ON ∇ - G - FOLIATIONS

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In recent years a lot of attention has been paid to the study of Riemannian foliations. A great deal is known about their characteristic classes, the geometry of leaves and the base-like cohomology. In this short note we would like to point out that ∇-G-foliations have many properties similar to the properties of Riemannian foliations. The given examples are going to show that they are met naturally in the study of certain geometrical objects on manifolds. All the objects considered in this note are smooth i.e. of class C^∞ .

1. Definitions and examples.

Let M be a smooth manifold of dimension n, and N be a smooth manifold of dimension $q \leq n$. Let B(N,G) be a G-structure on N, and ∇ a G-connection. By Γ_G denote the groupoid of germs of local diffeomorphisms of N which are both affine transformations of the connection ∇ and automorphisms of the G-structure B(N,G).

Definition. A codimension q foliation F on the manifold M defined by a Γ_G-Haefliger structure $\underline{F}$ is called a ∇-G-foliation modelled on B(N,G).

Remark 1. Actually, a ∇-G-foliation is given by the following cocycle $\{U_i, f_i, g_{ij}\}$, where $\{U_i\}$ forms a covering of M, $f_i : U_i \to N$ is a submersion for each i, g_{ij} are local diffeomorphisms, $g_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ which are at the same time both affine transformations of the connection ∇ and automorphisms of the G--structure B(N,G).

Remark 2. A codimension q, Riemannian foliation F is a ∇-O(q)--foliation for some O(q) structure on a manifold N and the Riemannian connection ∇ of this structure (cf. [6]).

Proposition 1. Let F be a ∇-G-foliation modelled on B(N,G). The normal bundle of the foliation F admits a G-reduction and its Chern-Weil homomorphism annihilates $I^r(G)$ for $r > [q/2]$.

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Proof. Let $\{U_i, f_i, g_{ij}\}$ be a cocycle defining the foliation F. The normal bundle N(F) restricted to U_i is the inverse image by f_i of the tangent bundle TN i.e. $N(F)|U_i \cong f_i^{*}TN$. Thus, on each U_i, N(F) admits a G-reduction. Since g_{ij} are automorphisms of B(N,G), the G-reductions of N(F) glue together to give a global one. The connections $f_i^{*}\nabla$ are connections in the corresponing transverse G-structures (cf. [19]), and since g_{ij} are affine transformations of the connection ∇, these connections glue together and define a transversely projectible G-connection. Its curvature form Ω is transversely projectible, thus $\Omega^r = 0$ for $r>[q/2]$. Hence the Chern-Weil homomorphism annihilates $I^r(G)$ for $r>[q/2]$.

Example 1. Let F be a G-foliation modelled on B(N,G). We shall show that if the first prolongation $g^{(1)}$ of the Lie algebra g of the group G is trivial, then in many cases the foliation F is a ∇-G-foliation.

Proposition 2. Let F be a G-foliation modelled on B(N,G) whose structure tensor vanishes and $g^{(1)}$ is trivial. Then F is a ∇-G-foliation for a torsionless connection ∇.

Proof. Since $g^{(1)}$ is trivial, a G-connection is determined by its torsion (cf. [9]). Choose ∇ to be the torsionless connection in B(N,G). Since the image of a torsionless connection by an automorphism is a torsionless connection, any automorphism of the G-structure is an affine transformation of the connection ∇, thus the foliation F is a ∇-G-foliation.

The above proposition can be generalized in the following way.

Proposition 3. Let F be a G-foliation modelled on B(N,G). Assume that the space $Hom(R^q \wedge R^q, R^q)$ admits a subspace C such that $\rho(G)(C) = C$, where ρ is the natural representation of the group $G \subset GL(R^q)$ onto the vector space $Hom(R^q \wedge R^q, R^q)$, and $Hom(R^q \wedge R^q, R^q) = C \oplus \mathrm{im}\,\partial$, where ∂ is the antisymmetrization $\partial: Hom(R^q, g) \to Hom(R^q \wedge R^q, R^q)$. If the first prolongation $g^{(1)}$ of the Lie algebra g is trivial, then the foliation F is a ∇-G-foliation.

Proof. According to [8,9] , in this case, there exists a connection ∇ which is left invariant by any automorphism of the G-structure B(N,G). This means precisely that F is a ∇-G-foliation.

Proposition 3 can be used to furnish a much simpler proof of a result due to G.Cairns (cf.[3]).

Theorem 1. Let F be an oriented Riemannian foliation. If the foliation F admits an odd, q'-codimensional extension F', then the base-like Euler class vanishes.

Proof. The fact that the foliation F is a Riemannian foliation means that F admits a bundle-like metric g. Let F' be the given extension of F i.e. (F, F') is a flag structure. Let $\{(U_i, \varphi_i)\}$ be an adapted atlas to this flag structure such that $\varphi_i(U_i) = I_i x I_i^q$, where I_i, I_i^q denote n-q dimensional, q dimensional cube, respectively. For each i the bundle-like metric g induces a metric on I_i^q such that the transition transformations are local isometries. But these transformations preserve the natural foliation F_i' of I_i^q. Thus, they also map F_i' into F_j'. Hence the foliation F is a $G = SO(q-q') x SO(q')$-foliation. But the group G is semi-simple, so its natural representation on $Hom(R^q \wedge R^q, R^q)$ is semi-simple. And, since $g^{(1)} = 0$, all the assumptions of Proposition 3 are satisfied. Therefore, the base-like Euler class of the foliation F is given by the image of the polynomial $i^*Pf \in I(SO(q-q') x SO(q'))$ via the Chern-Weil homomorphism of the connection ∇, where Pf is the Pfaffian, i the natural inclusion, and ∇ the connection of Proposition 3. Directly from the definition of Pf (cf. [7]) this class vanishes.

Example 2. A.Goetz devised a way of inducing G-connections from a given linear connection for some Lie groups G. Let H be a reductive subgroup of G. Then the Lie algebra g of the group G can be decomposed into the direct sum $g = h \oplus m$, where h is the Lie algebra of H and $Ad_v m \subset m$ for any $v \in H$. There is one-to-one correspondence between the invariant projections of g onto h and the decompositions $g = h \oplus m$.

Let $B(N,H)$ be an H-reduction of a G-structure $B_0(N,G)$. Let w be a connection on B_0 and i the inclusion of B into B_0. Then $w' = pi^*w$ is a connection on B (cf. [10,11]).

Proposition 4. Let F be a ∇-G-foliation, H a reductive subgroup of the group G. If the transition transformations g_{ij} are automorphisms of an H-structure, then F is a ∇'-H-foliation.

Proof. Let w be the G-connection. Then $w' = pi^*w$ is an H-connection. One can easily check that the transition transformations g_{ij} are affine transformations of the connection w'.

We shall apply the above to the following (cf. [24]).

Let $V = (V_1, .., V_r)$ be a set of smooth distributions of constant dimension, assume that the set V is closed under intersections. The set of indices $\{1,..,r\}$ is partially ordered by $i \leq j$ iff $V_i \subset V_j$. Define $\overline{V}_i = \sum_{j<i} V_j$ or $\overline{V}_i = 0$ if i is minimal. Let $n_i = \dim V_i/\overline{V}_i$. The set V will be called an independent system of

distributions if $V_1+\ldots+V_r = TN$ and $n_i>0$, $\sum n_i = n$ - the dimension of the ambient manifold. Let D_i be a complementary distribution to $\overline{V}_i$ in V_i, then $V_i = \sum_{j<i} D_j$. Such a system D of distributions will be called a basis for V. Having a basis D one can define a system of projectors p_i, $p_i: TN \longrightarrow D_i$, $p_i^2 = p_i$, $p_i p_j=0$ for $i \neq j$.

To relate Goetz's construction to our case of a basis D consider the following operator. As $n = \sum n_i$, D defines an almost product structure and therefore a $GL(n_1,\ldots,n_r)$-structure. Let p_i be the projectors defined by the structure. Then for any linear connection w, the form $w'=\sum p_i w$ considered on a $GL(n_1,\ldots,n_r)$-structure defines a $GL(n_1,\ldots,n_r)$ connection, as the projection $\sum p_i$ of $gl(n)$ onto $gl(n_1,\ldots,n_r)$ is $GL(n_1,\ldots,n_r)$- invariant. Therefore in this case we can apply Goetz's construction and as a corollary of Proposition 4 we obtain the following proposition:

Proposition 5. Let V be an independent system of distributions on a manifold N, F be a V-foliation (i.e. the transition transformations g_{ij} preserve V) and ∇-GL(q)-foliation. If the transition transformations g_{ij} preserve a basis of V, then F is a ∇-V-foliation.

Example 3. Degenerate Riemannian foliations. The theory of degenerate Riemannian structures differs a great deal from the Riemannian one. The most important difference, for our purpose, is that there exists no canonical metric connection for such a structure. Fortunately, V.Oproiu found a general formula for connections in these structures (cf.[20]). Using this formula, after long and tedious computations one can prove the following proposition.

Proposition 6. Let F be a ∇-GL(q)-foliation and g be a degenerate Riemannian metric on a manifold N. Let F be defined by a cocycle $\{U_i,f_i,h_{ij}\}$ such that h_{ij} are affine transformations of the connection ∇ and automorphisms of the degenerate Riemannian metric g. Let there exist locally 1-forms w_k defining a distribution H supplementary to kerg such that $h_{ij}^{*}w_k=w_k$. Then there exists a degenerate metric connection ∇' such that the foliation F is a ∇'-degenerate Riemannian foliation.

Example 4. Let V be an almost-multifoliate structure on a manifold N. In [21] I.Vaisman proves the existence and uniqueness of a linear connection on the manifold N given by the following conditions:

Let D be a basis of V defined by taking the orthogonal complements of $\overline{V}_i$ to V_i, i) $\nabla D_i \subset D_i$; ii) $\nabla_Z g(X,Y) = 0$ for any $X,Y,Z \in D_i$; $T^i(X,Y)=0$ for $X,Y \in D_i$, where T^i is the D_i-component of the torsion tensor. g denotes a Riemannian metric on the manifold N compatible with the almost-multifoliate structure V; the basis D has been taken with respect to this metric.

This connection allows us to prove the following proposition.

Proposition 7. Let F be a Riemannian almost-multifoliate foliation, i.e. the foliation F is defined by a cocycle $\{U_i, f_i, g_{ij}\}$ such that the local diffeomorphisms g_{ij} preserve an almost-multifoliate structure V and they are isometries of a Riemannian metric compatible with the almost-multifoliate structure V. Then the Chern-Weil homomorphisms of the normal bundle of the foliation F factorizes through $I(O(q) \cap G_V)_{[q/2]}$, where G_V is the structure group of the almost-multifoliate structure V.

Proof. The local diffeomorphisms g_{ij} are isometries of the Riemannian metric, hence they preserve the distributions D_i. Therefore they are affine transformations of the Vaisman connection ∇ of the almost-multifoliate structure V. From the definition ∇ is a connection in the $O(q) \cap G_V$-reduction of the frame bundle of the manifold N. Thus the foliation F is a ∇- $O(q) \cap G_V$-foliation, which together with Proposition 1 yields the result.

2. Secondary characteristic classes.

The theory of secondary characteristic classes of ∇-G-foliations is very similar to the corresponding one for Riemannian foliations developped by Lazarov-Pasternack and contained in [15]. We shall consider a ∇-G-foliation with the trivial normal bundle. The characteristic classes considered will measure how much the foliation differs from a transversely parallelisable one.

Let G be a connected reductive Lie group. Then $I(G)$ is isomorphic to the polynomial algebra $R[c_1,\dots,c_r]$, where c_i are transgressions of a basis of the space of the primitive elements of the Lie algebra g. Let $WG_q = R[c_1,\dots,c_r]_{[q/2]} \otimes \Lambda\{h_j\}_{j=1}^{r}$. The cohomology of this complex is equal to the cohomology of the complex $W(J(>q/2), J(>0))$ (cf.[16]). The homomorphism $\Delta(w,w_S)$: $WG_q \longrightarrow A(M)$ is defined by a connection w - the transversely projectible connection in the frame bundle of the foliation F induced by the connection ∇, and the flat connection w_S defined by a trivialization S of the G-reduced normal bundle. The homomorphism $\Delta(w,w_S)$ defines a morphism in cohomology denoted by $\Delta^*(F,S)$.

Theorem 6.3 of [16] provides us with a basis of the space $H(WG_q)$. We leave to the reader to restate this theorem for the complex WG_q.

Let t be the transgression $t:H^r(BG;R) \longrightarrow H^{r-1}(G;R)$, $r\geqslant 1$. The mapping t is a homomorphism of the additive structures, maps primitive elements into primitive elements and products into 0. The definition and basic properties can be found in[2]. A polynomial $P\in I^r(G)$ can be considered as an element of the space $H^{2r}(BG,R)$, because of the Weil homomorphism (cf.[7],[2]). We can write explicitly a form $\check{t}P$ on the group G which represents tP in cohomology

$$\check{t}P = (-1/2)^{(r-1)} \frac{r!\,(r-1)!}{(2r-1)!} P(w,[w,w]\ ,\dots,[w,w]\ ),$$

where w is the Maurer-Cartan form on the group G.

Theorem 2. Let F be a ∇-G-foliation and $S = \{s_1,\dots,s_q\}$ and $S' = \{s'_1,\dots,s'_q\}$ be two trivializations of the reduced G-normal bundle and $s'_i = \sum g^j_i s_j$.

Then

i/ $\Delta^*(F,S)h_j - \Delta^*(F,S')h_j = g^*tc_j$,

ii/ for $P = c_{i_1}\dots c_{i_p} \otimes h_{j_1}\wedge\dots\wedge h_{j_s}$, where $2(i_1+\dots+i_p) < q$, $j_0 = \min\{j_1,\dots,j_s\}$, $2(i_1+\dots+i_p+j_0) > q$,

$$\Delta^*(F,S)P - \Delta^*(F,S')P = \sum_k (-1)^{k-1} g^* tc_{j_k}\cdot$$

$$\Big[\Delta(F,S')c_{i_1}\dots c_{i_p}\cdot \prod_{\substack{1\leqslant k'\leqslant s\\ k\neq k'}} (\Delta(F,S')h_{j_{k'}} + g^*\check{t}c_{j_{k'}})\Big].$$

The proof is the same as that of Theorem 4.2 of [15].

Proposition 8. Let (F_t,S_t), $t\in I$, be a differentiable family of ∇-G-foliations with trivial normal bundles. Then

$$\Delta^*(F_0,S_0)P = \Delta^*(F_t,S_t)P \text{ for any } t\in I,$$

where the polynomial is of the form $P = c_I h_J$, $2(i_1+\dots+i_p+j_0) > q+2$.

Proof. Since all the reduced normal bundles are isomorphic - the trivializations define these isomorphisms - we can reduce the above problem to the case of one trivial G-bundle and a differentiable family of G-connections. Therefore we can apply results of [14]. We use them to show that the form $\partial/\partial s\,\Delta(F_s,S_s)P = \partial/\partial s\,\Delta(W_s, w^0)P$ is an exact one. The computations are long and tedious but straightforward, and we shall leave them to the reader. The important condition is that $2(i_1+\dots+i_p+j_0)-2>q$.

Theorem 3. Let F be a given ∇-G-foliation, ∇_t a differentiable family of G-connections such that the foliation F is a ∇_t-G-foliation. Let S_t be a differentiable family of trivializations of the reduced normal bundle. Let w_t be the connection of the normal bundle induced by the connection ∇_t, and w_{S_t} be the flat connection induced by the trivialization S_t. Then for any $t\in I$

$$\Delta^*(w_t,w_{S_t})P=\Delta^*(w_0,w_{S_0})P ,$$

for $P=c_ih_j$ such that i/ $2(i_1+\dots+i_p+j_0)>q+2$, or ii/ $j=(j_1)$ and $2(i_1+\dots+i_p+j_1)>q+1$.

Proof. If $2(i_1+\dots+i_p+j_0)>q+2$, we obtain from Proposition 7 what we want. If $j=(j_1)$, then $\Delta^*(w_t,w_{S_t})$ does not depend on the trivialization, and therefore we have only to consider $\Delta^*(w_t,w_{S_0})$ and applying Theorem 4 of [14] as in the proof of Proposition 7 we get that $\partial/\partial t\ \Delta^*(w_tw_{S_0})P=0$ if $2(i_1+\dots+i_p+j_0)>q+1$, as the form f_t is locally the inverse image of a 1-form on the manifold M.

Theorem 4. Let G,H be two reductive connected Lie groups, G being a closed subgroup of the group H. Let F be a given q-codimensional foliation with trivial normal bundle. Assume that the foliation F is a ∇_t-G-foliation for a differentiable family of connections ∇_t of an H-structure on a manifold N, and G-connections for some differentiable family of G-substructures of the H-structure. Let S_t be a differentiable family of trivializations of the H-reduced normal bundle being at the same time trivializations of the family of the G-reductions of the normal bundle. Then, if $i^*:I(H)\to I(G)$ is the homomorphism induced by the inclusion $i:G\to H$

$$\Delta^*(w_t,w_{S_t})\ P=\Delta^*(w_0,w_{S_0}) ,$$

for $P\in\operatorname{im} i^*$ and of the form i, ii of Theorem 3 or $P=c_{i_1}h_{i_2}$ if $2(i_1+i_2)>q+1$ and $c_{i_1}c_{i_2}\in\operatorname{im} i^*$.

Proof. Since the considered G-structures are reductions of a given H-structure and the homomorphism Δ^* is functorial in the structure groups, the first part of the theorem is a corollary of Theorem 3. The second part results from the following equality proved by S-S. Chern and J.Simons (cf.[4]). Let $c=c_ic_j\in I(H)$, then

$$\Delta(w,w_0)h = c_i \wedge \Delta(w,w_0)h_j + dW_0 = c_j \wedge \Delta(w,w_0)\, h_i + dW_1,$$

where W_0, W_1 are some forms. Therefore applying the above equality and the first part of the proof we get the result.

3. Normal bundles of order r and characteristic classes.

Let F_0 - be a ∇_0-G-foliation modelled on B(N,G) and F_1 a ∇_1-G--foliation modelled on B(N,G). We say that the foliations F_0 and F_1 are affinely G-homotopic if on the manifold MxI there exists a G-foliation F modelled on B(N;G) such that

1) F is transverse to $M\times\{t\}$ for any $t\in I$,

2) $F|M\times\{0\} = F_0$, $F|M\times\{1\} = F_1$,

3) F is defined by a cocycle $\{U_i , f_i , g_{ij}\}$ such that if $\bar{g}_{ij}(x,t) = (g_{ij}(x),t)$ and $p:N\times I \longrightarrow N$ is the natural projection, in the principal fibre bundle $p^*B(N,G)$ there exists a connection ∇ of which $\bar{g}_{ij}$ are affine transformations and $\nabla|N\times\{0\} = \nabla_0$ and $\nabla|N\times\{1\} = \nabla_1$.

One can easily check that the characteristic classes of a ∇-G--foliation depend only on its affine G-homotopy class.

Let F be a ∇-G-foliation modelled on B(N,G), and $N^r(F)$ be its normal bundle of order r (cf. [22]). On $N^r(F)$ we have a ∇^r-G^r-foliation F^r of the same dimension as F such that on the zero section i of the bundle $N^r(F)$ the foliation F^r induces the foliation F. The inverse image of the ring of characteristic classes of the foliation F^r by the mapping i is equal to the ring of characteristic classes of the foliation F (cf. [22], Lemma 14). We say that a ∇_0-G-foliation F_0 modelled on B(N,G) and ∇_1-G-foliation F_1 modelled on B(N,G) are affinely G-homotopic of order r if the lifted foliations F_0^r and F_1^r modelled on $B^r(N,G^r)$ are affinely G^r-homotopic. Therefore the following theorem is true.

Theorem 5. The characteristic classes of ∇-G-foliations depend only on affine G-homotopy classes of order r.

Remark. Theorem 5 is a generalization of Cordero's result contained in [5] .

4. Structure theorem for ∇-G-foliations.

Let F be a ∇-G-foliation modelled on B(N,G). Let B(M,F;G) be the transverse G-structure and w the transversely projectible connection in B(M,F;G) induced by ∇ . On B(M,F;G) we have a canonical foliation F_G of the same dimension as F given by $\theta = 0$, $d\theta = 0$, where θ is the fundamental form of the G-structure B(M,F;G) (cf. [18], [19]). Let A_i , i=1,..,k be a basis of the Lie algebra g =

= Lie G. Then the vector fields A_i^* i=1,..,k and $B(e_j)$, j = 1,..,q form a trivialization of the normal bundle of the foliation F_G, where A_i^* is the fundamental vertical vector field defined by A_i and $B(e_j)$ the horizontal vector field defined by $\theta(B(e_j)) = e_j \in R^q$. One can easily check that because F is a ∇-G-foliation, these vector fields are locally projectible and these projections are just the vector fields A_i^* and $B(e_j)$ of the G-structure B(N,G) with the connection ∇. Therefore the foliation F_G is transversely parallelisable. Hence we can define the transverse central sheaf $\underline{C}$ of the foliation F (cf.[18],[19]). The sections of this sheaf are the lifts of transverse infinitesimal automorphisms of the transverse G-structure B(M,F;G) which are at the same time affine transformations of the connection w. Actually, let X be a section of the sheaf $\underline{C}$. Then $[X,A_i^*]= 0$ and $[X,B(e_j)] = 0$. This is equivalent to the condition that $L_X\theta = 0$ and $L_X w = 0$. This means exactly that X is a lift of a transverse vector field on M which is a transverse infinitesimal automorphism of B(M,F;G) and an infinitesimal affine transformation of the connection w. Locally, these vector fields are pull-backs of infinitesimal automorphisms of B(N,G) which are affine infinitesimal transformations of the connection ∇.

Proposition 9. The transverse central sheaf $\underline{C}$ of a ∇-G-foliation F is the sheaf of germs of lifts of transverse infinitesimal automorphisms of the transverse G-structure B(M,F;G) which are at the same time infinitesimal affine transformations of the connection w.

Remark. Proposition 9 corresponds to Theorem 1 contained in [17] proved by P.Molino for Riemannian foliations.

Definition. A ∇-G-foliation F on M modelled on B(N,G) is transversely complete if a transverse parallelism defined by the vector fields A_i^* , i = 1,..,k , $B(e_j)$, j = 1,..,q is complete.

Theorem 6. Let F be a transversely complete ∇-G-foliation on the manifold M. Then the closures of the leaves of the foliation F_G form a locally trivial fibre bundle called the basic fibre bundle.
Proof. It is an immediate consequence of Molino's structure lemma (cf.[18],[19]).

Let p: B(M,F;G) $\longrightarrow$ W be the basic fibre bundle of the foliation F. Since any central vector field must commute with the pull-back of any vector field on the manifold W, it is tangent to the fibre. Thus the transverse orbits of the transverse central sheaf $\underline{C}$ are the closures of the leaves of the foliation F_G. As for Riemannian foliations (cf. [18], [19]) for any leaf L of the foliation F_G, the foliation of $\bar{L}$ by leaves of the foliation F_G is a Lie foliation with dense leaves.

Remark. On a compact manifold M, for any compact Lie group G, any ∇-G-foliation F is transversely complete. It is so, because the total space of the transverse G-structure B(M,F;G) is compact.

Let F be a ∇-G-foliation modelled on B(N,G) and defined by a cocycle $\{U_i, f_i, g_{ij}\}$. By imF we understand a subset $\bigcup \operatorname{im} f_i$ of the manifold N.

Theorem 7. Let G be a compact Lie group and M be a compact manifold. Let F be a ∇-G-foliation modelled on B(N,G). If a stalk of the sheaf $\underline{B}$ of germs of infinitesimal automorphisms of B(N,G), being at the same time infinitesimal affine transformations of the connection ∇, is trivial at some point of imF, then all the leaves of the foliation F are compact.

Proof. The proof is almost the same as the proof of Theorem 1 of [17]. Let x be a point at which the stalk $\underline{B}_x$ is trivial. Thus for any point m of $f_i^{-1}(x)$ the stalk $\underline{C}_m$ is trivial, hence the leaf of the foliation F_G through m is compact; thus all the leaves of F_G are compact, so the leaves of the foliation F are compact.

One of the reasons that Riemannian foliations on compact manifolds have very nice topological and geometrical properties is that the lifted foliation to the transverse O(q)-structure is transversely complete. This can be proved because the group O(q) is compact and any Riemannian foliation is ∇-G-foliation for the Riemannian connection ∇. Very often we can reduce the structure group of the normal bundle to some compact group (cf. Theorem 1), but rather rarely the Riemannian connection is a connection of this reduced G-structure. This is equivalent to vanishing of the structure tensor of this reduced transverse G-structure. But fortunately, we can show that any G-foliation, with G being a compact group, is a ∇-G-foliation for some G-connection ∇. Therefore some theorems, for example, on base-like cohomology true for Riemannian foliations are true as well for such G-foliations. Thus, in many cases, we can find easier ways of computing the base-like cohomology of Riemannian foliations with some additional transverse structure.

Proposition 10. Let F be a G-foliation modelled on B(N,G) for some compact Lie group G. Then F is a ∇-G-foliation for some connection ∇ in B(N,G).

Proof. Since G is a compact group, the first prolongation of the Lie algebra g of the group G is trivial. Since all finite dimensional representations of the group G are semi-simple, so is the natural representation on $Hom(R^q \wedge R^q, R^q)$. Therefore the assumptions of Proposition 3 are fulfilled and F is a ∇-G-foliation.

Let F be a G-foliation with the group G connected and compact on a compact manifold M. Since the foliation F_G on the total space of the transverse G-structure B(M,F;G) is transversely parallelisable, using the same méthods as in [1] we can prove the following theorem.

Theorem 9. Let G be a compact Lie group and F a G-foliation on a compact manifold M. Then the basic Leray-Serre spectral sequence of the principal fibre bundle B(M,F,G) ---> M **has the second term** $E_2^{pq} = H^p(M/F) \otimes H^q(g)$ and is convergent to $H^*(B(M,F;G),F_G)$.

To complete this short note we would like to point out how the structure of the group of automorphisms of the model G-structure influences the behaviour of the leaves of a given G-foliation; for the proofs and more details see [23] . Let Γ be a pseudogroup of local diffeomorphisms of the manifold N. We say that the pseudogroup Γ has the property E_k , k any integer, if for any point x of the manifold N, the spaces $\{ j_x^k f : f \in \Gamma(U_m) \}$ are equal for some sequence of open subsets U_m such that $U_0 = N$ and $\bigcap U_m = \{x\}$. Let G be a closed subgroup of the linear group GL(q) of finite type k, let B(N,G) be a G-structure on the manifold N. Let Γ be a pseudogroup of automorphisms of the G-structure B(N,G) having the property E_k . Let F be a Γ-foliation on an n-manifold M in the sense of Haefliger (cf. [12]). Then we have the following.

Theorem 10. The lifted foliation $\widetilde{F}$ to the universal covering $\widetilde{M}$ of the manifold M is simple i.e. defined by a global submersion.

Theorem 11. Assume that the manifold M is compact. The growth of the leaves of the foliation F is dominated by the growth of the fundamental group $\pi_1(M)$ of the manifold M.

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