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CHARACTERIZATIONS OF THE COUNTABLE INFINITE PRODUCT OF RATIONALS AND SOME RELATED PROBLEMS

Fons van Engelen

All spaces under discussion are separable and metrizable.

In the beginning of this century, topological characterizations were obtained of such well-known homogeneous zero-dimensional absolute Borel sets as the Cantor set C, $C\setminus\{p\}$, the space of rationals $\mathbb{Q}$, the space of irrationals $\mathbb{P}$, and the product $\mathbb{Q}\times C$ (see [1],[3],and [18]); in fact, apart from the discrete spaces, these are the only homogeneous subsets of C that are either an F_σ (i.e. σ-compact) or a G_δ (i.e. completely metrizable, or, equivalently, (topologically) complete). In [13], van Mill characterized a homogeneous zero-dimensional space which is both an absolute $F_{\sigma\delta}$ and an absolute $G_{\delta\sigma}$, but neither complete nor σ-compact: he considered products of the "basic" spaces $\mathbb{Q}$, $\mathbb{P}$, and C, and noticed that $\mathbb{Q}\times\mathbb{P}$ was not yet characterized. Considering infinite products of these spaces led him to the question of finding a topological characterization of the countable infinite product of rationals $\mathbb{Q}^\omega$, which is an absolute $F_{\sigma\delta}$, but not an absolute $G_{\delta\sigma}$. Such a characterization can in fact be deduced from a theorem of Steel in [20], as was pointed out to the author by A.W. Miller. However, Steel uses deep results from descriptive set theory, viz. determinacy of certain games, which may be the reason that his paper went unnoticed by many topologists, including myself.

The aim of this paper is twofold: first, to give a completely elementary proof of the characterization of $\mathbb{Q}^\omega$ (and the other homogeneous zero-dimensional absolute Borel sets of exact class two) that was obtained before I learned of Steel's results; the techniques we use seem to be interesting in their own right (see e.g. van Mill [15]). And second, to point out some consequences of Steel's theorem that were never mentioned in the literature.

1. Preliminaries

For all undefined terms and notation, see Engelking [7], or Kuratowski [10]. $A\approx B$ means that A and B are homeomorphic. All metrics in this paper are denoted by d and assumed to be bounded by 1; the diameter of a set A is denoted by diam(A). C always denotes the Cantor set, $\mathbb{Q}$ the space of rationals, and $\mathbb{P}$ the space of irrationals.

A subset of a space X is *clopen* if it is both closed and open in X. A space X is *homogeneous* if for each $x,y \in X$, there exists a homeomorphism $h: X \to X$ such that $h(x) = y$; *homogeneous with respect to dense copies of* A if for all dense subspaces A_1, A_2 of X such that $A_1 \approx A \approx A_2$, there exists a homeomorphism $h: X \to X$ such that $h[A_1] = A_2$; *strongly homogeneous* if $U \approx X$ for each non-empty clopen subset U of X. It is easily seen that a strongly homogeneous zero-dimensional space is homogeneous.

By a *complete* space we mean a topologically complete space. A space X is *σ-complete* if $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is complete, i.e. if X is an absolute $G_{\delta\sigma}$. If P is a topological property, then a space X is *nowhere* P if no non-empty open subset of X has the property P.

Throughout this paper, M denotes the set of all finite sequences of natural numbers, including the empty sequence $\emptyset$. For $s = (i_1,\ldots,i_k) \in M$, "$s,i$" denotes $(i_1,\ldots,i_k,i)$, $|s| = k$, $\nu(s) = i_1 + \ldots + i_k$, $f(s) = i_k$, $s|\ell = (i_1,\ldots,i_\ell)$ if $\ell \le k$, and $\hat{s} = s|k-1$ if $k > 1$, $\hat{s} = \emptyset$ if $k = 1$; also, put $|\emptyset| = \nu(\emptyset) = 0$. If $s,t \in M$, we write "$s \le t$" if $t|k = s$ for some $k \le |t|$, or if $s = \emptyset$. If $\sigma = (i_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\omega}$, then $\sigma|k = (i_1,\ldots,i_k)$, and "$s < \sigma$" means that $s = \emptyset$ or $s = \sigma|k$ for some $k \in \mathbb{N}$.

In our proof, we will use the following criterion for convergence of homeomorphisms, which is a slight modification of a result of Anderson [2].

1.1 THEOREM: *Let* X *be compact, and for each* $n \in \mathbb{N}$, *let* $h_n: X \to X$ *be a homeomorphism such that* $d(h_{n+1},h_n) < \min\{2^{-n}, 3^{-n}.\min\{\min\{d(h_i(x),h_i(y)): d(x,y) \ge \frac{1}{n}\}: 1 \le i \le n\}\}$. *Then* $\lim_{n\to\infty} h_n$ *is an autohomeomorphism of* X.

We will also need an "estimated homeomorphism extension theorem" for the Cantor set, which is essentially due to van Mill [14] :

1.2 THEOREM: *Let* $g: C \to C$ *be a homeomorphism, let* $\varepsilon > 0$, *and let* A *be closed and nowhere dense in* C. *If* $h_0: A \to g[A]$ *is a homeomorphism such that* $d(g|A,h_0) < \varepsilon$, *then there exists an autohomeomorphism* h *of* C *such that* $h|A = h_0$, *and* $d(g,h) < \varepsilon$.

Some other known theorems that we will need:

1.3 THEOREM (Ostrovskiĭ [16]; see also [4]): *Let* A *be a strongly homogeneous zero-dimensional space, and suppose that* $X = \bigcup_{i=1}^{\infty} X_i$, *where each* X_i *is closed and nowhere dense in* X, *and* $X_i \approx A$ *for each* i. *Then* $X \approx \mathbb{Q} \times A$.

1.4 THEOREM (van Engelen [4]): *Let* A *be a (non-discrete) zero-dimensional strongly homogeneous space. If* C *is homogeneous with respect to dense copies of* A, *then* C *is also homogeneous with respect to dense copies of* $\mathbb{Q}\times A$.

Slightly different versions of the following lemma appear in [4] and [6]; the easy proof is omitted.

1.5 LEMMA: *Let* A *be a non-empty, compact, nowhere dense subset of* $B\subset X$, *and let* $(\varepsilon_n)_{n\in\mathbb{N}}$ *be a given sequence of positive numbers. Then there exists a countable discrete subset* $D=\{d_n : n\in\mathbb{N}\}$ *of* $B\setminus A$ *such that* $\mathrm{Cl}_X D = D\cup A$, *and* $d(d_n,A)<\varepsilon_n$ *for each* $n\in\mathbb{N}$.

The last theorem of this section is a special case of a result of Levi [11].

1.6 THEOREM: *Let* X *be an analytic space. Then* X *is Baire if and only if* X *contains a dense complete subspace.*

2. The first characterization of $\mathbb{Q}^\omega$

Recall the definition of M from section 1.

2.1 DEFINITION: $\mathcal{X}$ *is the class of all zero-dimensional spaces* X *for which there exist non-empty closed subspaces* X_s, *for each* $s\in M$, *satisfying*

(i) $X=X_\emptyset$, *and* $X_s=\bigcup_{i=1}^\infty X_{s,i}$ *for each* $s\in M$;

(ii) *for each* $i\in\mathbb{N}$, *and each* $s\in M$, $X_{s,i}$ *is nowhere dense in* X_s;

(iii) *if* $\sigma\in\mathbb{N}^\omega$, *and* $p_k\in X_{\sigma|k}$ *for each* $k\in\mathbb{N}$, *then the sequence* $(p_k)_{k\in\mathbb{N}}$ *converges.*

2.2 LEMMA: $\mathbb{Q}^\omega\in\mathcal{X}$.

Proof: Enumerate $\mathbb{Q}$ as $\{q_n : n\in\mathbb{N}\}$, and put $X_\emptyset=\mathbb{Q}^\omega$, $X_{i_1\ldots i_k}=(q_{i_1},\ldots,q_{i_k})\times\mathbb{Q}\times\mathbb{Q}\times\ldots$.□

Our aim is to prove that, up to homeomorphism, $\mathbb{Q}^\omega$ is the only element of $\mathcal{X}$. We first show that the sets X_s of definition 2.1 can be chosen to be disjoint. In fact, we will prove:

2.3 LEMMA: *Let* $X\in\mathcal{X}$ *be embedded in* C. *Then the sets* X_s *from definition 2.1 can be chosen to satisfy the additional property*

(iv) $\overline{X}_s\cap\overline{X}_t=\emptyset$ *if* $s,t\in M$, $|s|=|t|$, $s\neq t$.

Proof: For each $s\in M$, let $\mathcal{U}_s$ be a disjoint clopen cover of $\overline{X}_s\setminus\bigcup_{i<f(s)}\overline{X}_{\hat{s},i}$; enumerate $\mathcal{U}_s$ as $\{U_j : j\in E_s\}$, using pairwise disjoint indexing sets. If $j_n\in E_{s|n}$ for each $n\leq |s|=k$, then define a closed set

$$X(s,j_1\ldots j_k)=X_s\cap\bigcap_{n=1}^k U_{j_n}.$$

The reader can easily verify that,

(1) $\overline{X}(s,j_1\dots j_k) \cap \overline{X}(t,\ell_1\dots\ell_k) = \emptyset$ if $(s,j_1\dots j_k) \neq (t,\ell_1\dots\ell_k)$.
(2) $X = \cup\{X(s,j): |s| = 1,\ j \in E_s\}$, and $X(s,j)$ is nowhere dense in X;
(3) $X(s,j_1\dots j_k) = \cup\{X(t,j_1\dots j_k j): \hat{t} = s,\ j \in E_t\}$, and $X(t,j_1\dots j_k j)$ is nowhere dense in $X(s,j_1\dots j_k)$.

Thus, infinitely many $X(s,j)$, with $|s| = 1$, $j \in E_s$, are non-empty, and each non-empty $X(s,j_1\dots j_k)$ contains infinitely many non-empty $X(t,j_1\dots j_k j)$ with $\hat{t} = s$, $j \in E_t$. Furthermore, if $\sigma \in \mathbb{N}^\infty$, $j_k \in E_{\sigma|k}$ for each $k \in \mathbb{N}$, and $p_k \in X(\sigma|k,j_1\dots j_k)$, then $p_k \in X_{\sigma|k}$, and hence the sequence $(p_k)_{k\in\mathbb{N}}$ converges.□

2.4 LEMMA: *Let* $X,Y \in \mathcal{X}$ *be densely embedded in* C. *Then there exists a homeomorphism* $h: C \to C$ *such that* $h[X] = Y$.

Proof: Since $X,Y \in \mathcal{X}$, there exist closed non-empty subsets X_s of X, Y_s of Y, for each $s \in M$, satisfying properties (i), (ii), and (iii) of definition 2.1, and (iv) of lemma 2.3. Throughout this proof, (i), (ii), (iii) and (iv) will always refer to those properties.
We will construct, for each $n \in \mathbb{N}$, a homeomorphism $h_n: C \to C$, such that

(*) $d(h_n,h_{n+1}) < \varepsilon_n = \min\{2^{-n},3^{-n}.\min\{\min\{d(h_i(x),h_i(y)): d(x,y) \geq \frac{1}{n}\}: 1 \leq i \leq n\}\}$;
(**) $\forall s \in M: \forall x \in X_s: \exists t \in M: |t| = |s|$, and $\forall n \geq \nu(s): h_n(x) \in \overline{Y}_t$;
(***) $\forall s \in M: \forall y \in Y_s: \exists t \in M: |t| = |s|$, and $\forall n \geq \nu(s): h_n^{-1}(y) \in \overline{X}_t$.

Suppose this has been done; then $\lim_{n\to\infty} h_n$ is an autohomeomorphism of C by (*) and theorem 1.1. We claim that $h[X] = Y$.
Indeed, let $x \in X$, say $x \in \bigcap_{s<\sigma} X_s$. By (**), for each $k \in \mathbb{N}$ we can find $t(k) \in M$ such that $|t(k)| = k$, and such that $h_n(x) \in \overline{Y}_{t(k)}$ for each $n \geq \nu(\sigma|k)$; thus, if $k < \ell$, then $h_{\nu(\sigma|\ell)}(x) \in \overline{Y}_{t(k)} \cap \overline{Y}_{t(\ell)} \subset \overline{Y}_{t(k)} \cap \overline{Y}_{t(\ell)|k}$, and hence $t(\ell)|k = t(k)$ by (iv). So in fact, we can find $\tau \in \mathbb{N}^\omega$ such that $\tau|k = t(k)$. Then $h_{\nu(\sigma|k)}(x) \in \overline{Y}_{\tau|k}$; and if $p_k \in Y_{\tau|k}$ satisfies $d(h_{\nu(\sigma|k)}(x),p_k) < \frac{1}{k}$, then $(p_k)_k$ converges in Y by (iii), and hence $h(x) = \lim_{k\to\infty} h_{\nu(\sigma|k)}(x) = \lim_{k\to\infty} p_k \in Y$. A similar argument shows that (***) implies $h^{-1}[Y] \subset X$.
Thus, roughly speaking, if $x \in X_s$, and $\nu(s) = n$, then h_n determines the Y_t with $|t| = |s|$ which, at the end of the process, will contain $h(x)$; and if $y \in Y_s$, and $\nu(s) = n$, then h_n^{-1} determines the X_t with $|t| = |s|$ which, at the end of the process, will contain $h^{-1}(x)$.

We will construct the homeomorphisms h_n inductively, together with finite collections $\mathcal{A}_s = \{A_\alpha : \alpha \in E_s\}$, $\mathcal{B}_s = \{B_\alpha : \alpha \in E_s\}$ (using pairwise disjoint indexing sets), each consisting of pairwise disjoint Cantor sets in C, such that the following hold for each $n \in \mathbb{N} \cup \{0\}$, and each $s \in M$ with $\nu(s) = n$:

(1) if $n \geq 2$, then $d(h_n, h_{n-1}) < \varepsilon_{n-1}$;

(2) if $\nu(t) \leq n$, and $\alpha \in E_t$, then $h_n[A_\alpha] = B_\alpha$;

(3) if $|t| = |s|$, $t \neq s$, and $\nu(t) \leq n$, then $\cup\mathcal{A}_s \cap \cup\mathcal{A}_t = \emptyset$ $= \cup\mathcal{B}_s \cap \cup\mathcal{B}_t$;

(4) $\overline{X}_s \subset \cup\{\cup\mathcal{A}_t : |t| = |s|, \nu(t) < n\} \cup \cup\mathcal{A}_s$;
$\overline{Y}_s \subset \cup\{\cup\mathcal{B}_t : |t| = |s|, \nu(t) < n\} \cup \cup\mathcal{B}_s$;

(5) if $\alpha \in E_s$, then there exist $t_1, t_2 \in M$ with $|t_1| = |s| = |t_2|$, such that A_α is a clopen subset of $\overline{X}_{t_1}$, and B_α is a clopen subset of $\overline{Y}_{t_2}$.

(6) if $\alpha \in E_{\hat{s}}$, $|t| = |\hat{s}|$, and $A_\alpha \subset \overline{X}_t$, then $A_\alpha \cap \overline{X}_{t,f(s)} \subset \cup\{\cup\mathcal{A}_{\hat{s},j} : j \leq f(s)\}$;
if $\alpha \in E_{\hat{s}}$, $|t| = |\hat{s}|$, and $B_\alpha \subset \overline{Y}_t$, then $B_\alpha \cap \overline{Y}_{t,f(s)} \subset \cup\{\cup\mathcal{B}_{\hat{s},j} : j \leq f(s)\}$;

(7) if $\alpha \in E_s$, then for some $\beta \in E_{\hat{s}}$, A_α is a nowhere dense subset of A_β, and B_α is a nowhere dense subset of B_β.

First note that from (ii) it follows that no X_s or Y_s can contain isolated points (in the relative topology), so that $\overline{A} \approx C$ for any non-empty clopen subset A of X_s or Y_s.
Put $\mathcal{A}_\emptyset = \{C\} = \mathcal{B}_\emptyset$. Since $\overline{X}_1, \overline{Y}_1$ are nowhere dense in C, we can define a homeomorphism $h_1 : C \to C$ such that $h_1[\overline{X}_1] = \overline{Y}_1$; if we put $\mathcal{A}_1 = \{\overline{X}_1\}$, $\mathcal{B}_1 = \{\overline{Y}_1\}$, then (1) - (7) are satisfied.
So suppose that h_m, $\mathcal{A}_s$, and $\mathcal{B}_s$, satisfying (1) - (7), have been constructed for $m \leq n$, $\nu(s) \leq n$ (≥ 1).
Fix $s \in M$ with $\nu(s) = n+1$, and fix $\alpha \in E_{\hat{s}}$. By (5), there exist $t_1, t_2 \in M$ with $|t_1| = |\hat{s}| = |t_2|$, such that $A_\alpha \subset \overline{X}_{t_1}$, $B_\alpha \subset \overline{Y}_{t_2}$. Put $s_1 = (t_1, f(s))$, $s_2 = (t_2, f(s))$. By (5), $\overline{Y}_{s_2} \setminus \cup\{\cup\mathcal{B}_{\hat{s},i} : i < f(s)\}$ is closed in $\overline{Y}_{s_2}$, so we can find a clopen V' in B_α, satisfying

$$B_\alpha \cap \cup\{\cup\mathcal{B}_{\hat{s},i} : i < f(s)\} \subset V' \subset B_\alpha \setminus (\overline{Y}_{s_2} \setminus \cup\{\cup\mathcal{B}_{\hat{s},i} : i < f(s)\}).$$

Since $\nu(\hat{s}) \leq n$, $h_n[A_\alpha] = B_\alpha$ by (2), and also by (2), $h_n^{-1}[\cup\{\cup\mathcal{B}_{\hat{s},i} : i < f(s)\}]$ $= \cup\{\cup\mathcal{A}_{\hat{s},i} : i < f(s)\}$ since $\nu(\hat{s},i) \leq n$ for each $i < f(s)$. Thus,

$$h_n[A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s},i} : i < f(s)\}] = B_\alpha \cap \cup\{\cup\mathcal{B}_{\hat{s},i} : i < f(s)\},$$

and since $\overline{X}_{s_1} \setminus \cup\{\cup\mathcal{A}_{\hat{s},i} : i < f(s)\}$ is closed in $\overline{X}_{s_1}$ by (5), we can find a clopen $U_{s,\alpha}$ in A_α such that

$$A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s},i} : i < f(s)\} \subset U_{s,\alpha} \subset A_\alpha \setminus (\overline{X}_{s_1} \setminus \cup\{\cup\mathcal{A}_{\hat{s},i} : i < f(s)\}),$$

while moreover

$$h_n[U_{s,\alpha}] = V_{s,\alpha} \subset V'.$$

Since $\mathcal{A}_{\hat{s}}$ is pairwise disjoint, $A_\alpha \cap \bigcup\{\bigcup\mathcal{A}_{\hat{s},i}: i < f(s)\}$ is nowhere dense in A_α by (7), so we may assume that $A_\alpha \backslash U_{s,\alpha} \neq \emptyset$, and hence $B_\alpha \backslash V_{s,\alpha} \neq \emptyset$. Let $\mathcal{V}_{s,\alpha}$ be a clopen disjoint cover of $B_\alpha \backslash V_{s,\alpha}$ by non-empty sets of diameter less than ε_n. For each $W \in \mathcal{V}_{s,\alpha}$, put

$$p_W = \min\{p: h_n^{-1}[W] \cap \overline{X}_{t_1,p} \neq \emptyset\},\ q_W = \min\{q: W \cap \overline{Y}_{t_2,q} \neq \emptyset\},$$

and

$$A(W,s,\alpha) = h_n^{-1}[W] \cap \overline{X}_{t_1,p_W},\ B(W,s,\alpha) = W \cap \overline{Y}_{t_2,q_W}.$$

Now define

$$\mathcal{A}_s = \{A(W,s,\alpha): W \in \mathcal{V}_{s,\alpha},\ \alpha \in E_{\hat{s}}\},\ \mathcal{B}_s = \{B(W,s,\alpha): W \in V_{s,\alpha},\ \alpha \in E_{\hat{s}}\},$$

and put $\mathcal{A}_s = \{A_\beta: \beta \in E_s\}$, $\mathcal{B}_s = \{B_\beta: \beta \in E_s\}$, such that if $A_\beta = A(W,s,\alpha)$, then $B_\beta = B(W,s,\alpha)$.

Before defining h_{n+1}, we will show that (3) - (7) are satisfied for each $s \in M$ with $\nu(s) = n+1$. Fix $s \in M$ with $\nu(s) = n+1$.

To prove (3), let $|t| = |s|$, $t \neq s$, and $\nu(t) \leq n+1$. If $\hat{s} \neq \hat{t}$, then since $\nu(\hat{s}),\nu(\hat{t}) \leq n$, we have $\bigcup\mathcal{A}_{\hat{s}} \cap \bigcup\mathcal{A}_{\hat{t}} = \emptyset$, and hence by (7), $\bigcup\mathcal{A}_s \cap \bigcup\mathcal{A}_t = \emptyset$; if $\hat{s} = \hat{t}$, then $\bigcup\mathcal{A}_t \subset \bigcup\{\bigcup\mathcal{A}_{\hat{s},i}: i < f(s)\} \subset C\backslash\bigcup\mathcal{A}_s$ by the construction of $\mathcal{A}_s$. Similarly, $\bigcup\mathcal{B}_s \cap \bigcup\mathcal{B}_t = \emptyset$.

For (4), fix $x \in \overline{X}_s$. Then $x \in \overline{X}_{\hat{s}}$, so by (4), $x \in \bigcup\{\bigcup\mathcal{A}_t: |t| = |\hat{s}|,\ \nu(t) < \nu(\hat{s})\} \cup \bigcup\mathcal{A}_{\hat{s}}$. First suppose that $x \in A$ for some $A \in \mathcal{A}_{t'}$, for some $t' \in M$ with $|t'| = |\hat{s}|$, $\nu(t') < \nu(\hat{s})$. By (5), $A \subset \overline{X}_{t_1}$ for some $t_1 \in M$ with $|t_1| = |t'|$, and since $A \cap \overline{X}_{\hat{s}} \neq \emptyset$, we must have $t_1 = \hat{s}$. Put $t = (t',f(s))$. Since $\nu(t) = \nu(t') + f(s) < \nu(\hat{s}) + f(s) = \nu(s) = n+1$, we can apply (6), and obtain that $x \in A \cap \overline{X}_s = A \cap \overline{X}_{\hat{s},f(t)} \subset \bigcup\{\bigcup\mathcal{A}_{\hat{t},j}: j \leq f(t)\} = \bigcup\{\bigcup\mathcal{A}_{t',j}: j \leq f(t)\} \subset \bigcup\{\bigcup\mathcal{A}_r: |r| = |s|,\ \nu(r) < \nu(s) = n+1\}$.

Second, suppose that $x \notin \bigcup\{\bigcup\mathcal{A}_r: |r| = |s|,\ \nu(r) < n+1\}$; consequently, by the first case, $x \notin \bigcup\{\bigcup\mathcal{A}_t: |t| = |\hat{s}|,\ \nu(t) < \nu(\hat{s})\}$, and hence $x \in A_\alpha$ for some $\alpha \in E_{\hat{s}}$. Using notation as in the construction of $\mathcal{A}_s$, we find that $t_1 = \hat{s}$, and hence $(t_1,f(s)) = s_1 = s$. Since $\bigcup\{\bigcup\mathcal{A}_r: |r| = |s|,\ \nu(r) < n+1\} \supset \bigcup\{\bigcup\mathcal{A}_{\hat{s},i}: i < f(s)\}$, we have $x \in A_\alpha \cap (\overline{X}_{s_1}\backslash\bigcup\{\bigcup\mathcal{A}_{\hat{s},i}: i < f(s)\})$, whence $x \in h_n^{-1}[W]$ for some $W \in \mathcal{V}_{s,\alpha}$. We claim that $x \in A(W,s,\alpha)$; since $x \in h_n^{-1}[W] \cap \overline{X}_{t_1,f(s)}$, it suffices to show that $h_n^{-1}[W] \cap \overline{X}_{t_1,p} = \emptyset$ if $p < f(s)$. So take $p < f(s)$; then $\nu(t_1,p) \leq n$, so by (4) and (7), $\overline{X}_{t_1,p} \subset \bigcup\{\bigcup\mathcal{A}_t: |t| = |(t_1,p)|,\ \nu(t) < n\} \cup \bigcup\mathcal{A}_{t_1,p} \subset \bigcup\{\bigcup\mathcal{A}_t: |t| = |\hat{s}|,\ t \neq \hat{s}\} \cup \bigcup\{\bigcup\mathcal{A}_{\hat{s},i}: i < f(s)\}$. Now $\bigcup\mathcal{A}_t \cap A_\alpha = \emptyset$ if $|t| = |\hat{s}|$, $t \neq \hat{s}$, by (3); and $h_n^{-1}[W] \subset A_\alpha\backslash\bigcup\{\bigcup\mathcal{A}_{\hat{s},i}: i < f(s)\}$.

This proves the claim. The proof that $\overline{Y}_s \subset \cup\{\cup\mathcal{B}_t: |t| = |s|, \nu(t) < n+1\} \cup \cup\mathcal{B}_s$ is similar, so (4) holds. (5) is trivial, and so is (7). It remains to check (6).

Let $\alpha \in E_{\hat{s}}$, and suppose that $t \in M$ is such that $|t| = |\hat{s}|$, and $A_\alpha \subset \overline{X}_t$. Then $A_\alpha \cap (\overline{X}_{t,f(s)} \backslash \cup\{\cup A_{\hat{s},j}: j < f(s)\}) \subset A_\alpha \backslash U_{s,\alpha} = h_n^{-1}[\cup V_{s,\alpha}]$, where $U_{s,\alpha}$, $V_{s,\alpha}$ are as in the construction of A_s. If $p < f(s)$, then $\nu(\hat{s},p) \le n$, so by (6), $A_\alpha \cap \overline{X}_{t,p} \subset \cup\{\cup A_{\hat{s},j}: j \le p\} \subset \cup\{\cup A_{\hat{s},j}: j < f(s)\}$, whence $h_n^{-1}[W] \cap \overline{X}_{t,p} = \emptyset$ for each $W \in V_{s,\alpha}$. Thus, for each $W \in V_{s,\alpha}$, if $h_n^{-1}[W] \cap \overline{X}_{t,f(s)} \ne \emptyset$, then $A(W,s,\alpha) = h_n^{-1}[W] \cap \overline{X}_{t,f(s)}$; so $\cup A_s \supset A_\alpha \cap (\overline{X}_{t,f(s)} \backslash \cup\{\cup A_{\hat{s},j}: j < f(s)\})$. This completes the proof of (6).

We will now define h_{n+1} satisfying (1) and (2).

Since $A(W,s,\alpha) \approx B(W,s,\alpha) \approx h_n^{-1}[W] \approx W \approx C$ for each $W \in V_{s,\alpha}$, each $s \in M$ with $\nu(s) = n+1$, and each $\alpha \in E_{\hat{s}}$, and since $A(W,s,\alpha)$ (resp. $B(W,s,\alpha)$) is closed and nowhere dense in $h_n^{-1}[W]$ (resp. W), there exist homeomorphisms $g(W,s,\alpha): h_n^{-1}[W] \to W$ such that $g(W,s,\alpha)[A(W,s,\alpha)] = B(W,s,\alpha)$. Since $V_{s,\alpha}$ is a disjoint clopen cover of $B_\alpha \backslash V_{s,\alpha}$, we can define a homeomorphism $g_{s,\alpha}: A_\alpha \backslash U_{s,\alpha} \to B_\alpha \backslash V_{s,\alpha}$ by

$$g_{s,\alpha} = \cup\{g(W,s,\alpha): W \in V_{s,\alpha}\}.$$

Note that $d(g_{s,\alpha}, h_n|(A_\alpha \backslash U_{s,\alpha})) < \varepsilon_n$ since $\mathrm{diam}(W) < \varepsilon_n$ for each $W \in V_{s,\alpha}$. Now put $I_j = \{s \in M: \nu(s) = n+1, f(s) = j\}$, for each $j \in \{1,\dots,n+1\}$. Using induction on j, we will define for each $s \in I_j$, and each $\alpha \in E_{\hat{s}}$, a homeomorphism $h_{s,\alpha}: A_\alpha \to B_\alpha$, such that

(I) $h_{s,\alpha}|(A_\alpha \backslash U_{s,\alpha}) = g_{s,\alpha}$;

(II) if $\ell \le j$, $t \in I_\ell$, $\beta \in E_{\hat{t}}$, and $A_\beta \subset A_\alpha$, then $h_{s,\alpha}|A_\beta = h_{t,\beta}$;

(III) $d(h_{s,\alpha}, h_n|A_\alpha) < \varepsilon_n$.

Suppose the $h_{s,\alpha}$ can be constructed. Let $s_0 \in M$ be the sequence $(n+1)$, and let α_0 be the unique element of $E_{\hat{s}} = E_\emptyset$; then $A_{\alpha_0} = C = B_{\alpha_0}$, so h_{s_0,α_0} is an autohomeomorphism of C. We claim that $h_{n+1} = h_{s_0,\alpha_0}$ is as required. Indeed, by (III), h_{n+1} clearly satisfies (1). To prove (2), let $t \in M$ with $\nu(t) \le n+1$, and let $\gamma \in E_t$. If $\nu(t) = n+1$, then $A_\gamma = A(W,t,\beta)$ for some $\beta \in E_{\hat{t}}$, and some $W \in V_{t,\beta}$. Hence $A_\gamma \subset A_\beta \backslash U_{t,\beta} \subset A_\beta$, so applying (II) (for $\ell = f(t)$, $j = n+1$, $\alpha = \alpha_0$, and $s = s_0$), we find that $h_{n+1}[A_\gamma] = (h_{s_0,\alpha_0}|A_\beta)[A_\gamma] = h_{t,\beta}[A_\gamma]$, and by (I), $h_{t,\beta}[A_\gamma] = g_{t,\beta}[A_\gamma] = B(W,t,\beta) = B_\gamma$. If $\nu(t) \le n$, then $t = \hat{s}$ for some $s \in M$ with $\nu(s) = n+1$; hence by (II) (for $\ell = f(s)$, $j = n+1$, $t = s$, $\beta = \gamma$, $\alpha = \alpha_0$, $s = s_0$), we find that $h_{n+1}[A_\gamma] = h_{s,\gamma}[A_\gamma] = B_\gamma$.

The homeomorphisms $h_{s,\alpha}$ are constructed as follows.

For $s \in I_1$, $\alpha \in E_{\hat{s}}$, define $h_{s,\alpha}$ by

$$h_{s,\alpha}|U_{s,\alpha} = h_n|U_{s,\alpha};$$
$$h_{s,\alpha}|(A_\alpha\backslash U_{s,\alpha}) = g_{s,\alpha}.$$

Since $g_{s,\alpha}[A_\alpha\backslash U_{s,\alpha}] = h_n[A_\alpha\backslash U_{s,\alpha}]$, and $h_n[A_\alpha] = B_\alpha$, $h_{s,\alpha}$ maps A_α onto B_α, and $h_{s,\alpha}$ is a homeomorphism since $U_{s,\alpha}$ is clopen in A_α. Clearly, (I) and (III) are satisfied. For (II), note that from (3) and (7) it follows that $A_\beta \subset A_\alpha$ for some $\beta \in E_{\hat{t}}$, $t \in I_1$, can only occur if $\hat{s} \leq \hat{t}$; since $\nu(s) = \nu(t)$, and $f(s) = f(t)$, we have $\nu(\hat{s}) = \nu(\hat{t})$, and hence $\hat{s} = \hat{t}$, so $s = t$. Then $\alpha = \beta$, and we are done.

Now suppose $h_{t,\beta}$ has been defined for $t \in \bigcup_{\ell=1}^{j} I_\ell$, $\beta \in E_{\hat{t}}$, such that (I), (II), and (III) are satisfied, and fix $s \in I_{j+1}$, $\alpha \in E_{\hat{s}}$. For $1 \leq \ell < f(s) = j + 1$, put $s = (\hat{s},\ell,(j + 1) - \ell)$. Then $s_\ell \in I_{(j+1)-\ell}$, and $(j + 1) - \ell \leq j$, so $h_{s_\ell,\gamma}\colon A_\gamma \to B_\gamma$ has been defined for each $\gamma \in E_{\hat{s}_\ell}$. Let $E'_{\hat{s}_\ell} = \{\gamma \in E_{\hat{s}_\ell}\colon A_\gamma \subset A_\alpha\} = \{\gamma \in E_{\hat{s}_\ell}\colon B_\gamma \subset B_\alpha\}$. Then

$$g_\ell = \bigcup\{h_{s_\ell,\gamma}\colon \gamma \in E'_{\hat{s}_\ell}\}\colon \bigcup\{A_\gamma\colon \gamma \in E'_{\hat{s}_\ell}\} \to \bigcup\{B_\gamma\colon \gamma \in E'_{\hat{s}_\ell}\}$$

is a well-defined homeomorphism since $\mathcal{A}_{\hat{s}_\ell}$ and $\mathcal{B}_{\hat{s}_\ell}$ consist of pairwise disjoint sets; and

$$g = \bigcup_{\ell=1}^{j} g_\ell\colon \bigcup\{A_\gamma\colon \gamma \in E'_{\hat{s}_\ell},\ 1 \leq \ell \leq j\} \to \bigcup\{B_\gamma\colon \gamma \in E'_{\hat{s}_\ell},\ 1 \leq \ell \leq j\}$$

is a well-defined homeomorphism since by (3), $\bigcup\mathcal{A}_{\hat{s}_\ell} \cap \bigcup\mathcal{A}_{\hat{s}_{\ell'}} = \emptyset = \bigcup\mathcal{B}_{\hat{s}_\ell} \cap \mathcal{B}_{\hat{s}_{\ell'}}$, if $\ell \neq \ell'$. Let D_1 denote the domain, and D_2 the range of g. Then $D_1 \subset A_\alpha \cap \bigcup\{\bigcup\mathcal{A}_{\hat{s},i}\colon i < f(s)\} \subset U_{s,\alpha}$, $D_2 \subset B_\alpha \cap \bigcup\{\bigcup\mathcal{B}_{\hat{s},i}\colon i < f(s)\} \subset V_{s,\alpha}$, $D_1 \approx C$ (resp. $D_2 \approx C$) is nowhere dense in $U_{s,\alpha} \approx C$ (resp. $V_{s,\alpha} \approx C$) by (7), and $d(g,h_n|D_1) < \varepsilon_n$ by (III). So by theorem 1.2, there exists a homeomorphism $\tilde{g}\colon U_{s,\alpha} \to V_{s,\alpha}$ such that $\tilde{g}|D_1 = g$, and $d(\tilde{g},h_n|U_{s,\alpha}) < \varepsilon_n$. Define $h_{s,\alpha}\colon A_\alpha \to B_\alpha$ by

$$h_{s,\alpha}|U_{s,\alpha} = \tilde{g};$$
$$h_{s,\alpha}|(A_\alpha\backslash U_{s,\alpha}) = g_{s,\alpha}.$$

Then $h_{s,\alpha}$ satisfies (I) and (III). If $\ell \leq j + 1$, $t \in I_\ell$, $\beta \in E_{\hat{t}}$, and $A_\beta \subset A_\alpha$, then by (3) and (7), $\hat{s} \leq \hat{t}$. If $\hat{s} = \hat{t}$, then $s = t$, $\alpha = \beta$, and we are done. If $\hat{s} < \hat{t}$, then for some $1 \leq k \leq j$, we have $\hat{s} < \hat{s}_k \leq \hat{t}$. By (7), there exist $\gamma \in E_{\hat{s}_k}$, $\delta \in E_{\hat{s}}$, such that $A_\beta \subset A_\gamma \subset A_\delta$. Since $\mathcal{A}_{\hat{s}}$ consist of pairwise disjoint sets, we have $\delta = \alpha$. Hence, $h_{s,\alpha}|A_\beta = (h_{s,\alpha}|A_\gamma)|A_\beta = (\tilde{g}|A_\gamma)|A_\beta = (g_k|A_\gamma)|A_\beta = h_{s_k,\gamma}|A_\beta$. Since (II) holds for $j = f(s_k)$, $h_{s_k,\gamma}|A_\beta = h_{t,\beta}$, and we are done.

This completes the inductive construction of the homeomorphism $h_{s,\alpha}$, and hence of the autohomeomorphisms h_n of C. To complete the proof of the lemma, we must show that the conditions (*), (**), and (***), at the begin of this proof, follow from (1) - (7). Now (*) is clear from (1), and since (***) is similar to (**), we will only prove (**).

Let $s \in M$, and $x \in \overline{X}_s$. By (4), $x \in A_\alpha$ for some $\alpha \in E_{t'}$, for some $t' \in M$ with $|t'| = |s|$, $\nu(t') \le \nu(s)$. Hence by (2), $h_n(x) \in B_\alpha$ for each $n \ge \nu(t')$, in particular for each $n \ge \nu(s)$. By (5), $B_\alpha \subset \overline{Y}_t$ for some $t \in M$ with $|t| = |t'|$, so $|t| = |s|$, and $h_n(x) \in \overline{Y}_t$ for each $n \ge \nu(s)$.□

2.5 THWOREM: *Up to homeomorphism,* $\mathbb{Q}^\omega$ *is the unique element of* X.

Proof: $\mathbb{Q}^\omega \in X$ by lemma 2.2; and if $X \in X$, then X contains no isolated points, so X can be densely embedded in C. Now apply lemma 2.4.□

2.6 COROLLARY: *The Cantor set is homogeneous with respect to dense copies of* $\mathbb{Q}^\omega$.

In [12], Luzin "effectively" described an absolute $F_{\sigma\delta}$ which is not an absolute $G_{\delta\sigma}$, viz. the subspace of $\mathbb{P} \approx \mathbb{N}^\omega$ consisting of all sequences of natural numbers which converge to infinity. As a corollary to our first characterization, we will show that in fact this space is homeomorphic to $\mathbb{Q}^\omega$.

2.7 THEOREM: *Let* $X = \{(x_i)_{i\in\mathbb{N}} \in \mathbb{N}^\omega : \lim_{i\to\infty} x_i = \infty\}$. *Then* $X \approx \mathbb{Q}^\omega$.

Proof: Note that X consists of those sequences of natural numbers which, for each $n \in \mathbb{N}$, take the value n at only finitely many coordinates. Let $\{E_i : i \in \mathbb{N}\}$ be an enumeration of the collection of finite subsets of $\mathbb{N}$.
For $s,t \in M$, if $|s| = |t| \ge 1$, $s = (i_1,\dots,i_k)$, put

$$X(s,t) = \{\sigma = (x_m)_m \in X\colon t < \sigma, \text{ and for each } n \in \{1,\dots,k\},\ x_m = n \text{ if and only if } m \in E_{i_n}\}.$$

Then $X(s,t)$ is closed in X. If we also put $X(\emptyset,\emptyset) = X$, then it is easily seen that, for each $s_0,t_0 \in M$ with $|s_0| = |t_0| \ge 1$,

$$X(s_0,t_0) = \bigcup\{X(s,t)\colon s,t \in M,\ \hat{s} = s_0,\ \hat{t} = t_0\},$$

and that $X(s,t)$ is nowhere dense in $X(s_0,t_0)$ if $\hat{s} = s_0$, $\hat{t} = t_0$. Finally, if $\sigma,\tau \in \mathbb{N}^\omega$, and $p_k = (p_i^k)_{i\in\mathbb{N}} \in X(\sigma|k,\tau|k)$ for each $k \in \mathbb{N}$, then $p_i^k = p_i^{k+1}$ if $i \le k$, so $(p_k)_k$ converges to a point of X.□

3. The second characterization of $\mathbb{Q}^\omega$

Throughout this section, X_1 denotes the class of all zero-dimensional absolute $F_{\sigma\delta}$-spaces which are nowhere σ-complete and of the first category. Using theorem 2.5, we will show that, up to homeomorphism, $\mathbb{Q}^\omega$ is the unique element of X_1.

3.1 LEMMA: *If* X *is an analytic space which is not σ-complete, then* X *contains a closed nowhere σ-complete subspace* Y *which is nowhere dense in* X.

Proof: First note that any non-σ-complete space A contains a nowhere σ-complete closed subspace B, viz. $B = A\setminus\bigcup\{U\colon U$ is an open σ-complete subset of $A\}$. So we may assume that X is nowhere σ-complete. If X is Baire, then by theorem 1.6, X contains a dense complete subset G. Since G is an absolute G_δ, we can write $X\backslash G = \bigcup_{i=1}^\infty F_i$, with F_i closed in X. Then for some j, F_j is not σ-complete. By the above remark, F_j contains a closed nowhere σ-complete subspace Y; then Y is as required. If X is not Baire, then there exist a non-empty open set U, and closed nowhere dense sets A_i in X, such that $U \subset \bigcup_{i=1}^\infty A_i$. Since U is an F_σ in X, and since U is not σ-complete, U contains a subset F which is not σ-complete, and closed in X. Then $F = \bigcup_{i=1}^\infty (A_i \cap F)$, and hence some $A_j \cap F$ is not σ-complete; again, if Y is nowhere σ-complete, and closed in $A_j \cap F$, then Y is as required.□

The following lemma is the key to our second characterization; the proof is inspired by a result of Saint-Raymond ([17]; see also [4] and [6]).

3.2 LEMMA: *Let* A *be a Borel set in* C *which is not* σ*-complete, and let* F *be a* σ*-compact space such that* $A \subset F \subset C$. *Then* A *contains a closed nowhere dense subset* Y *which is nowhere* σ*-complete and first category, such that* $Cl_C Y \subset F$.

Proof: We let $\overline{}$ denote closure in C. Since $F\backslash A$ is Borel in C, there exists a continuous surjection $\phi\colon \mathbb{P} \to F\backslash A$. Let $W = \{x \in \mathbb{P}:$ there exists a neighborhood V_x of x in $\mathbb{P}$, and a σ-compact subset E_x of F, such that $\phi[V_x] \subset E_x$, and $E_x \cap A$ is σ-complete$\}$. Then W is open in $\mathbb{P}$, so there exist countably many open V_i in $\mathbb{P}$, and σ-compact E_i in F, such that $W = \bigcup_{i=1}^\infty V_i$, $\phi[V_i] \subset E_i$, and $E_i \cap A$ is σ-complete. Suppose that $F\backslash A \subset E = \bigcup_{i=1}^\infty E_i$; then $A = (E \cap A) \cup (F\backslash E)$ is σ-complete, a contradiction. So $G = \mathbb{P}\backslash\phi^{-1}[E\backslash A]$ is non-empty, and a G_δ in $\mathbb{P}$, whence complete. If $\emptyset \neq U$ is open in G, say $U = U' \cap G$, with U' open in $\mathbb{P}$, then $\phi[U'] = \phi[U] \cup \phi[U'\backslash U] \subset (\overline{\phi[U]} \cap F) \cup E$, which is a σ-compact subset of F. Since $U' \not\subset W$, $((\overline{\phi[U]} \cap F) \cup E) \cap A$ is not σ-complete; but $E \cap A$ is σ-complete, so $\overline{\phi[U]} \cap A$ is not σ-complete.

Now write $F = \bigcup_{i=1}^\infty F_i$, with F_i compact, and let $\{B_i : i \in \omega\}$ be a basis for the topology of A. We will construct compact sets K_s, open subsets U_s of C, open subsets W_s of G, and points $x_i \in B_i$, for each $s \in M$ and each $i \in \omega$, such that:

(1) $K_s \subset \overline{\phi[W_s]} \subset U_s$;
(2) for each $n \in \mathbb{N}$: $\overline{U}_{s,n} \cap K_s = \emptyset$;
(3) for each $n,m \in \mathbb{N}$: $\overline{U}_{s,n} \cap \overline{U}_{s,m} = \emptyset$ if $n \neq m$;
(4) for each $n \in \mathbb{N}$: $Cl_G(W_{s,n}) \subset W_s$;
(5) for each $n \in \mathbb{N}$: $\overline{U}_{s,n} \subset U_s$;
(6) $\mathrm{diam}(W_s) \leq 2^{-|s|}$ (with respect to a complete metric on G);
(7) $\mathrm{diam}(U_s) \leq 2^{-\nu(s)}$;

(8) for each $n \in \mathbb{N}$: $d(K_s, K_{s,n}) \le 2^{1-\nu(s,n)}$;

(9) $K_s \cap A$ is nowhere σ-complete, and nowhere dense in $\overline{\phi[W_s]} \cap A$;

(10) $K_s = \overline{K_s \cap A}$ is contained in some F_j;

(11) $Z_k = \bigcup_{|s| \le k} K_s$ is compact, and $Z_k \cap A$ is nowhere dense in A;

(12) for each $i \le k$: $x_i \notin Z_k$.

We use induction on $|s|$. First, put $W_\emptyset = G$, $U_\emptyset = C$. Then $\overline{\phi[W_\emptyset]} \cap A = \bigcup_{i=1}^\infty (\overline{\phi[W_\emptyset]} \cap A \cap F_i)$ is not σ-complete, so some $\overline{\phi[W_\emptyset]} \cap A \cap F_j$ is not σ-complete. By lemma 3.1, $\overline{\phi[W_\emptyset]} \cap A \cap F_j$ contains a nowhere σ-complete, closed nowhere dense subset $H_\emptyset$; put $K_\emptyset = \overline{H}_\emptyset$. Since $H_\emptyset$ is nowhere dense in A, $B_0 \not\subset H_\emptyset$, say $x_0 \in B_0 \setminus H_\emptyset$. Then (1), (9) - (12) are satisfied, and so are (6) and (7) since all metrics are assumed to be bounded by 1. Next, suppose that K_s, U_s, W_s, and x_i have been defined for $|s| \le k$, $i \le k$, in accordance with conditions (1) - (12). Fix $s \in M$ with $|s| = k$. From (1), (9), and (10), it easily follows that K_s is nowhere dense in $K_s \cup \phi[W_s]$, so by lemma 1.5, there exists a countable discrete subset $D_s = \{y_{s,n} : n \in \mathbb{N}\}$ of $\phi[W_s] \setminus K_s$, such that $\overline{D}_s = D_s \cup K_s$, and $d(y_{s,n}, K_s) \le 2^{-\nu(s,n)}$ for each $n \in \mathbb{N}$. Now let $U_{s,n}$ be an open neighborhood of $y_{s,n}$ such that $\overline{U}_{s,n} \subset U_s$, $\overline{U}_{s,n} \cap K_s = \emptyset$, $\overline{U}_{s,n} \cap \overline{U}_{s,m} = \emptyset$ if $n \ne m$, and $\mathrm{diam}(U_{s,n}) \le 2^{-\nu(s,n)}$, for each $n,m \in \mathbb{N}$. Since $y_{s,n} \in \phi[W_s]$, $y_{s,n} = \phi(x_{s,n})$ for some $x_{s,n} \in W_s$; hence there is an open neighborhood $W_{s,n}$ of $x_{s,n}$ in G such that $\mathrm{Cl}_G(W_{s,n}) \subset W_s$, $\mathrm{diam}(W_{s,n}) \le 2^{-|s|-1}$, and $\overline{\phi[W_{s,n}]} \subset U_{s,n}$. Then $\overline{\phi[W_{s,n}]} \cap A$ is not σ-complete, so as above, $\overline{\phi[W_{s,n}]} \cap A$ contains a nowhere σ-complete, closed nowhere dense subset $H'_{s,n}$ which is contained in some F_j; let $H_{s,n}$ be a non-empty clopen subset of $H'_{s,n}$ which is disjoint from $\{x_i : i \le k\}$, and put $K_{s,n} = \overline{H}_{s,n}$. Then (1) - (7), (9), and (10) are satisfied. To prove (8), note that $d(K_s, K_{s,n}) \le d(K_s, U_{s,n}) + \mathrm{diam}(U_{s,n}) \le d(K_s, y_{s,n}) + 2^{-\nu(s,n)} \le 2^{1-\nu(s,n)}$. We will now show that $Z_{k+1} = \bigcup_{|s| \le k+1} K_s$ is closed in C. For each $\varepsilon > 0$, put $Z_k^\varepsilon = \{x \in C : d(x, Z_k) \le \varepsilon\}$, and $M_k^\varepsilon = \{s \in M : |s| = k+1,\ K_s \not\subset Z_k^\varepsilon\}$. Then each Z_k^ε is compact, and M_k^ε is finite by (1), (7), and (8). Since Z_k is compact by the inductive hypothesis, we have $Z_k = \bigcap_{\varepsilon>0} Z_k^\varepsilon$, and $Z_{k+1} = \bigcap_{\varepsilon>0} (Z_k^\varepsilon \cup \bigcup\{K_s : s \in M_k^\varepsilon\})$ is compact, being the intersection of compacta. To prove the second part of (11), suppose that V is a non-empty open subset of A which is contained in Z_{k+1}. Since $Z_k \cap A$ is closed and nowhere dense in A, $V \setminus Z_k$ is a non-empty open subset of A, contained in $\bigcup_{|s|=k+1} K_s$. So for some $s \in M$ with $|s| = k+1$, $(V \setminus Z_k) \cap K_s \ne \emptyset$; however, by (1), (3), and (5), $(V \setminus Z_k) \cap K_s = (V \setminus Z_k) \cap U_s$, contradicting the fact that $K_s \cap A = H_s$ is nowhere dense in A. Hence (11) holds. In particular, $B_{k+1} \not\subset Z_{k+1} \cap A$, so we can find a point $x_{k+1} \in B_{k+1} \setminus Z_{k+1}$; then (12) is also satisfied. This completes the induction.

Now put $Y = \bigcup_{i=0}^{\infty} (Z_i \cap A)$; we claim that Y is as required. We will first show that $\overline{Y}\setminus\bigcup_{i=0}^{\infty} Z_i \subset F\setminus A$; so suppose that $x \in \overline{Y}\setminus\bigcup_{i=0}^{\infty} Z_i$, and fix $i \in \omega$. Since $x \notin Z_i$, $x \notin Z_i^{\varepsilon}$ for some $\varepsilon > 0$. From (1) and (4) it follows that $\bigcup_{i=0}^{\infty} Z_i \subset Z_i \cup \bigcup_{|s|=i} \overline{\phi[W_s]}$, and from (1) and (7) that $\overline{\phi[W_s]} \subset Z_i^{\varepsilon}$ for all but finitely many $s \in M$ with $|s| = i$. Hence for some finite $M_0 \subset \{s \in M: |s| = i\}$, we have $\overline{Y} \subset Z_i^{\varepsilon} \cup \bigcup_{s\in M_0} \overline{\phi[W_s]}$. Then $x \in \overline{\phi[W_s]}$ for some $s \in M_0$, and this s is unique with $|s| = i$ by (1), (3), and (5). So by (4), there exists an infinite sequence σ of natural numbers such that $x \in \bigcap_{s<\sigma} \overline{\phi[W_s]}$ which is a one point-set by (1) and (&). Also, $\bigcap_{s<\sigma} \overline{W}_s = \bigcap_{s<\sigma} W_s$ is a one point-set by (5), and by completeness of the metric on G. Hence, if $z \in \bigcap_{s<\sigma} W_s$, then $\phi(z) = x$, so $x \in \phi[\mathbb{P}] = F\setminus A$. Thus, $\overline{Y} \subset \bigcup_{i=0}^{\infty} Z_i \cup F\setminus A \subset F$ by (10), and $\overline{Y} \cap A = \bigcup_{i=0}^{\infty} (Z_i \cap A) = Y$. By (12), $B_j \not\subset Y$ for each $j \in \omega$, so Y is closed and nowhere dense in A. Since Y is clearly nowhere σ-complete by (9), to complete the proof it suffices to show that each $K_s \cap A$ is nowhere dense in Y. So let $x \in K_s \cap A$, and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ so large that $2^{-\nu(s,n)} < \frac{1}{2}\varepsilon$. Since, in the construction, $\overline{D}_s = D_s \cup K_s$, $y_{s,m} \in B(x,2^{-\nu(s,n)})$ for some $m > n$; and since $y_{s,m} \in U_{s,m}$, and $\operatorname{diam}(U_{s,m}) \le 2^{-\nu(s,m)}$, we have $U_{s,m} \subset B(x,\varepsilon)$, so $K_{s,m} \subset B(x,\varepsilon)$. By (2), $K_{s,m} \cap K_s = \emptyset$, so $B(x,\varepsilon) \cap Y\setminus(K_s \cap A) \supset K_{s,m} \cap A \neq \emptyset$.□

3.3 LEMMA: *Let* $X \in \mathcal{X}_1$, *let* F *be a σ-compact space such that* $X \subset F \subset C$, *and let* $\varepsilon > 0$. *Then there exist closed nowhere dense subsets* X_i *of* X *such that*

(i) $X = \bigcup_{i=1}^{\infty} X_i$;

(ii) $X_i \in \mathcal{X}_1$ *for each* $i \in \mathbb{N}$;

(iii) $\mathrm{Cl}_C(X_i) \subset F$;

(iv) $\operatorname{diam}(X_i) < \varepsilon$.

Proof: Again, let $\overline{}$ denote closure in C. If $F = \bigcup_{i=1}^{\infty} F_i$, with F_i compact, and $X = \bigcup_{i=1}^{\infty} Y_i$, with Y_i closed and nowhere dense in X, then $X = \bigcup_{i,j=1}^{\infty} (Y_i \cap F_j)$, i.e. we can write $X = \bigcup_{i=1}^{\infty} A_i$, where A_i is closed and nowhere dense in X, and $\overline{A}_i \subset F$; of course we may assume that each A_i is non-empty. Fix $i \in \mathbb{N}$, and let $\mathcal{D}$ be a cover of $\overline{X}\setminus\overline{A}_i$ by non-empty disjoint clopen subsets of $\overline{X}$, such that $\operatorname{diam}(D) < d(D,\overline{A}_i)$ for each $D \in \mathcal{D}$. Since $D \cap X \neq \emptyset$ is not σ-complete, and $D \cap X \subset F \subset C$, we can apply lemma 3.2 to obtain, for each $D \in \mathcal{D}$, a closed nowhere dense subset $E(D)$ of $D \cap X$ which is nowhere σ-complete and first category, such that $\overline{E(D)} \subset F$. Put $B_i = A_i \cup \bigcup_{D\in\mathcal{D}} E(D)$. Since $\overline{X}\setminus(\overline{A}_i \cup \bigcup_{D\in\mathcal{D}} \overline{E(D)}) = \bigcup_{D\in\mathcal{D}} (D\setminus\overline{E(D)})$ is open in $\overline{X}$, we have $\overline{B}_i = \overline{A}_i \cup \bigcup_{D\in\mathcal{D}} \overline{E(D)} \subset F$, and B_i is closed in X. From the diameter condition on the elements of $\mathcal{D}$ it follows that A_i is nowhere dense in B_i; thus, since each $E(D)$ is first category, B_i is first category. Also, if U is a non-empty open subset of B_i, then $U \cap E(D) \neq \emptyset$ for some $D \in \mathcal{D}$, so U is not σ-complete, i.e. B_i is nowhere σ-complete, whence $B_i \in \mathcal{X}_1$.

Finally, B_i is nowhere dense in X: if V is non-empty and open in X, and $V \subset B_i$, then $V \cap D = V \cap E(D) \neq \emptyset$ for some $D \in \mathcal{D}$, contradicting the fact that E(D) is nowhere dense in X. Now let $\mathcal{U}_i$ be a clopen disjoint cover of B_i by non-empty sets of diameter less than ε, and enumerate $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ as $\{X_i : i \in \mathbb{N}\}$; then the sets X_i are as required.□

We are now ready to prove the main theorem of this section.

3.4 THEOREM: *Up to homeomorphism,* $\mathbb{Q}^{\omega}$ *is the only element of* $\mathcal{X}_1$.

Proof: Being a product of σ-compacta, $\mathbb{Q}^{\omega}$ is an absolute $F_{\sigma\delta}$, and clearly it is first category. That $\mathbb{Q}^{\omega}$ is nowhere σ-complete follows from a result of Sikorski ([19]; see section 4 of this paper). So now suppose that $X \in \mathcal{X}_1$; embed X in C, and let $\{F_k : k \in \mathbb{N}\}$ be a family of σ-compact subsets of C such that $\bigcap_{k=1}^{\infty} F_k$, and put $F_0 = C$. We will construct closed subspaces X_s of X, for each $s \in M$, satisfying conditions (i) and (ii) of definition 2.1, as well as

(*) for each $s \in M$, $X_s \in \mathcal{X}_1$;

(**) for each $s \in M$, $\mathrm{diam}(X_s) < (|s|+1)^{-1}$;

(***) for each $s \in M$, $\overline{X}_s \subset F_{|s|}$ (closure in C).

The construction is a triviality: Put $X_{\emptyset} = X$, and if X_s has been defined for all $s \in M$ with $|s| \leq k$, then we obtain the sets $X_{s,i}$ by applying lemma 3.3 to $X_s \subset F_{|s|+1} \subset C$, $\varepsilon = (|s|+2)^{-1}$. We claim that the sets X_s satisfy condition (iii) of definition 2.1. Indeed, let $\sigma \in \mathbb{N}^{\omega}$. Since $\overline{X}_{\sigma|1} \supset \overline{X}_{\sigma|2} \supset \ldots$ is a decreasing sequence of compacta, $\bigcap_{k=1}^{\infty} \overline{X}_{\sigma|k} = \emptyset$, say $x \in \bigcap_{k=1}^{\infty} \overline{X}_{\sigma|k}$. By (***), $x \in \bigcap_{k=1}^{\infty} F_k = X$. Thus, $x \in \bigcap_{k=1}^{\infty} X_{\sigma|k}$, and if U is any open neighborhood of x in X, then by (**), $X_{\sigma|n} \subset U$ for some $n \in \mathbb{N}$. Hence, if $p_k \in X_{\sigma|k}$ for each $k \in \mathbb{N}$, then $p_k \in U$ for $k \geq n$, so $(p_k)_k$ converges to x.□

From this characterization of $\mathbb{Q}^{\omega}$, we can, by elementary methods, obtain characterizations of all zero-dimensional homogeneous absolute Borel sets of exact class two (i.e. they are either an absolute $F_{\sigma\delta}$, or an absolute $G_{\delta\sigma}$, but not both). Let $\mathcal{X}_2$ be the class of all zero-dimensional nowhere σ-complete absolute $F_{\sigma\delta}$ spaces that are Baire, and let $\mathcal{Y}_1$ (resp. $\mathcal{Y}_2$) denote the class of all zero-dimensional σ-complete spaces that are first category (resp. Baire), and nowhere an absolute $F_{\sigma\delta}$. We will show that, up to homeomorphism, each of $\mathcal{X}_2$, $\mathcal{Y}_1$, $\mathcal{Y}_2$ contains exactly one element (which is homogeneous), and also that, if X is a homogeneous zero-dimensional absolute Borel set of exact class two, then $X \in \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2$.

3.5 LEMMA: *Let* X *be dense and co-dense in* C. *Then* $X \in \mathcal{X}_1$ *if and only if* $C\backslash X \in \mathcal{Y}_2$, *and* $X \in \mathcal{X}_2$ *if and only if* $C\backslash X \in \mathcal{Y}_1$.

Proof: It suffices to remark that if U is a clopen subset of C, then $U \cap X$ is an absolute $F_{\sigma\delta}$ (resp. an absolute $G_{\delta\sigma}$) if and only if $U\backslash X$ is an absolute $G_{\delta\sigma}$ (resp. an absolute $F_{\sigma\delta}$), and that by theorem 1.6, X is Baire (resp. first category) if and only if $C\backslash X$ is first category (resp. Baire).□

3.6 THEOREM: *Let* $\mathbb{Q}^\omega$ *be densely embedded in* C. *Then up to homeomorphism,* $C\backslash\mathbb{Q}^\omega$ *is the only element of* $\mathcal{Y}_2$; *furthermore,* C *is homogeneous with respect to dense copies of* $C\backslash\mathbb{Q}^\omega$.

Proof: By lemma 3.5, $C\backslash\mathbb{Q}^\omega \in \mathcal{Y}_2$; and if $A,B \in \mathcal{Y}_2$ are densely embedded in C, then by lemma 3.5 and theorem 3.4, $C\backslash A \approx \mathbb{Q}^\omega \approx C\backslash B$, so by corollary 2.6, there exists an autohomeomorphism h of C such that $h[C\backslash A] = C\backslash B$, whence $h[A] = B$.□

Since all dense embeddings of $\mathbb{Q}^\omega$ in C are equivalent (corollary 2.6), we will just write $C\backslash\mathbb{Q}^\omega$ for the unique element of $\mathcal{Y}_2$.

3.7 THEOREM: *Up to homeomorphism,* $\mathbb{Q}\times(C\backslash\mathbb{Q}^\omega)$ *is the unique element of* $\mathcal{Y}_1$; *furthermore,* C *is homogeneous with respect to dense copies of* $\mathbb{Q}\times(C\backslash\mathbb{Q}^\omega)$.

It is clear that $\mathbb{Q}\times(C\backslash\mathbb{Q}^\omega) \in \mathcal{Y}_2$. So suppose that $X \in \mathcal{Y}_1$, say $X = \bigcup_{i=1}^\infty X_i$, with X_i closed and nowhere dense in X. Fix $i \in \mathbb{N}$, and let $\mathcal{D}$ be a cover of $X\backslash X_i$ by non-empty clopen disjoint subsets of X such that $\operatorname{diam}(D) < d(D,X_i)$ for each $D \in \mathcal{D}$. If we embed D densely in C, then since D is σ-complete and not an absolute $F_{\sigma\delta}$, $C\backslash D$ is an absolute $F_{\sigma\delta}$ which is not σ-complete. By lemma 3.2, $C\backslash D$ contains a closed nowhere dense subset Y such that $Y \in \mathcal{X}_1$, i.e. $Y \approx \mathbb{Q}^\omega$; then $E(D) = \overline{Y}\backslash Y \approx C\backslash\mathbb{Q}^\omega$. Note that $E(D)$ is closed and nowhere dense in D. Put $A_i = X_i \cup \bigcup_{D\in\mathcal{D}} E(D)$; then A_i is closed and nowhere dense in X. By theorem 1.6, each $E(D)$ contains a dense complete subset $G(D)$; then $\bigcup_{D\in\mathcal{D}} G(D) = \bigoplus_{D\in\mathcal{D}} G(D)$ is complete and dense in A_i, so A_i is Baire. Clearly, A_i is σ-complete, and A_i is nowhere an absolute $F_{\sigma\delta}$ since every non-empty open subset of A_i intersects some $E(D)$. Hence $A_i \approx C\backslash\mathbb{Q}^\omega$, so by theorem 1.3, $\bigcup_{i=1}^\infty A_i = X \approx \mathbb{Q}\times(C\backslash\mathbb{Q}^\omega)$. The last statement of the theorem follows immediately from theorems 3.6 and 1.4.□

3.8 THEOREM: *Let* $\mathbb{Q}\times(C\backslash\mathbb{Q}^\omega)$ *be densely embedded in* C. *Then up to homeomorphism,* $C\backslash(\mathbb{Q}\times(C\backslash\mathbb{Q}^\omega))$ *is the only element of* $\mathcal{X}_2$; *furthermore,* C *is homogeneous with respect to dense copies of* $C\backslash(\mathbb{Q}\times(C\backslash\mathbb{Q}^\omega))$.

Proof: Same as the proof of theorem 3.6.□

3.9 THEOREM: *Let* X *be zero-dimensional and homogeneous.*

(a) *If* X *is an absolute* $F_{\sigma\delta}$ *but not* σ*-complete, then* $X \in \mathcal{X}_1 \cup \mathcal{X}_2$.

(b) *If* X *is* σ*-complete but not an absolute* $F_{\sigma\delta}$, *then* $X \in \mathcal{Y}_1 \cup \mathcal{Y}_2$.

Proof: (a) Suppose U is non-empty and clopen in X, and σ-complete. Let $x \in U$, and for each $y \in X$, let $h_y: X \to X$ be a homeomorphism such that $h_y(x) = y$, and put $U_y = h_y[U]$. If $\{U_i: i \in \mathbb{N}\}$ is a countable subcover of $\{U_y: y \in X\}$, then $X = \bigcup_{i=1}^{\infty} U_i$ is σ-complete, a contradiction. So X is nowhere σ-complete. If X is Baire, then $X \in \mathcal{X}_2$; if X is not Baire, then some non-empty clopen subset U of X is first category, and as above, this implies that X is first category. The proof of (b) is similar.□

4. Some consequences of a theorem of Steel

In this section, it will be convenient to denote the Cantor set by 2^ω, where 2 is the two point discrete space.

The following definitions and theorem are taken from Steel [20]. Let $Q_0 = \{x \in 2^\omega: \exists n:\forall m\geq n: x_n = 0\}$, $Q_1 = \{x \in 2^\omega: \exists n:\forall m\geq n: x_n = 1\}$. If $x \in 2^\omega\setminus(Q_0 \cup Q_1)$, then x consists of blocks of zeros separated by blocks of ones; define $\phi: 2^\omega\setminus(Q_0 \cup Q_1) \to 2^\omega$ by $\phi(x)(n) = 0$ (resp. 1) if the n^{th} block of zeros in x has even (resp. odd) length. Note that ϕ is continuous.

4.1 DEFINITION: (a) $\Gamma \subset P(2^\omega)$ *is a reasonably closed pointclass if* $\phi^{-1}[A] \cup Q_0 \in \Gamma$ *for each* $A \in \Gamma$, *and* $f^{-1}[A] \in \Gamma$ *for each* $A \in \Gamma$ *and each continuous* $f: 2^\omega \to 2^\omega$.

(b) $A \subset 2^\omega$ *is everywhere properly* Γ *if for each non-empty open* U *in* X *we have* $U \cap A \in \Gamma$, $2^\omega\setminus(U \cap A) \notin \Gamma$.

4.2 THEOREM (Steel [20]): *If* Γ *is a reasonably closed pointclass of Borel sets, and* $A,B \subset 2^\omega$ *are everywhere properly* Γ, *and either both meager or both comeager, then* $h[A] = B$ *for some autohomeomorphism* h *of* X.

Now for $\alpha \in [1,\omega_1)$, let $\mathcal{A}_\alpha,\mathcal{M}_\alpha$ denote the classes of Borel sets in 2^ω of, respectively, the additive class α and the multiplicative class α (recall that $\mathcal{A}_1 = F_\sigma$, $\mathcal{M}_1 = G_\delta$, $\mathcal{A}_2 = G_{\delta\sigma}$, etc.).

4.3 LEMMA: *If* $\alpha \geq 2$, *then* $\mathcal{A}_\alpha$ *and* $\mathcal{M}_\alpha$ *are reasonably closed pointclasses.*

Proof: Take e.g. $A \in \mathcal{A}_\alpha$. If $f: 2^\omega \to 2^\omega$ is continuous, then clearly $f^{-1}[A] \in \mathcal{A}_\alpha$, so we only have to show that $\phi^{-1}[A] \cup Q_0 \in \mathcal{A}_\alpha$. Now ϕ is a continuous map on $2^\omega\setminus(Q_0 \cup Q_1)$, so $\phi^{-1}[A]$ is of additive class α in $2^\omega\setminus(Q_0 \cup Q_1)$; hence we can find $B \subset 2^\omega$, $B \in \mathcal{A}_\alpha$, such that $B \cap 2^\omega\setminus(Q_0 \cup Q_1) = \phi^{-1}[A]$. Since $\alpha \geq 2$ and $2^\omega\setminus(Q_0 \cup Q_1)$ is a G_δ in 2^ω, also $\phi^{-1}[A] \in \mathcal{A}_\alpha$, and thus $\phi^{-1}[A] \cup Q_0 \in \mathcal{A}_\alpha$ since Q_0 is countable.□

For $\alpha \geq 2$, let $\mathcal{X}_1^\alpha$ (resp. $\mathcal{X}_2^\alpha$) be the class of all zero-dimensional Borel sets that are absolutely of multiplicative class α, nowhere absolutely of additive class α, and first category (resp. Baire). Similarly, define $\mathcal{Y}_\alpha^1$ (resp. $\mathcal{Y}_\alpha^2$).

to be the class of all zero-dimensional Borel sets that are absolutely of additive class α, nowhere absolutely of multiplicative class α, and first category (resp. Baire).

4.4 LEMMA: *For all* $\alpha \geq 2$, *each of* $X_1^\alpha, X_2^\alpha, Y_1^\alpha, Y_2^\alpha$ *contains at most one element, up to homeomorphism, and this element, if it exists, is strongly homogeneous, hence homogeneous, and* C *is homogeneous with respect to dense copies of it.*

Proof: If e.g. $X \in X_1^\alpha$, then X can be densely embedded in the Cantor set; this dense embedding is everywhere properly M_α and meager, so we can apply lemma 4.3 and theorem 4.2; the other cases are proved similarly. Strong homogeneity follows from the observation that, if X is in one of the classes, and U is a non-empty clopen subset of X, then U is in the same class, whence homeomorphic to X.Γ

For $\alpha = 2$, the classes described above are just the classes considered in the preceding section; in particular, for $\alpha = 2$, they are non-empty. We will now show that they are in fact non-empty for each α.
For this, we recall the very elegant construction of Borel sets of exact class as given by Sikorski in [19] (see also [8]): Let $p \in 2^\omega$, and put $M_0 = \{p\}$, $A_0 = 2^\omega \setminus M_0$; if $\alpha \in [1,\omega_1)$, and A_β, M_β have been defined for $\beta < \alpha$, then put $M_\alpha = \Pi_{i=1}^\infty A_\gamma \subset \Pi_{i=0}^\infty 2^\omega \approx 2^\omega$ if $\alpha = \gamma + 1$, $M_\alpha = \Pi_{\beta<\alpha} A_\beta \subset \Pi_{\beta<\alpha} 2^\omega \approx 2^\omega$ if $\lim(\alpha)$, and in both cases put $A_\alpha = 2^\omega \setminus M_\alpha$.
Sikorski showed that $M_\alpha \in M_\alpha \setminus A_\alpha$, and $A_\alpha \in A_\alpha \setminus M_\alpha$. It is easily verified that $M_2 \approx \mathbb{Q}^\omega$, and that each M_α, A_α is dense in 2^ω for $\alpha \geq 1$.

4.5 LEMMA: *Let* $\alpha \geq 2$. *If* α *is even, then* $M_\alpha \in X_1^\alpha$, $A_\alpha \in Y_2^\alpha$; *if* α *is odd, then* $M_\alpha \in X_2^\alpha$, $A_\alpha \in Y_1^\alpha$.

Proof: For $\alpha = 2$, this follows from the results of section 3, so suppose the theorem has been proved for $\beta < \alpha$. Suppose e.g. that α is a limit (the other cases are entirely similar); then α is even, $M_\alpha = \Pi_{\beta<\alpha} A_\beta$. Since $A_\beta \in X_1^\beta \cup X_2^\beta$, A_β is strongly homogeneous by lemma 4.4. So if U is a non-empty basic clopen subset of M_α, then $U \approx M_\alpha \notin A_\alpha$; hence M_α is nowhere absolutely of additive class α. Since A_β is first category for odd β, M_α is first category, so $M_\alpha \in X_1^\alpha$. As in the proof of lemma 3.5 it is shown that this implies $A_\alpha \in Y_2^\alpha$.□

4.6 THEOREM: *If* $\alpha \geq 2$, *then up to homeomorphism, each of* $X_1^\alpha, X_2^\alpha, Y_1^\alpha, Y_2^\alpha$ *contains exactly one element.*

Proof: By lemma 4.4 it suffices to show that each class is non-empty. If α is even, then $M_\alpha \in X_1^\alpha$, $A_\alpha \in Y_2^\alpha$; it is easily checked that $\mathbb{Q} \times A_\alpha \in Y_1^\alpha$, and if this space is densely embedded in 2^ω, then its complement is in X_2^α. Similarly if α is odd.□

Thus, as in theorem 3.9, we conclude that there are exactly four homogeneous zero-dimensional absolute Borel sets of exact class α, for each $\alpha \geq 2$, that there are very simple and elegant characterizations of these spaces, and also that it is very easy to construct them "from below". For descriptions and characterizations of *all* homogeneous zero-dimensional absolute Borel sets, see [4] and [5].

The construction of the sets M_α and A_α naturally led Sikorski to the following question (Coll. Math. problem P.215):

QUESTION: Let A_n be a Borel subset of additive class α in a metric space X_n, but not of multiplicative class α in X_n $(n = 1,2,\ldots)$. Prove or disprove that the set $A = A_1 \times A_2 \times \ldots$ (which is, of course, of the multiplicative class $\alpha + 1$) is not of the additive class $\alpha + 1$ in the space $X = X_1 \times X_2 \times \ldots$.

We will give a partial answer to this question using the so-called "Wadge lemma" (see Wadge [21]): If A,B are Borel sets in 2^ω, then either there is a continuous $f\colon 2^\omega \to 2^\omega$ with $A = f^{-1}[B]$, or there is a continuous $g\colon 2^\omega \to 2^\omega$ with $B = g^{-1}[2^\omega \backslash A]$.

4.7 THEOREM: *If in the above question* A_n *is absolutely Borel of additive class* α, *and separable, then* $\Pi_{n=1}^\infty A_n$ *is not absolutely of additive class* $\alpha + 1$.

Proof: By a theorem of Kunen and Miller [9], each A_n contains a closed subset B_n which is zero-dimensional and not absolutely of multiplicative class α; consider B_n as a subset of 2^ω. If there were a continuous $g\colon 2^\omega \to 2^\omega$ such that $B_n = g^{-1}[P_\alpha]$, then B_n is of multiplicative class α in 2^ω, a contradiction. So by the Wadge lemma, for each $n \in \mathbb{N}$, there is a continuous $f_n\colon 2^\omega \to 2^\omega$ such that $S_\alpha = f_n^{-1}[B_n]$. Define $f\colon \Pi_{n=1}^\infty 2^\omega \to \Pi_{n=1}^\infty 2^\omega$ by $f(x)(n) = f_n(x_n)$. Then $f^{-1}[\Pi_{n=1}^\infty B_n] = \Pi_{n=1}^\infty S_\alpha = P_{\alpha+1}$, so since $P_{\alpha+1}$ is not of additive class $\alpha+1$, neither is $\Pi_{n=1}^\infty B_n$, and hence $\Pi_{n=1}^\infty A_n$ cannot be absolutely of additive class $\alpha + 1$.$\square$

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