Wojciech Bartoszek; Tomasz Downarowicz

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In: Zdeněk Frolík and Vladimír Souček and Jiří Vinárek (eds.): Proceedings of the 13th Winter School on Abstract Analysis, Section of Analysis. Circolo Matematico di Palermo, Palermo, 1985. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 10. pp. [13]--16.

Persistent URL: http://dml.cz/dmlcz/701856

COMPACTNESS OF TRAJECTORIES OF DYNAMICAL SYSTEMS IN COMPLETE UNIFORM SPACES

Wojciech Bartoszek and Tomasz Downarowicz

In this paper we investigate the asymptotic behaviour of trajectory $\{\varphi_t(x)\}_{t\geq 0}$, for a semigroup of mappings $\{\varphi_t\}_{t\geq 0}$ of a Hausdorff space X into itself. More precisely : the main subject of our interest is to establish conditions equivalent to precompactness of the trajectory and of the set of limit points for a given point $x \in X$. This topic has already been studied in [7], [8] and [6]. In our case the space X is in addition equipped with a complete uniform structure $\mathcal{U}$ (see [4] for definition). We also assume the following four conditions for the family $\{\varphi_t\}$:

(i) $\varphi_0(x) = x$, for all $x \in X$

(ii) $\varphi_t \circ \varphi_s = \varphi_{t+s}$, for all $t,s \in R_+$

(iii) $\lim_{t \to s} \varphi_t(x) = \varphi_s(x)$, for all $s \in R_+$

(iv) for every $W \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that for all x, y with $(x,y) \in V$ and all $t \geq 0$ we have $(\varphi_t(x), \varphi_t(y)) \in W$.

The first three of the above conditions mean that the mappings φ_t form an one-parameter continuous semigroup acting on X. The last condition establishes its equicontinuity.

For a fixed element W of $\mathcal{U}$ by $\mathcal{U}_W$ we will denote the collection of all the elements V which fulfill (iv). We also write W_x instead of $\{ y : (x,y) \in W\}$. For contraction semigroups acting on subsets of Banach spaces the condition (iv) may be replaced by an adequate norm - condition. In this case many interesting results were obtained, dealing with limit properties of the trajectory $\gamma(x) = \{\varphi_t(x) : t \geq 0\}$ (see [1], [3], [5]). Subsequently in [2] were obtained some analogous results for nonextending semigroups acting

This paper is in final form and no version of it will be submitted for publication elsewhere.

on Polish spaces. The methods of proofs used there have let hope that further generalisations are possible.

We start by proving the following lemma, which is an adaptation of a well known result from [3] (Theorem 1) .

Lemma 1. For every $x \in X$, the set of limit points $w(x) = \bigcap_{s \geq 0} \overline{\{\varphi_t(x) : t \geq s\}}$ is either minimal or empty.

Proof. We have to show that $\overline{\gamma(y)} = w(x)$, for every $y \in w(x)$. The inclusion $\subseteq$ is immediate because for every $t \geq 0$, the point $\varphi_t(y)$ is a limit of $\varphi_t(x)$. Now let $z \in w(x)$ and U be an open neighbourhood of z. There exists an element W of the structure $\mathcal{U}$ such that for all α large enough $(\varphi_{t_\alpha}(x), v) \in W$ implies $v \in U$, where $t_\alpha \to \infty$ is some fixed net satysfying $\varphi_{t_\alpha}(x) \to z$. We can also easily find a net $s_\alpha \to \infty$ with $\varphi_{t_\alpha - s_\alpha}(x) \to y$. For $V \in \mathcal{U}_W$ we have $(\varphi_{t_\alpha - s_\alpha}(x), y) \in V$ for some α . Thus $(\varphi_{t_\alpha}(x), \varphi_{s_\alpha}(y)) \in W$, so $\varphi_{s_\alpha}(y) \in U$, hence $w(x) \subseteq \overline{\gamma(y)}$, and the minimality is proved.

By X_0 we shall denote the set of all $x \in X$ such that the trajectory $\gamma(x)$ is precompact.

Lemma 2. The set X_0 is closed and φ_t-invariant.

Proof. Let $x_\alpha \to x$, where $x_\alpha \in X_0$. For the precompactness of $\gamma(x)$ it is enough to show that $\gamma(x)$ is totally bounded with respect to $\mathcal{U}$ (see [4]). For $U \in \mathcal{U}$ let $W \in \mathcal{U}$ be such that $(b,a) \in W$, $(b,c) \in W$ and $(c,d) \in W$ imply $(a,d) \in U$. By equicontinuity, for some α we have $(\varphi_t(x_\alpha), \varphi_t(x)) \in W$ for every $t \geq 0$. Now $\gamma(x_\alpha)$ is precompact, thus there exists a finite set of points $y_n = \varphi_{t_n}(x_\alpha)$ such that W_{y_n} cover $\gamma(x_\alpha)$. Hence, for fixed $t \in R_+$, $(\varphi_t(x_\alpha), \varphi_{t_n}(x_\alpha)) \in W$ for some n. Also $(\varphi_{t_n}(x_\alpha), \varphi_{t_n}(x)) \in W$ and thus $(\varphi_t(x), \varphi_{t_n}(x)) \in U$. We have obtained a finite covering U_{z_n} of $\gamma(x)$, where $z_n = \varphi_{t_n}(x)$, so the precompactness of $\gamma(x)$ is proved. The invariantness of X_0 is obvious and so the proof is complete.

Theorem. Let $\{\varphi_t\}_{t \geq 0}$ be an equicontinuous semigroup acting on a complete uniform space X. Then the following conditions are equivalent :

a) $x \in X_0$

b) $w(x)$ is nonempty and compact

c) there exists a φ_t-invariant probability measure μ_x on $w(x)$

d) for every continuous function $F : X \to E$ (E is a Banach space) the Bochner integrals

$T^{-1} \int_0^T F(\varphi_t(x))\,dt$ are convergent for $T \to \infty$ to a limit

$\bar{F}(x) \in E$.

If the above holds then $\bar{F}$ is a continuous invariant function on X_0 and it equals $\int_{w(x)} F(y)\mu_x(dy)$, where μ_x is the unique invariant probability measure on $w(x)$.

Proof. a)$\Rightarrow$b) is obvious by the definition of X_0 and $w(x)$. b)$\Rightarrow$c) is the well known corollary of the Markov - Kakutani theorem. c)$\Rightarrow$b). Suppose that $w(x)$ is non-compact. Then it is not totally bounded and thus there exists an infinite collection of nonempty pairwise disjoint open sets of the form W_{z_n}, where $z_n \in w(x)$, and $W \in \mathcal{U}$. Since $w(x)$ is minimal we may (changing if necessary the set W) choose the points z_n of the form $\varphi_{t_n}(z_0)$ for some $z_0 \in w(x)$. Now, for $V \in \mathcal{U}_W$ we have $\varphi_{t_n}(V_{z_0}) \subseteq W_{z_n}$. By invariantness of μ_x the measures of the sets W_{z_n} are at least $\mu_x(V_{z_0})$. This is a contradiction since by minimality of $w(x)$ $\mu_x(V_{z_0}) > 0$ and, on the other hand, μ_x is finite. b)$\Rightarrow$d) see [1] Th. 3.2 and Corollary 3.1 .
d)$\Rightarrow$a). Suppose $x \notin X_0$, i.e. $\gamma(x)$ is not totally bounded. An easy argument using the uniform structure allows as to find an infinite collection of open pairwise disjoint neighbourhoods U_n of certain points $x_n = \varphi_{t_n}(x)$ such that every convergent net is (starting from some index) contained in at most one of U_n's. We may also assume that for every n the set $U_n \cap \{\varphi_t(x),\ t \leqslant t_n\}$ is of the form $\{\varphi_t(x),\ t \in (t_n - \varepsilon,\ t_n]\}$. Let F_n be continuous functions on X with $F_n(x_n) = 1$, $F_n = 0$ out of U_n (see [4] for the existence of Uhryson functions on uniform spaces). The function $F = \sum_{n=1}^{\infty} \beta_n \cdot F_n$ is continuous which contradicts d) whenever β_n increases rapidly enough. To prove the last assertion of the Theorem consider $E = R$ and restrict all the functions F to the compact set $\gamma(x)$. Now observe that the map $F \to \bar{F}(x)$ is a linear nonnegativ functional on $C(\overline{\gamma(x)})$. So, by the Riesz theorem it is represented by a Radon measure ν_x on $\overline{\gamma(x)}$, i.e. we can write $\bar{F}(x) = \langle F, \nu_x \rangle$. Taking $F \equiv 1$ we obtain that ν_x is a probability measure. To see the invariantness of ν_x denote $F_s = F \circ \varphi_s$ for $F \in C(\overline{\gamma(x)})$ and $s \geqslant 0$ and calculate :

$T^{-1} \int_0^T F_s(\varphi_t(x))\,dt \longrightarrow \langle F_s, \nu_x \rangle = \langle F, \nu_x \circ \varphi_s^{-1} \rangle$. On the other hand

$$T^{-1} \int_0^T F_s(\varphi_t(x))\,dt = T^{-1} \int_s^{T+s} F(\varphi_t(x))\,dt =$$

$$= T^{-1}(T+s)(T+s)^{-1}\int_0^{T+s} F(\varphi_t(x))\,dt - T^{-1}\int_0^{s} F(\varphi_t(x))\,dt \longrightarrow \langle F, \nu_x\rangle .$$

Our last step is to check that ν_x is supported by w(x). Let $y \notin w(x)$. There exists $W \in \mathcal{U}$ such that y does not belong to the set $U = \bigcup_{z\in w(x)} W_z$ together with its open neighbourhood. But $\varphi_t(x) \in U$ for big t, hence for any continuous $F : X \to R$ with $F(y) = 1$ and $F = 0$ on U we have $F_s = 0$ on $\overline{\gamma(x)}$ and $\langle F, \nu_x\rangle = \langle F_s, \nu_x\rangle = 0$ for s big enough. The uniqueness of the measure ν_x follows from the well known Halmos - von Neumann theorem. We omit the easy standard approximation argument for proving the continuity of $\overline{F}$ on X_0 .

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WOJCIECH BARTOSZEK
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, WROCŁAW BRANCH
KOPERNIKA 18, 51-617 WROCŁAW

TOMASZ DOWNAROWICZ
INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27, 50-370 WROCŁAW