dense. Then so is $(f_i \oplus 1_B : A \oplus B \longrightarrow A_i \oplus B)_J$.

<u>Proof</u>: If $u \in A \oplus B$, $u \neq 0$ then there are $a, b \neq 0$ such that $a \oplus b \leq u$.

Thus, $(f_i \oplus 1)(u) \geq (f_i \oplus 1)(a \oplus b) = f_i(a) \oplus b \neq 0$ for some i. □

2.5. Lemma: Let $x = 1 \oplus u$ be complemented in $A \oplus B$. Then u is complemented and $\bar{x} = 1 \oplus \bar{u}$.
Proof: We have $(1 \oplus \bar{u}) \wedge x = 0$ and hence $1 \oplus \bar{u} \leq \bar{x}$. On the other hand, write $\bar{x} = \bigvee_{m \in M} y_m \oplus v_m$ with $y_m \neq 0$. Since $x \wedge \bar{x} = 0$, we have $y_m \oplus (v_m \wedge u) = 0$, hence $v_m \wedge u = 0$ so that $v_m \leq \bar{u}$. Thus, $\bar{x} \leq 1 \oplus \bar{u}$. □

3. Connectedness and local connectedness. Regular cuts

3.1. A non-trivial locale A is said to be connected if the only complemented elements in A are $0(A)$ and $1(A)$. An element $a \in A$ is said to be connected if $a \neq 0$ and there is no decomposition $a = a_1 \vee a_2$ with $a_i \neq 0$ and $a_1 \wedge a_2 = 0$.

Observation: The element a is connected iff the locale $[a]$ is connected.

3.2. Lemma: If $\emptyset \neq U \subseteq A$ is a chained set of connected elements then $\bigvee U$ is connected.
Proof: Standard: if $\bigvee U = a \vee b$, $a \wedge b = 0$, we have, for any $u \in U$, $u = u \wedge (a \vee b) = (u \wedge a) \vee (u \wedge b)$, hence either $u \wedge a = 0$ or $u \wedge b = 0$ so that finally $u \leq b$ or $u \leq a$. Now if $u \leq a$ and $v \leq b$, there is obviously no U-chain between u and v. Thus, either $\bigvee U = a$ or $\bigvee U = b$. □

3.3. Corollary: For each connected $x \in A$ there is the largest connected $c(x)$ such that $x \leq c(x)$ (namely, $\bigvee \{u \mid u \text{ connected}, u \geq x\}$). For any two non-void x,y either $c(x) = c(y)$ or $c(x) \wedge c(y) = 0$. □

3.4. Corollary: If A has a cover U consisting of connected elements, it has a disjoint cover consisting of connected elements. □

3.5. From 3.2. and 1.8 we immediately obtain
Corollary: (1) For any cover consisting of connected elements and any connected $x \in A$ we have $c(x,U) = c(x)$.

(2) If A is connected then any cover consisting of connected elements is chained.

3.6. A locale is said to be locally connected if it has a basis consisting of connected elements.

3.7. Observation: Let A be locally connected. Then for any $a \in A$, a is locally connected. □

3.8. From 3.4 we immediately obtain

Corollary: Let A be locally connected. Then there is a system $(a_i)_J$ of connected elements of A such that

$$(1)\ \bigvee_J a_i = 1(A)$$

$$(2)\ i \neq j \Rightarrow a_i \wedge a_j = 0\ .\ \square$$

3.9. A couple of non-trivial locales (A,B) is said to have regular cuts if each complemented x in $A \oplus B$ is of the form $1 \oplus u$.

3.10. Remarks: (1) Obviously, if (A,B) has regular cuts then A is connected.

(2) Equally obviously, A is connected iff (A, $\mathbf{2}$) has regular cuts.

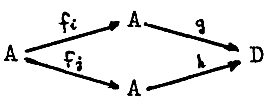
(3) In classical topology, whenever X is connected then the clopen sets in $X \times Y$ are of the form $X \times U$ with U clopen in Y. Thus, the property of regular cuts is contained in the connectedness of X. The situation in general locales is different. There exist connected A such that (A,B) do not always have the regular cuts. An example is rather complicated and will be presented elsewhere. The purpose of this article is mainly to show that the products behave well with respect to connectedness at least in the locally connected case.

3.11. Theorem: Let there be given a collectionwise dense chained system $f_i : A \longrightarrow A_i$ $(i \in J)$ of sublocales of A. Let (A_i,B) have regular cuts. Then (A,B) has regular cuts.

In particular (recall 3.10.(2)), if A_i are connected, A is.

Proof: Let $x \in A \oplus B$ be complemented. Thus, obviously, $(f_i \oplus 1)(x)$ are complemented in $A_i \oplus B$ and hence equal to $1(A_i) \oplus u_i$ for some (complemented) u_i in B.

Now consider f_i, f_j which meet so that there is a commutative diagram



with non-trivial D. We obtain

$$1(D) \oplus u_i = (g \oplus 1)(f_i \oplus 1)(x) = (h \oplus 1)(f_j \oplus 1)(x) = 1(D) \oplus u_j$$

and hence $(1(D) \neq 0(D))$ $u_i = u_j$. Taking into account that $(f_i)_J$ is chained, we infer that $u_i = u$ for all i. Thus, $\forall i$ $(f_i \oplus 1)(x) = 1(A_i) \oplus u = (f_i \oplus 1)(1(A) \oplus u)$ and hence $x = 1(A) \oplus u$ by 1.10. $\square$

3.12. Theorem: Let A be a product of connected spatial locales, B a spatial locale. Then (A,B) has regular cuts.

Proof: Consider $A = \bigoplus_J A_i$, $A_i = \mathcal{O}(X_i)$, X_i connected, $B = \mathcal{O}(Y)$. Recall that the natural projection

$$\pi : \bigoplus_J \mathcal{O}(X_i) \oplus \mathcal{O}(Y) \longrightarrow \mathcal{O}(\mathsf{X}\, X_i \times Y)$$

obviously satisfies

$$\pi(\bigoplus_J a_i \oplus b) = \mathsf{X}\, a_i \times b$$

and hence π is dense. Now let x be complemented in $A \oplus B$. Then $\pi(x)$ is clopen in $\mathsf{X}\, X_i \times Y$ and since $\mathsf{X}\, X_i$ is connected, $\pi(x) = A \times u$ for a u clopen in Y. Thus,

$$\pi(1(A) \oplus u) = A \times u = \pi(x)$$

and hence, by 1.10, $x = 1 \oplus u$. □

4. Products of connected locally connected locales

4.1. Throughout the following paragraphs 4.1 - 4.8, A_i $(i \in J)$ are connected locally connected locales, B a non-trivial locale, $A = \bigoplus_J A_i$, and x is an arbitrary but fixed complemented element of $A \oplus B$.

Sometimes we will wish to point out a particular "coordinate" of a basic object $\bigoplus_J a_i \oplus b$. Then we write

$$\bigoplus_{J \setminus \{j\}} a_i \oplus a_j \oplus b.$$

4.2. An element $\bigoplus_J a_i \oplus b$ is said to be exact if there are b_1, b_2 such that $b = b_1 \vee b_2$ and

$$\bigoplus_J a_i \oplus b_1 \leq x \text{ and } \bigoplus_J a_i \oplus b_2 \leq \bar{x}.$$

4.3. Lemma: Let $\bigoplus_{J \setminus \{j\}} a_i \oplus a_j^m \oplus b$ $(m \in M)$ be exact and let the system $\{a_j^m \mid m \in M\}$ be chained. Then $\bigoplus_{J \setminus \{j\}} a_i \oplus \bigvee_M a_j^m \oplus b$ is exact.

Proof: We have $b = b_1^m \vee b_2^m$ such that

$$\bigoplus a_i \oplus a_j^m \oplus b_1^m \leq x, \quad \bigoplus a_i \oplus a_j^m \oplus b_2 \leq \bar{x}.$$

Hence,

$$(\bigoplus a_i \oplus (a_j^m \wedge a_j^n) \oplus b) \wedge x = (\bigoplus a_i \oplus a_j^m \oplus b) \wedge ((\bigoplus a_i \oplus a_j^n \oplus b) \wedge x) =$$
$$= (\bigoplus a_i \oplus a_j^m \oplus b) \wedge (\bigoplus a_i \oplus a_j^n \oplus b_1^n) = \bigoplus a_i \oplus (a_j^m \wedge a_j^n) \oplus b_1^n.$$

On the other hand, since $\wedge$ is commutative we can reverse the rôles of m and n to obtain that

$$(\bigoplus a_i \oplus (a_j^m \wedge a_j^n) \oplus b) \wedge x = \bigoplus a_i \oplus (a_j^m \wedge a_j^n) \oplus b_1^m.$$

Comparing the right hand sides and recalling 2.2.(δ) we see that if $a_j^m \wedge a_j^n \neq 0$ then $b_1^m = b_1^n$. Consequently, since $\{a_j^m \mid m \in M\}$ is chained, all the b_1^m are equal to a unique b_1. Similarly we see that $b_2^m = b_2$ for all m. Thus,

$$\bigoplus a_i \oplus \bigvee_M a_j^m \oplus b_1 \leqslant x, \quad \bigoplus a_i \oplus \bigvee_M a_j^m \oplus b_2 \leqslant \bar{x}. \square$$

4.4. We will say that $\bigoplus_J a_i \oplus b$ is c-exact if for any connected $c_i \leqslant a_i$, $\bigoplus_J c_i \oplus b$ is exact.

4.5. Observation: If $\bigoplus a_i \oplus b$ is c-exact, $a_i' \leqslant a$ (and all but finitely many a_i' equal to 1) and $b' \leqslant b$, then $\bigoplus a_i' \oplus b'$ is c-exact . $\square$

4.6. Lemma: (1) If $\bigoplus_J a_i \oplus b^m (m \in M)$ are c-exact then $\bigoplus_J a_i \oplus (\bigvee_M b^m)$ is c-exact.

(2) If $\bigoplus_{J\setminus\{j\}} a \oplus a_j^m \oplus b$ $(m \in M)$ are c-exact then $\bigoplus_{J\setminus\{j\}} a_i \oplus \bigvee_{m\in M} a_j^m \oplus b$ is c-exact.

Proof: (1) is obvious.

(2): Here we will use the local connectedness of the locales A_i. Put $a_j = \bigvee_M a_j^m$. Let $c_i \leqslant a_i$ be connected. Write

$$c_j \wedge a_j^m = \bigvee\{d_k^m \mid k \in K(m)\}$$

with d_k^m connected. Thus,

$$c_j = \bigvee\{d_k^m \mid m \in M, k \in K(m)\}.$$

Since c_j is connected, $\{d_k^m \mid m \in M, k \in K(m)\}$ has to be chained (recall 1.8) and consequently the statement follows from 4.5 and 4.3. $\square$

4.7. Theorem: Let $A_i (i \in J)$ be connected locally connected locales. Put $A = \bigoplus_J A_i$. Then, for each non-trivial locale B, (A,B) has regular cuts.

Proof: Let x be complemented in $A \oplus B$. Write

$$x = \bigvee_{r\in R} (\bigoplus_{i\in J} a_{ir} \oplus b_r) , \quad \bar{x} = \bigvee_{r\in S} (\bigoplus_{i\in J} a_{ir} \oplus b_r).$$

Thus, all the $\bigoplus_{i\in J} a_{ir} \oplus b_r$ with $r \in R \cup S$ are c-exact (in fact, exact) and $1(A \oplus B) = x \vee \bar{x} = \bigvee_{r\in R\cup S} (\bigoplus_i a_{ir} \oplus b_r)$. Recall 2.2.($\beta$) and use 4.5. and 4.6 to obtain that $1(B) = b_1 \vee b_2$ such that $1(A) \oplus b_1 \leqslant x$ and $1(A) \oplus b_2 \leqslant \bar{x}$ which immediately yields $x = 1 \oplus b_1$. $\square$

4.8. Thus, recalling 3.10.(1) we immediately see that the product of connected locally connected locales is connected.

In fact, we have

Theorem: The product of any system of connected locally connected locales is connected locally connected.

Proof: It remains to prove the local connectedness. Let $\mathfrak{B}_i$ be basis of A_i consisting of connected elements. Since $A = \bigoplus_J A_i$ is generated by all the $\bigoplus_J a_i$ with all but finitely many a_i equal to $1(A_i)$, we see easily that A is generated by the elements

$$\bigoplus_J b_i$$

with $b_i \in \mathcal{B}_i$ for finitely many i and $b_i = 1(A_i)$ in the remaining cases. By the Observation in 3.1 and by 2.3 it suffices to show that $\bigoplus_J [b_i]$ are connected. This follows from 3.7 and 4.7. □

4.9. Lemma: Let $1(A) = \bigvee_{i\in J} a_i$ with mutually disjoint connected a_i. Then each complemented x in A has the form $\bigvee_{i\in K} a_i$ with some $K \subseteq J$.
Proof: Let x be complemented. Since a_i is connected and $a_i = (a_i \wedge x) \vee \vee (a_i \wedge \bar{x})$ we have either $a_i \leq x$ or $a_i \leq \bar{x}$. Put $K = \{i \mid a_i \leq x\}$. □

4.1o. Proposition: Let A be connected and either A or B locally connected. Then (A,B) has regular cuts.
Proof: If A is locally connected, the statement follows from 4.7. Let B be locally connected. By 3.8 we have a system $(b_i)_J$ of connected $b_i \in B$ such that $\bigvee b_i = 1$ and $b_i \wedge b_j = 0$ for $i \neq j$. Then, $1(A \oplus B) = \bigvee_{i\in J} 1(A) \oplus b_i$ with disjoint summands. Since each $A \oplus [b_i]$ is connected by 3.7 and 4.7 (with the roles of A and B reversed), $A \oplus [b_i]$ is connected and hence finally $1 \oplus b_i$ is. Now if x is complemented, we have by 4.9 $x = \bigvee_{i\in K} 1 \oplus b_i = 1 \oplus \bigvee_K b_i$. □

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