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ON COMPLETENESS OF THE REPRESENTATION SPACE OF J. W. CALKIN

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Abstract: The representation space of J. W. Calkin [1] depends on a state ω on l_∞ with $\omega(c_o) = \{0\}$. We prove that this space is complete if and only if ω is a finite convex combination of ultrafilter-limits.

For a Hilbert space H (supposed infinite-dimensional, but not necessarily separable), let B(H) denote the algebra of all bounded linear mappings of H into itself. Let K(H) denote the ideal of all compact operators in B(H). J. W. Calkin [1] has constructed a representation of the factor algebra B(H)/K(H) in a certain scalar product space $\mathcal{L}'$. The space $\mathcal{L}'$ depends on a state ω on l_∞ . It is known that $\mathcal{L}'$ is complete if ω is a limit with respect to a free ultrafilter [3] . Thus, the assertion that $\mathcal{L}'$ is never complete ([1] , theorem 4.1.) is not true. In this note we obtain a necessary and sufficient condition for the completeness of $\mathcal{L}'$. In particular, $\mathcal{L}'$ is not always complete.

We now fix some notations and recall the definition of $\mathcal{L}'$. Let ω be a positive linear functional on l_∞ such that $\omega(c_o) = \{0\}$ and ω ((1,1,1,...)) = 1. By [2] II §2 (8), there is a positive finitely additive measure μ on the σ-algebra of all subsets of $\mathbb{N}$ such that

$$\omega((x_n)) = \int_{\mathbb{N}} x_n \, d\mu(n)$$

for all sequences (x_n) in l_∞ . Denote by $\mathcal{L}''$ the space of all H-valued sequences (f_n) tending weakly to zero. The linear structure of $\mathcal{L}''$ is defined by

$$a\,(f_n) + b\,(g_n) = (\,a\,f_n + b\,g_n).$$

We consider the linear subspace

$$\mathcal{N}_\omega := \{(f_n) \in \mathcal{L}'' : \omega(((f_n,f_n))) = 0\}$$

and the factor space

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$$\mathcal{L}'_\omega := \mathcal{L}''/\mathcal{N}_\omega .$$

The image of (f_n) under the quotient map $\mathcal{L}'' \longrightarrow \mathcal{L}'_\omega$ is denoted by $(f_n)_\omega$. For (f_n), $(g_n) \in \mathcal{L}''$, we define

(1) $\quad ((f_n)_\omega , (g_n)_\omega) := \omega(((f_n,g_n))),$

(2) $\quad \| (f_n)_\omega \| := ((f_n)_\omega ,(f_n)_\omega)^{1/2},$

(3) $\quad |||(f_n)||| := \sup \{ \| f_n \| : n \in \mathbb{N} \},$

(4) $\quad |||(f_n)_\omega ||| := \inf \{ |||(f_n) - (h_n)||| : (h_n) \in \mathcal{N}_\omega \}.$

It is easy to see that $\mathcal{L}'', \mathcal{N}_\omega$ are closed subspaces of the space of all bounded H-valued sequences with respect to the norm (3). Consequently, the spaces $(\mathcal{L}'', |||\cdot|||)$ and $(\mathcal{L}'_\omega, |||\cdot|||)$ are Banach spaces. If $(g_n) \in \mathcal{N}_\omega$, then

$$\|(f_n)_\omega \|^2 = \|(f_n-g_n)_\omega \|^2 = \omega((\| f_n-g_n \|^2)) \leq |||(f_n-g_n)|||^2$$

for all $(f_n) \in \mathcal{L}''$. This implies

$$\| F \| \leq ||| F |||$$

for all $F \in \mathcal{L}'_\omega$. Now the open mapping theorem yields the following criterion.

<u>Lemma:</u> The normed space $(\mathcal{L}'_\omega, \|\cdot\|)$ is complete if and only if there exists a positive δ such that

(5) $\quad \delta ||| F ||| \leq \| F \|$

for all $F \in \mathcal{L}'_\omega$.

We now are in a position to prove the necessary and sufficient condition for the completeness of $\mathcal{L}'_\omega$.

<u>Theorem:</u> The space $\mathcal{L}'_\omega$ endowed with the scalar product (1) is a Hilbert space if and only if ω is a finite convex combination of limits with respect to free ultrafilters.

<u>Proof:</u> Suppose $(\mathcal{L}'_\omega,(.,.))$ is a Hilbert space, i. e., $(\mathcal{L}'_\omega, \|\cdot\|)$ is complete. Let (e_n) be an orthonormal sequence in H. Denote by $\chi_n(M)$ the caracteristic function of $M \subset \mathbb{N}$. Then $(\chi_n(M)e_n) \in \mathcal{L}''$ for all subsets $M \subset \mathbb{N}$. If $\mu(M) > 0$ and $(h_n) \in \mathcal{N}_\omega$, then

$$\inf \{ \| h_n \|^2 : n \in M \} \leq (\mu(M))^{-1} \omega((\chi_n(M) \| h_n \|^2)) = 0.$$

This implies $|||(\chi_n(M)e_n)_\omega ||| = 1$. On the other hand,

$$\|(\chi_n(M)e_n)_\omega \|^2 = \omega((\chi_n(M))) = \mu(M).$$

According to the lemma, this implies

$$0 < \inf \{ \mu(M) : M \subset \mathbb{N} \text{ and } \mu(M) > 0 \}.$$

Standart arguments yield the existence of pairwise disjoint subsets

$M_1, M_2, \ldots, M_l$ of $\mathbb{N}$ satisfying the following conditions:

$$a_k := \mu(M_k) > 0, \quad \sum_{k=1}^{l} a_k = 1, \text{ and } M \subset M_k \text{ implies } \mu(M) \in \{0, a_k\}.$$

For $1 \leq k \leq l$, we define ultrafilters $\mathcal{U}_k$ by

$$\mathcal{U}_k = \{ M \subset \mathbb{N} : \mu(M \cap M_k) = a_k \}.$$

Since $\omega(c_o) = \{0\}$, $\mu(M) = 0$ if M is finite. Consequently the ultrafilters $\mathcal{U}_k$ are free. For each subset $M \subset \mathbb{N}$ we have

$$\mu(M \setminus \bigcup M_k) \leq \mu(\mathbb{N} \setminus \bigcup M_k) = 1 - \sum a_k = 0,$$
$$\omega((\chi_n(M))) = \mu(M) = \mu(M \setminus \bigcup M_k) + \sum \mu(M \cap M_k) =$$
$$= \sum a_k \lim_{\mathcal{U}_k} \chi_n(M).$$

This implies

(6) $$\omega((x_n)) = \sum_{k=1}^{l} a_k \lim_{\mathcal{U}_k} x_n$$

for all $(x_n) \in l_\infty$.

Conversely, suppose ω has the representation (6), whereat $a_k > 0$, $\sum a_k = 1$, and $\mathcal{U}_k$ are free ultrafilters. Put $\delta = (\min\{a_k : 1 \leq k \leq l\})^{1/2}$. By the lemma, it suffices to verify (5) for all $F = (f_n)_\omega \in \mathcal{X}'_\omega$. For, fix $(f_n) \in \mathcal{X}''$ and $\varepsilon > 0$. Denote

$$b_k = \lim_{\mathcal{U}_k} \| f_n \|, \quad b = \max\{ b_k : 1 \leq k \leq l \}.$$

By the definition of the limit with respect to an ultrafilter, there are sets $M_k \in \mathcal{U}_k$ such that $| \|f_n\| - b_k | < \varepsilon$ for all $n \in M_k$. Define $g_n := \chi_n(M_o) f_n$, where $M_o := \bigcup M_k$. Then $(g_n - f_n) \in \mathcal{N}_\omega$ and $|||(g_n)||| \leq \varepsilon + b$. This implies

$$\delta |||(f_n)_\omega||| \leq \delta(\varepsilon + b) \leq \delta\varepsilon + (\sum a_k b_k^2)^{1/2} = \delta\varepsilon + \|(f_n)_\omega\|.$$

Letting $\varepsilon \longrightarrow 0$, we get (5), which completes the proof.

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