D. G. Keselman

On some problems of Choquet theory connected with potential theory

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ON SOME PROBLEMS OF CHOQUET THEORY CONNECTED WITH POTENTIAL THEORY.

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1. We shall introduce some necessary notations and definitions.

$C(Q)$ - the space of all real continuous functions on a compact space Q ;

$\mathbb{1}$ - the function equal to one on the whole of Q ;

$A(T)$ - the space of all real continuous affine functions on a convex compact set T ;

$E(T)$ - the set of all boundary points of T, or - as it is also called - the extremal boundary of T ;

$Sh_T = \overline{E(T)} \setminus E(T)$ - the increment of the set $E(T)$;

$\tau(\nu) = x$ - this means that the point x is a barycenter of the probability measure ν on T , i.e. for all $a \in A(T)$ the equality $a(x) = \nu(a)$ holds;

another terminology: the measure represents the point x, or is representative for the point x ;

$M_x^+(G)$ - the set of all probability measures which represent the point x and are centred on a Borel subset $G \subset T$;

$\sup \nu$ - the closed support of the probability measure ν on T ;

face (x) - the smallest face T containing the point x ;

S - the Choquet simplex; Q_x-the smallest closed face T contain

$_x$ - maximal measure on S representing the point ;

$= \frac{2}{x_1^2} + \frac{2}{x_2^2} + \ldots + \frac{2}{x_n^2}$ - Laplace operator in R^n

DEFINITION. We shall say that a set F $C(Q)$ separates points of Q if for any two different points x,y Q there is a function f F such that $f(x) \neq f(y)$.

DEFINITION. A linear subspace H $C(Q)$ is called a functional system if H contains and separates points of Q .

Let H' be the set of all real continuous linear functionals on H .

By the state space of H we shall call the set $S_H = L$ H' $L( ) = 1 = L$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

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$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2}$ - Laplace operator in R^n .

DEFINITION. We shall say that a set $F \subset C(Q)$ separates points of Q if for any two different points $x, y \in Q$ there is a function $f \in F$ such that $f(x) \neq f(y)$.

DEFINITION. A linear subspace $H \subseteq C(Q)$ is called a functional system if H contains $\mathbb{1}$ and separates points of Q .

Let H' be the set of all real continuous linear functionals on H .

By the state space of H we shall call the set $S_H = \{L \in H' \mid L(\mathbb{1}) = 1 = \|L\|\}$.

If H' is considered in the weak topology, then S_H is a non-empty compact convex subset of the locally convex space H'.

DEFINITION. If a point y belongs to Q, then φy denotes the element of S_H defined by the equality

$$(\varphi y)(f) = f(y) \quad \text{for } f \in H .$$

Notice that the map φ is continuous.

As H separates points of Q, φ is a one-to-one map, and is, therefore, a homeomorphism embedding Q into S_H as a compact subset. This fact allows us to consider the initial compact space Q as a subset of the state space.

DEFINITION. We shall denote by ∂_H the set of all $y \in Q$ for which $\varphi y \in E(S_H)$. We shall call ∂_H the Choquet boundary of H.

DEFINITION. A subset $G \subset Q$ is called a boundary of H if for each $f \in H$ there is a point $x \in G$ such that

$$|f(x)| = \|f\| \; (= \sup\{|f(y)| \mid y \in Q\}).$$

DEFINITION. The smallest closed boundary of H is called the Šilov boundary.

It is known that the Choquet boundary ∂_H is a boundary of H, and its closure coincides with the Šilov boundary.

We shall assume that the functional system is closed (henceforward we shall consider only the closed systems).

We shall denote by $h \to \tilde{h}$ the isometry H on $A(S_N)$, where $\tilde{h}(s) = s(h)$, $s \in S_H$.

Thus a realization of the closed functional system H on the compact space Q is gotten, in the form of the space of all real continuous affine functions on the convex compact space S_H.

As the initial compact Q can be considered as a closed subset of S_N, the isometry $h \to \tilde{h}$ is in fact the operator of extension of the functions from H from the set Q to continuous affine functions defined already on the whole state space.

DEFINITION. A functional system is called simplicial if its state space is the Choquet simplex.

2. Let ω be a relatively compact domain in R^n $(n \geq 2)$. Consider the functional system

$$H(\bar{\omega}) = \{f \in C(\bar{\omega}) \mid \Delta f = 0 \text{ in } \omega\}.$$

In 1970 Effros and Kazdan proved that $H(\ )$ is simplicial.

Sparring ourselves of the plain corrections when studying the D...chlet problem for the Laplace equation, we shall assume the topological boundary of the open relatively compact set ω to be regular, i.e. that any portion of it has a positive capacity.

Let Y be a metrizable locally compact space, and H^* a sheaf of the hyperharmonic functions on Y such that (Y/H^*) P is a harmonic space.

Consider an open, relatively compact subset $\omega \subset Y$ and a system of functions

$H(\bar{\omega}) = \{ f \in C(\bar{\omega}) \mid f$ is a harmonic function on $\omega \}$.

In 1974-76 Blitnèr and Hansen investigated the simpliciality +) of a system of a similar sort and in particular, proved that if $H(\bar{\omega})$ contains constant functions and separates points of $\bar{\omega}$, i.e. if it is a functional system, then it is simplicial.

In the light of these results the following problem arises:

Let ω_1 and ω_2 be two open, relatively compact subsets of R^{n_1} and R^{n_2}, respectively, where $n_1 > n_2 \geq 2$. Consider two simplicial functional systems:

$H(\omega_i) = \{ f \in C(\bar{\omega}_i) \mid f$ is harmonic in $\omega_i \}$, $i = 1, 2$.

Let S_1 and S_2 be their state spaces. Do the infinitely dimensional simplexes S_1 and S_2 distinguish the dimensions of the sets $\bar{\omega}_1$ and $\bar{\omega}_2$, i.e. can the simplexes S_1 and S_2 be affinely homeomorphic?

The answer to this question is given by the result stating that it is impossible, as the condition of the affine homeomorphism of two simplexes S_1 and S_2 is equivalent to the existence of a homeomorphism

$$\varphi : \overline{E(S_1)} \quad \overline{E(S_2)},$$

satisfying the conditions:

1) $\varphi(E(S_1)) = E(S_2)$

2) $(\forall x \in Sh_{S_2})$ and $\tilde{x} = \varphi(x)$

the equality $\tilde{\mu}_{\tilde{x}} = \mu_x \circ \varphi^{-1}$ holds, where $\tilde{\mu}_{\tilde{x}}$ is the maximal measure on S_2 representing the point $\tilde{x}$.

Further, Effros and Kazdan proved, when studying the simplicial character of the class of the harmonic functions, that if a relatively compact domain $\omega \subset R^n$ $(n > 2)$ has an irregular point, then the state space of the functional system $H(\bar{\omega})$ is a prime

+) In the Choquet theory usually the simpliciality of the cones of functions is investigated. But as in this paper only functional systems are considered, all the definitions and results are stated so that they be applicable to functional systems.

simplex, i.e. a simplex which is no convex span of its two proper closed boundaries. By the way, it was also proved that a necessary condition for the primarity of a simplex S is the existence of a point x in it, which is from Sh_S, for which $Q_x = S$.

In connection with it the following problem arises:

Let S be an arbitrary prime simplex,

$$K_S = \{ y \in S \mid Q_y = S \} .$$

Have K_S and Sh_S always a nonempty intersection ?
If not, then what intrinsic characterization of a simplex yields its primarity ? The answer to this problem is given by the following results:

1) It is proved that if the increment Sh_S consists of a finite number of points, then always $Sh_S \cap K_S \neq \emptyset$, and there is a point $x \in Sh_S$ for which $Q_x = S$;
2) In the general case it turns out to be untrue: an example of a prime simplex is mentioned, for which

$$Sh_S \cap K = \emptyset .$$

In the course of achieving these results in a general simplex S, the set K_S is being investigated. It is proved by the way that, in particular, in the infinitely dimensional simplexes there are no circled points (let alone the inner ones).

Further, inasmuch as the trying to get the conditions equivaent to the primarity of a simplex in the terms of the intersection of K_S with Sh_S has proved to be a failure, the study of the prime simplexes goes on, however, in the terms of the facial topology. It has been proved in the same time that any nonempty set open in the facial topology is an intersection of the neighborhoods of the points from Sh_S with E(S).

In this way we succeed in getting a necessary and sufficient condition of the primarity of a simplex S, it is namely the presence of a direction $\{\varphi_\alpha\}$ in the extremal boundary E(S), which in the facial topology converges to all boundary points.

Again, consider an open, relatively compact set ω in $R^n (n \geq 2)$ and a harmonic functional system

$$H(\bar{\omega}) = \{ h \in C(\bar{\omega}) \mid \Delta h = 0 \quad \text{in } \omega \} \quad (0.1)$$

Let S be its state space, and we shall assume that the compact set $\bar{\omega}$ is embedded in S. In the process of solution of the classical Dirichlet problem it is traditional to investigate the behavior of the solution in the inner, as well as in the boundary points of $\bar{\omega}$. However, if $\bar{\omega}$ is embedded in infinitely dimensional simplex, the interior of ω is no more its interior for S, because in the latter the interior is empty.

A natural question sets in : how to introduce an analogue of interior in infinitely dimensional simplex ?

Let S be a metric Choquet simplex. We define the Dirichlet operator $f \longrightarrow U_f$ on the space $C(\overline{E(S)})$ by setting $U_f(x) = \mu_x(f)$. The affine function U_f will be called a solution of the Dirichlet problem with continuous boundary data f. For each function f from $C(\overline{E(S)})$ let us consider the set $A_f = \{x \in S \mid U_f$ is continuous at the point $x\}$.

Let $A = \bigcap_{f \in C(\overline{E(S)})} A_f$.

It is plain that A is the set of the points of continuity of the solution of any Dirichlet problem with the continuous boundary function.

THEOREM 1. In the metric simplex S the set A has the following properties:

1) $A = \{x \in S \mid M_x^+(\overline{E(S)}) = \{\mu_x\}\}$; ρ ,

2) There is a continuous function ρ, convex on S, such that $A = A_\rho$; $\rho = \rho / \overline{E(S)}$;

3) A is of the type G_δ and is dense in S.

If the increment $Sh_S \neq \emptyset$, then for the set $I = S - A$ the following theorem holds:

THEOREM 2.

1) The set I in convex;

2) The point $x \in I \Leftrightarrow \mathrm{face}(x) \cap I \neq \emptyset$;

3) for any two points: $y \in S$ and $x \in I$ we have $]y;x] \subset I$.

Let us assume now, that S is the state space of the functional system (0,1). If the topological boundary Γ_ω contains some irregular points, then by the embedding of $\bar{\omega}$ in S these points become a subset of the set I ; i.e. the set I can be assumed as a generalization of the set of the irregular boundary points of to a Choquet simplex. On the other hand, as it follows from the classical method of Perron-Brelot, we have

$$A \cap \bar{\omega} = \omega \cup \partial_{H(\bar{\omega})} .$$

From this reason, the role of an analogue of the interior of fits ill to the set A.

Let us define the Dirichlet operator $f \rightarrow U_f$ already on the set B, of all real bounded Borel functions on $E(S)$, setting

$$U_f(x) = \int f \, d\mu_x \qquad x \in S .$$

Thus gotten affine function U_f we shall also call a solution of the Dirichlet problem with the boundary data f.

Let

$A_f = \{x \in S \mid U_f$ is continuous at $x\}$.

Let us consider the sets

$A_o = \bigcap_{f \in B} A_f$ and $I_o = S - A_o$.

THEOREM 3. For all functions f from B the following formula holds.

$$A_f = \{x \in S \mid \hat{f}(x) = \check{f}(x)\},$$

where

$\hat{f} = \inf_{a \in A(S), a/_{E(S)} \geq f} a$ and $\check{f} = \sup_{a \in A(S), a/_{E(S)} \leq f} a$

THEOREM 4. The following assertions hold:

1) $x \in A_o \Leftrightarrow \text{face}(x) \subset A_o$;

if $I_o \neq \emptyset$, then

2) I_o is convex;

3) for any points $y \in S$, $x \in I_o$ the set $[x;y[\subset I_o$;

4) $\bar{I}_o = S$.

COROLLARY. If there are no isolated points in the extremal boundary of the simplex S then for all $x \in A_o$ the maximal measure μ_x has no atomic components.

Finally, if we assume that S is the state space of the functional system (0,1), then, using the aforementioned and the method of Perron-Brelot, we get that

$$A_o \cap \bar{\omega} = \omega.$$

This fact allows to consider the set A_o as an analogue of the interior of ω and to call it a quasiinterior of the simplex S.

Let us return back to the assertion proved by Effros and Kazdan: if ω is arelatively compact domain of R^n, which has irregular points, then the state space S of the functional system $H(\bar{\omega})$ is a primary simplex.

By the embedding of $\bar{\omega}$ in S, the domain ω passes by virtue of the Harnack inequality, into a part +) ++) of the simplex belonging to its quasiinterior, which is, moreover, dense, because $\bar{\omega} \supset E(S)$.

+) Recal that two points $x,y \in S$ pass into the same loble if face(x) = face(x).

++) The author uses in Russian the word "dolya".

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The primarity of the state space at Effros and Kazdan followed from the fact that any irregular point x of the domain ω alwass has a probability measure, which represents x and charges ω, and therefore, charger a dense part ++) of A_o.

In connection with it a problem appears: how to characterize the dense parts++) in a simplex S ?

THEOREM 5. A part++) $\mathcal{D}$ is dense in a simplex S iff for each point $x \in \mathcal{D}$ we have

$$\sup \mu_x = \overline{E(S)} .$$

From this theorem it follows that if for a point $x \in Sh_S$ there is a probability measure $\nu_x \in M_x^+(S)$, which charges a dense part++) of a simplex, then the simplex is prime.

As in the elliptic case such a measure charges a dense part++) of A_o, a conjecture might appear, that in any simplex S in which the increment is nonempty, the existence of a dense part++) in A_o yields the primarity of S. However, an example can be mentioned to show that it is not true.

Let G be a closed subset of S which is a closed boundary for a functional system $A(S)$.

DEFINITION. We shall say that a point x from G has a system of weak peak points on G if for any neighborhood $\mathcal{V}(x) \subset G$ of the point x there is a quasivertex, i.e. such a function $h \in A(S)$, that $h(x) > 0$ and $h/_{G - \mathcal{V}(x)} < 0$.

A characteristics of the points which have a system of quasivertices: It is shown that such points can belong only to the Šilov boundary $\overline{E(S)}$, and that x has a system of quasivertices on G iff for each measure $\nu_x \in M_x^+(G)$ the point x belongs to $\operatorname{supp} \nu_x$.

As a corollary to this assertion a property of irregular points of open relatively compact sets $\omega \in R^n$ $(n > 2)$ is obtained namely: every its irregular point has a system of weak peak points on $\bar{\omega}$, i.e. for every its neighborhood $\mathcal{V}(x) \subset \bar{\omega}$ there is such a function $h \in H(\bar{\omega})$ that $h(x) > 0$ and $h/_{\bar{\omega} \setminus \mathcal{V}(x)} < 0$.

By the way of application to any convex xompact set T we get, that the points having a system of weak peak points on the whole T are exactly the boundary points.

++) The author uses in Russian the word "dolya" .

The main results are published in the following papers:

1. KESEL'MAN D.G. "Several properties of the Choquet simplexes", Izvěstija SKNG VS, ser.estestvennyje nauki 1979 No 4,p.15-18.

2. KESEL'MAN D.G. "On some problems of Choquet theory.Proceedings: Applications of Functional Analysis in the Approximation Theory", Kalinin 1979, p.41-51.
3. KESEL'MAN D.G. "On the affine homeomorphism of the Choquet simplexes", Sib. Mat. J. 1982, 22 No 4, p.114-117.
4. KESEL'MAN D.G. "On Quasiinterior of Choquet Symplexes in the Points of Increment", Rostov n-D. 1981, 13 p. Manuscript presented at the Rost. inž. stroit. in-tom.Dep.v. VINITI 10.Juni 1981, No 2819-81.

ASTRACHANSKIJ TECHNIČESKIJ INSTITUT RYBNOJ PROMYŠLENNOSTI I CHOZJAJSTVA, KAFEDRA VYŠŠEJ MATEMATIKI,ULICA TATIŠČEVA 16, 414025 ASTRACHAŇ, USSR.