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PLANE ELLIPTIC SYSTEMS AND MONOGENIC FUNCTIONS IN SYMMETRIC DOMAINS

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Abstract In this paper we introduce generalized Laurent expansions for monogenic functions defined in open domains of R^{m+1} which are invariant under certain subgroups of SO(m+1). In this way we generalize the plane elliptic system introduced by P. Lounesto and P. Bergh in [4] in order to study axially symmetric monogenic functions.

Introduction Let A be the Clifford algebra constructed over C^m. Then in [1] and [3] A-valued functions f in open subsets Ω of R^{m+1} were investigated which satisfy $Df = 0$ or $fD = 0$ in Ω, $D=\sum_{j=o}^{m} e_j\frac{\partial}{\partial x_j}$ being a generalized Cauchy-Riemann operator. They were called left or right monogenic functions in Ω. Moreover in [6] we proved a Laurent type expansion for left monogenic functions in annular domains of the form $B(0,R)\setminus\overline{B}(0,r)$, $0<r<R\leq\infty$, and we introduced the so called spherical monogenic functions, which are a refinement of the spherical harmonics.

The ideas behind this paper emerge from group representation theory. Let G be a subgroup of SO(m+1) and let $\Omega\subseteq R^{m+1}$ be open and invariant under G. Then G acts in a natural way on the space $M_{(r)}(\Omega;A)$ of left monogenic functions in Ω (see [7]) and so we obtain a representation of G. The generalized Laurent expansion for monogenic functions in Ω then arises from splitting this representation into irreducible pieces.

When G = SO(m+1), Ω is an annular domain and the Laurent expansion according to G is the expansion of $f\in M_{(r)}(\Omega;A)$ into spherical monogenic functions.

The first section contains a result about restrictions to S^{m-1} of left monogenic functions in a neighbourhood of S^{m-1} in R^{m+1}. It is based on the use of Lie balls and complex harmonic functions (see

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[5]).
In section 2 we consider the case where $G = SO(m)$. The corresponding open set Ω is then called axially symmetric and we obtain a Laurent expansion of the form $f = \sum_{k=o}^{\infty} \Pi_k f$, where $\Pi_k f$ are so called axial monogenic functions, satisfying special plane elliptic systems.
For k=0 we obtain the system introduced by P. Lounesto and P. Bergh (see [4]).
In section 3 we consider so called toral symmetry, where G is of the form $SO(m_1)\times\ldots\times SO(m_k)$. We restrict our attention to the case $G = SO(m_1)\times SO(m_2)$, where we obtain an expansion into "toral monogenic functions" which again satisfy certain plane elliptic systems, and to the case $G = SO(2)\times\ldots\times SO(2)$, which provides a new link between monogenic functions and A-valued holomorphic functions of several complex variables.

Preliminaries. Throughout this paper A denotes the Clifford algebra over C^m. A basis for A is given by $\{e_A : A\subseteq\{1,\ldots,m\}\}$, where $e_i = e_{\{i\}}$ $(i=1,\ldots,m)$, $e_o=e_\phi=1$, $e_i^2 = -e_o$ $(i=1,\ldots,m)$ and $e_ie_j + e_je_i = 0$ $(i\neq j,\ i,j = 1,\ldots,m)$ and where $e_A = e_{\alpha_1}\ldots e_{\alpha_h}$ when $A = \{\alpha_1,\ldots,\alpha_h\}$ with $\alpha_1<\ldots<\alpha_h$. For the definition of the involutions and the norm on A we refer to [1].
Arbitrary elements of R^m and R^{m+1} will be denoted by $\vec{x}=(x_1,\ldots,x_m)$ and $x=(x_o,x_1,\ldots,x_m)$ and will be identified with the Clifford numbers $\vec{x} = \sum_{j=1}^{m} x_je_j$ and $x = \sum_{j=o}^{m} x_je_j = x_o+\vec{x}$.
Let $\Omega\subseteq R^{m+1}$ be open. Then a function $f\in C_1(\Omega;A)$ is called left (right) monogenic in Ω if it satisfies $Df = 0$ $(fD = 0)$, where $D = \sum_{j=o}^{m} e_j \frac{\partial}{\partial x_j}$ generalizes the classical Cauchy-Riemann operator.
We also consider $\Omega\subseteq R^m$ open and functions $f\in C_1(\Omega;A)$ satifying $D_0f=0$ in Ω, $D_0 = \sum_{j=1}^{m} e_j \frac{\partial}{\partial x_j}$ being the Dirac operator. Those functions will also be called left monogenic and their corresponding right A-modules are denoted by $M_{(r),m}(\Omega;A)$.
For $\Omega\subseteq C^{m+1}$ open, $H(\Omega;A)$ denotes the space of A-valued holomorphic functions in Ω.

For $\Omega \subseteq \mathbb{R}^{m+1}$ open, $\mathcal{D}_{(1)}(\Omega;A)$ and $\mathcal{E}_{(1)}(\Omega;A)$ are the left A-modules of A-valued testfunctions and C_∞-functions in Ω. $\mathcal{D}'_{(1)}(\Omega;A)$ is the right module of left A-distributions in Ω. $B(0,R)$ (resp. $B_m(0,R)$) denotes the ball in $\mathbb{R}^{m+1}$ (resp. $\mathbb{R}^m$) with radius R and ω_m is the area of S^{m-1}.
Finally by $d\sigma_y$ (resp. $d\vec{\sigma}_y$) we mean the hypercomplex surface element in $\mathbb{R}^{m+1}$ (resp. $\mathbb{R}^m$) introduced in [1].

1. *RESTRICTIONS OF MONOGENIC FUNCTIONS*

Let $f \in M_{(r)}(S^{m-1}+B(0,R);A)$, $0<R<1$. Then $f|S^{m-1}$ is an analytic functions on S^{m-1}. Hence, in view of the Cauchy-Kowalewski extension theorem in [1] and [6], there exists an extension $r_m(f)$ of $f|S^{m-1}$, defined in an annulus of the form $B_m(0,R')\setminus \overline{B}_m(0,\frac{1}{R'})$, $R'>0$, and satisfying $D_0 r_m(f) = 0$.
Using results from [1], [5], [6] we have

Lemma 1. Let $0<R<1$. Then there exists $R'>1$, only depending upon R, such that the map r_m is continuous from $M_{(r)}(S^{m-1}+B(0,R);A)$ into $M_{(r),m}(B_m(0,R')\setminus \overline{B}_m(0,R'^{-1});A)$.

Remark. Let $y \in \mathbb{R}^{m+1}\setminus S^{m-1}$. Then the function $K(x,y) = \frac{1}{\omega_{m+1}} \frac{\overline{y}-\overline{x}}{|y-x|^{m+1}}$ belongs to $M_{(r)}(S^{m-1}+B(0,R);A)$, $R = |y - \frac{\vec{y}}{|\vec{y}|}|$. Hence for some $R'(y)>1$, $r_m(K(\cdot,y))(\vec{x}) = L(\vec{x},y)$ belongs to $M_{(r),m}(B(0,R'(y))\setminus \overline{B}(0,R'(y)^{-1})A)$.
Furthermore in view of Lemma 1, $L(\vec{x},y)D_y=0$ and for every $f \in M_1(S^{m-1}+\overline{B}(0,R);A)$,

$$r_m(f)(\vec{x}) = \int_{\partial(S^{m-1}+B(0,R))} L(\vec{x},y)\,d\sigma_y\,f(y).$$

2. *GENERALIZED LAURENT EXPANSION IN AXIAL SYMMETRIC DOMAINS*

Let $\mathbb{R}^m = \{x \in \mathbb{R}^{m+1}\ x_0=0\}$ and let SO(m) be the group of rotations leaving the x_0-axis invariant. Then a domain $\Omega \subseteq \mathbb{R}^{m+1}$ is called axial symmetric if it is invariant under SO(m).
Hence for every $x \in \Omega$ the sphere $S_{x_0,|\vec{x}|} = \{y \in \mathbb{R}^{m+1} : y_0 = x_0 \text{ and } |\vec{y}|=|\vec{x}|\}$ is contained in Ω and so we may introduce cilindrical coordinates (x_0,ρ,ω) in Ω by means of the formulae $x=x_0+\rho\omega$, $|\vec{x}|=\rho$, $\omega=\frac{\vec{x}}{|\vec{x}|}$.
Furthermore we use the notation $\vec{e}_\rho$ instead of ω to denote the unit

normal on S^{m-1}.
We know from [6] that the Cauchy-Riemann operator D may be written in the form

$$D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\vec{e}_\rho \Gamma_\omega,$$

Γ_ω being the spherical Cauchy-Riemann operator.
Furthermore there exists an open domain $\tilde{\Omega}$ in the halfplane $\{(x_0,\rho) : \rho>0\}$ such that for every $\omega\in S^{m-1}$, $S_\omega\tilde{\Omega}=(\Omega\setminus\{x:\vec{x}=0\}) \cap \{x : \vec{x} = |\vec{x}|\omega\}$, where $S_\omega\tilde{\Omega} = \{x_0+\rho\omega : (x_0,\rho)\in\tilde{\Omega}\}$.
Let $f\in M_{(r)}(\Omega\setminus\{x : \vec{x} = 0\};A)$. Then $f\in M_{(r)}(\Omega;A)$ if and only if f admits a continuous extension to Ω (see [1]). Hence we may restrict ourselves to the case $\Omega\cap\{x : \vec{x} = 0\} = \phi$.
Let $f\in M_{(r)}(\tilde{\Omega};A)$ and express f in cilinder coordinates $f(x) = f(x_0,\rho,\omega)$, $(x_0,\rho)\in\tilde{\Omega}$, $\omega\in S^{m-1}$.
Then in view of [6], for each $(x_0,\rho)\in\tilde{\Omega}$ fixed we may expand f into a series of spherical monogenics on S^{m-1}.

$$f(x_0,\rho,\omega) = \sum_{k=o}^{\infty}(P_k f_{(x_0,\rho)}(\omega) + Q_k f_{(x_0,\rho)}(\omega))$$

and, using Lemma 1 we have

<u>Lemma 2</u>. The series

$$f(x_0,\rho,\omega) = \sum_{k=o}^{\infty}(P_k f_{(x_0,\rho)}(\omega) + Q_k f_{(x_0,\rho)}(\omega))$$

converges uniformly on the compact sets of Ω.

Let $\phi(x_0,\rho)$ be a testfunction in $\tilde{\Omega}$ and let T be a distribution in $\mathcal{D}'_{(1)}(\Omega;A)$. Then by $\langle T,\phi(x_0,\rho)\rangle$ the distribution on S^{m-1} is meant which maps every $\psi\in E_{(1)}(S^{m-1};A)$ onto $\langle T,\psi(\omega)\phi(x_0,\rho)\rangle$.
Furthermore we say that a distribution T has degree k when for every $\phi(x_0,\rho)$, $\langle T,\phi(x_0,\rho)\rangle$ is the sum of an inner and an outer spherical monogenic of degree k.
We now have

<u>Lemma 3</u>. Let $(T_k)_{k\in N}$ be a sequence of distributions in Ω such that T_k has degree k and $\sum_{k=o}^{\infty} T_k = 0$ in $\mathcal{D}'_{(1)}(\Omega;A)$. Then for every $k\in N$, $T_k = 0$.

In the sequel we use the notation $\Pi_k f(x)$ for $P_k f_{(x_0,\rho)}(\omega) + Q_k f_{(x_0,\rho)}(\omega)$. A function of the form $\Pi_k f$, $f \in M_{(r)}(\Omega;A)$, will be called axial monogenic of degree k.
We now come to

Theorem 1. (Generalized Laurent expansion)
The functions $\Pi_k f$ are left monogenic in Ω. Furthermore the operators Π_k are projection operators on $M_{(r)}(\Omega;A)$ and the series $f(x) = \sum_{k=o}^{\infty} \Pi_k f(x)$ converges in $M_{(r)}(\Omega;A)$.

Proof. From Lemma 2 it follows that uniformly on compact sets in Ω, $f(x) = \sum_{k=o}^{\infty} \Pi_k f(x)$.
Hence we have that in distributional sense

$$0 = \sum_{k=o}^{\infty} D\Pi_k f(x).$$

Using the decomposition $D = \frac{\partial}{\partial x_0} + \vec{e}_\rho \frac{\partial}{\partial \rho} + \frac{1}{\rho} \vec{e}_\rho \Gamma_\omega$, the fact that $\Pi_k f$ is a distribution of degree k, the properties of Γ_ω and the spherical transform (see [6]) it is easy to see that $D\Pi_k f$ is a distribution of degree k. Hence, by Lemma 3, $D\Pi_k f = 0$ for all $k \in N$ and so $\Pi_k f$ is left monogenic in Ω and $f = \sum_{k=o}^{\infty} \Pi_k f$ converges in $M_{(r)}(\Omega;A)$.
Furthermore it also follows from Lemma 3 that $\Pi_k \Pi_l = \delta_{kl} \Pi_l$. □
In order to study the nature of axial monogenic functions we introduce the notations

$$A_{k,\omega}(x_0,\rho) = P_k f_{(x_0,\rho)}(\omega) \text{ and } B_{k,\omega}(x_0,\rho) = -\vec{e}_\rho Q_k f_{(x_0,\rho)}(\omega)$$

Notice that for each (x_0,ρ) $\tilde{\Omega}$ fixed, $A_{k,\omega}(x_0,\rho)$ and $B_{k,\omega}(x_0,\rho)$ are inner spherical monogenics of degree k.
Furthermore, using the monogenicity of $\Pi_k f$, we obtain

Theorem 2. For each $\omega \in S^{m-1}$ fixed, $A_{k,\omega}$ and $B_{k,\omega}$ satisfy the equations

$$\text{(i)} \quad \frac{\partial}{\partial x_0} A_{k,\omega} - \frac{\partial}{\partial \rho} B_{k,\omega} = \frac{k+m-1}{\rho} B_{k,\omega}$$

$$\text{(ii)} \quad \frac{\partial}{\partial x_0} B_{k,\omega} + \frac{\partial}{\partial \rho} A_{k,\omega} = \frac{k}{\rho} A_{k,\omega}.$$

Remarks. (1) For k=0 the system (i), (ii) reduces to the system of generalized Cauchy-Riemann equations in the plane investigated by P. Lounesto and P. Bergh in [4]. It gives the description of the left monogenic functions which are invariant under SO(m).
(2) The system (i), (ii) may be rewritten as follows

$$\left(\frac{\partial}{\partial x_0}+i\frac{\partial}{\partial\rho}\right)(A+iB) = \frac{k+m-1}{\rho}B + i\frac{k}{\rho}A$$

$$= \frac{ik}{\rho}(A-iB) + i\frac{m-1}{2\rho}((A-iB)-(A+iB))$$

or putting $w = A + iB$,

$$\frac{\partial}{\partial\bar{z}}w = -i\frac{(m-1)}{4\rho}w + i\frac{k+\frac{m-1}{2}}{2\rho}\bar{w}.$$

Hence it is an equation of the form

$$\frac{\partial}{\partial\bar{z}}w + aw + b\bar{w} = 0,$$

which were investigated by Carleman [2] and Vekua [8] and so we may apply their theory to the above system. This leads to new theorems about the zeros of monogenic functions.

3. GENERALIZED LAURENT EXPANSION IN TORAL SYMMETRIC DOMAINS

3.1. The case $G = SO(m_1)\times SO(m_2)$

Let $D_1 = \sum_{j=1}^{m_1} e_j\frac{\partial}{\partial x_j}$ and $D_2 = \sum_{j=m_1+1}^{m_1+m_2} e_j\frac{\partial}{\partial x_j}$.

Then $D_1D_2 + D_2D_1 = 0$ and $D_0 = D_1+D_2$ is the Dirac operator in $R^{m_1+m_2}$. Let $SO(m_1)\times SO(m_2)$ be the subgroup of $SO(m_1+m_2)$ which rotates the variables $\vec{x}_1 = (x_1,\ldots,x_{m_1})$ and $\vec{x}_2 = (x_{m_1+1},\ldots,x_{m_1+m_2})$ independently. Then we shall now consider solutions of $D_0f=0$ in open subsets Ω of $R^{m_1+m_2}$ which are invariant under $SO(m_1)\times SO(m_2)$ and which, for reasons of simplicity, do not intersect the hyperplanes $\vec{x}_1=0$ and $\vec{x}_2=0$.
Furthermore we introduce so called toral coordinates in $R^{m_1+m_2}$ by putting $\vec{x}_j = r_j\omega_j$, $\omega_j = \frac{\vec{x}_j}{|\vec{x}_j|}$ and $r_j = |\vec{x}_j|$. Notice that in these notations $D_j = \vec{e}_{r_j}\frac{\partial}{\partial r_j} + \frac{1}{r_j}\vec{e}_{r_j}\Gamma_j$, where $\vec{e}_{r_j} = \omega_j$ and Γ_j is the spherical Cauchy-Riemann operator on S^{m_j-1}.

As $[\Gamma_1,\Gamma_2] = 0$ it makes sense to look for simultaneous eigenfunctions of Γ_1 and Γ_2 on $S^{m_1-1}\times S^{m_2-1}$ and the theory is similar to the theory of spherical monogenic functions.
We have the following

<u>Definition 1</u>. Let $(k,l)\in Z^2$. Then a function $P_{k,l}(\omega_1,\omega_2)$ is called simultaneous spherical monogenic of degree (k,l) in the following cases :

(i) if $(k,l)\in N^2$, $r_1^k r_2^l P_{k,l}(\omega_1,\omega_2)$ satisfies $D_1 f = D_2 f = 0$,

(ii) if $(k,l)\in (Z\setminus N)\times N$, $r_1^{-k-1} r_2^l\, \vec{e}_{r_1} P_{k,l}$ satisfies $D_1 f = D_2 f = 0$,

(iii) if $(k,l)\in N\times (Z\setminus N)$, $r_1^k r_2^{-l-1}\, \vec{e}_{r_2} P_{k,l}$ satisfies $D_1 f = D_2 f = 0$,

(iv) if $(k,l)\in (Z\setminus N)^2$, $r_1^{-k-1} r_2^{-l-1}\, \vec{e}_{r_1}\vec{e}_{r_2} P_{k,l}$ satisfies $D_1 f = D_2 f = 0$ in $R^{m_1+m_2}$.

Notice that in the cases (i)-(iv), the corresponding eigenvalues of (Γ_1,Γ_2) are respectively given by $(-k,-l)$, $(m_1-k-2,-l)$, $(-k,m_2-l-2)$, (m_1-k-2,m_2-l-2). We hereby make use of the fact that $[\Gamma_1,\vec{e}_{r_2}] = [\Gamma_2,\vec{e}_{r_1}] = 0$.
We now study the expansion of analytic functions on $R^{m_1+m_2}$ into simultaneous spherical monogenics.
Let $f(\omega_1,\omega_2)$ be analytic on $S^{m_1+m_2}$. Then we have

<u>Lemma 4</u>. There exists a unique function $\tilde{f}(x_1,x_2)$ satisfying $D_1\tilde{f} = D_2\tilde{f} = 0$ in a neighbourhoud of $S^{m_1-1}\times S^{m_2-1}$ in $R^{m_1+m_2}$, such that $\tilde{f}|S^{m_1-1}\times S^{m_2-1} = f$.

Using this Lemma we obtain

<u>Lemma 5</u>. There exist unique simultaneous spherical monogenics $P_{k,l}f$ such that on $S^{m_1-1}\times S^{m_2-1}$, $f = \sum_{k,l=-\infty}^{\infty} P_{k,l}f$.

Furthermore for some $C>0$ and $0<\delta<1$,

$$\sup_{\omega_1,\omega_2} |P_{k,l}f(\omega_1,\omega_2)| \leq C(1-\delta)^{|k|+|l|}, \text{ for all } (k,l)\in Z^2.$$

Let f be a solution of $(D_1+D_2)f = 0$ in an open set Ω which is invariant under $SO(m_1)\times SO(m_2)$.
Then there exists $\tilde{\Omega} \subseteq \{(r_1,r_2) : r_j>0\}$ such that $\Omega = \{r_1\omega_1+r_2\omega_2 : (r_1,r_2)\in\tilde{\Omega},\ \omega_j\in S^{m_j-1},\ j=1,2\}$.
Hence f gives rise to a set of analytic functions $f_{(r_1,r_2)}(\omega_1,\omega_2)$

on $S^{m_1-1}\times S^{m_2-1}$, parametrized by $(r_1,r_2)\in\tilde{\Omega}$ and determined by $f_{(r_1,r_2)}(\omega_1,\omega_2) = f(r_1\omega_1+r_2\omega_2)$.
Let $P_{k,l}f_{(r_1,r_2)}(\omega_1,\omega_2)$ be the simultaneous spherical monogenics associated to $f_{(r_1,r_2)}(\omega_1,\omega_2)$ and put for $(k,l)\in N^2$

$$\Pi_{k,l}f(x) = P_{k,l}f_{(r_1,r_2)}(\omega_1,\omega_2) + P_{-k-1,l}f_{(r_1,r_2)}(\omega_1,\omega_2)$$

$$+ P_{k,-l-1}f_{(r_1,r_2)}(\omega_1,\omega_2)+P_{-k-1,-l-1}f_{(r_1,r_2)}(\omega_1,\omega_2).$$

Then $\Pi_{k,l}f$ will be called a toral monogenic funciton of degree (k,l) and we have

Lemma 6. The series $f = \sum_{k,l=o}^{\infty} \Pi_{k,l}f$ converges uniformly on the compact subsets of Ω.

Using distributional techniques similar to the axial monogenic case, we now come to the toral version of the generalized Laurent expansion.

Theorem 3. The function $\Pi_{k,l}f$ are left monogenic in Ω. Furthermore the operators $\Pi_{k,l}$ are projection operators on $M_{(r),m_1+m_2}(\Omega;A)$ and the series $f = \sum_{k,l} \Pi_{k,l}f$ converges in $M_{(r),m_1+m_2}(\Omega;A)$.

In order to obtain the equations for $\Pi_{k,l}f$, we bring $\Pi_{k,l}f$ into the form

$$\Pi_{k,l}f = A_{k,l} + \vec{e}_{r_1}B_{k,l} + \vec{e}_{r_2}C_{k,l} + \vec{e}_{r_1}\vec{e}_{r_2}D_{k,l},$$

$A_{k,l}$, $B_{k,l}$, $C_{k,l}$ and $D_{k,l}$ being simultaneous spherical monogenics of degree $(k,l)\in N^2$.
We then obtain

Theorem 4. For each $(\omega_1,\omega_2)\in S^{m_1-1}\times S^{m_2-1}$ fixed, $A_{k,l}$, $B_{k,l}$, $C_{k,l}$ and $D_{k,l}$ satisfy the equations

$$\text{(i)}\quad \left(\frac{\partial}{\partial r_1} + \frac{k+m_1-1}{r_1}\right)B_{k,l} + \left(\frac{\partial}{\partial r_2} + \frac{l+m_2-1}{r_2}\right)C_{k,l} = 0$$

$$\text{(ii)}\quad \left(\frac{\partial}{\partial r_1} - \frac{k}{r_1}\right)C_{k,l} - \left(\frac{\partial}{\partial r_2} - \frac{l}{r_2}\right)B_{k,l} = 0$$

$$\text{(iii)}\quad \left(\frac{\partial}{\partial r_1} - \frac{k}{r_1}\right)A_{k,l} + \left(\frac{\partial}{\partial r_2} + \frac{l+m_2-1}{r_2}\right)D_{k,l} = 0$$

$$\text{(iv)}\quad \left(\frac{\partial}{\partial r_1} + \frac{k+m_1-1}{r_1}\right)D_{k,1} - \left(\frac{\partial}{\partial r_2} - \frac{1}{r_2}\right)A_{k,1} = 0.$$

Notice that the system (i)-(iv) splits into two separate systems (i),(ii) and (iii),(iv).
They are also plane elliptic systems of Vekua type.

3.2. The case $G = SO(2)\times\ldots\times SO(2)$

Consider $\vec{x}+i\vec{y}\in C^m$ and identify $SO(2)\times\ldots\times SO(2)$ with the group $\{(e^{i\theta_1},\ldots e^{i\theta_m}) : \theta_j\in[0,2\pi[\,,\ j=1,\ldots,m\}$.
Then an open set $\Omega\subseteq C^m$ which is invariant under G is called a Reinhardt domain and the orbits of G are polydics.

Let $D_0 = \sum_{j=1}^{m} \left(e_j \frac{\partial}{\partial x_j} + e_{j+m} \frac{\partial}{\partial y_j}\right)$ be the Dirac operator in $C^m = R^{2m}$ and put $i_j = -e_j e_{j+m}$.
Then $i_j^2 = -1$ and $[i_j, i_k] = 0$, $(j\neq k,\ j,k=1,\ldots,m)$.
Furthermore D_0 may be written in the form

$$D_0 = \sum_{j=1}^{m} e_j \left(\frac{\partial}{\partial x_j} + i_j \frac{\partial}{\partial y_j}\right).$$

Notice that special solutions to $D_0 f = 0$ are the solutions t. the system

$$\left(\frac{\partial}{\partial x_j} + i_j \frac{\partial}{\partial y_j}\right) f(\vec{x},\vec{y}) = 0,\ j=1,\ldots,m,$$

which gives rise to a theory which is quite similar to the theory of holomorphic functions of several complex variables.
When $z_j = x_j + i_j y_j$, these functions admit local Taylor series expansions of the form

$$f(\vec{x},\vec{y}) = \sum_{k_1,\ldots,k_m} z_1^{k_1} z_2^{k_2}\ldots z_m^{k_m}\, a_{k_1,\ldots,k_m},$$

where $a_{k_1,\ldots,k_m}\in A$.

We now introduce toral coordinates in C^m by putting

$$z_j = r_j e^{i_j\theta_j},\ r_j\in[0,+\infty[,\ \theta_j\in[0,2\pi[\,.$$

In these coordinates, $D_0 = \sum_{j=1}^{m} e_j e^{i_j\theta_j}\left(\frac{\partial}{\partial r_j} + \frac{i_j}{r_j}\frac{\partial}{\partial\theta_j}\right)$.

Let Ω be invariant under $SO(2)\times\ldots\times SO(2)$ and let $D_0 f = 0$ in Ω. Then we can expand f into a series of the form

$$f(x) = \sum_{k_1,\ldots,k_m} e^{i_1k_1\theta_1+\ldots+i_mk_m\theta_m} f_{k_1,\ldots,k_m}(r_1,\ldots,r_m).$$

and we have that

$$D_0 f(x) = \sum_{k_1,\ldots,k_m} \sum_{j=1}^{m} e_j e^{i_1k_1\theta_1+\ldots i_j(k_j+1)\theta_j+\ldots+i_mk_m\theta_m} \left(\frac{\partial}{\partial r_j} - \frac{k_j}{r_j}\right) f_{k_1,\ldots,k_m}(r_1,\ldots,r_m)$$

$$= \sum_{k_1,\ldots,k_m} \sum_{j=1}^{m} e^{i_1k_1\theta_1+\ldots-i_j(k_j+1)\theta_j+\ldots+i_mk_m\theta_m} e_j\left(\frac{\partial}{\partial r_j} - \frac{k_j}{r_j}\right) f_{k_1,\ldots,k_m}(r_1,\ldots,r_m)$$

$$=0.$$

If we now write $D_0 f(x)$ again into the form

$$D_0 f(x) = \sum_{k_1,\ldots,k_m} e^{i_1k_1\theta_1+\ldots+i_mk_m\theta_m} g_{k_1,\ldots,k_m}(r_1,\ldots,r_m),$$

we obtain that

$$g_{k\ ,\ldots,k_m} = \sum_{j=1}^{m} e_j\left(\frac{\partial}{\partial r_j} + \frac{k_j+1}{r_j}\right) f_{k_1,\ldots,-k_j-1,\ldots,k_m}.$$

Hence the equation $D_0 f = 0$ eventually leads to the system

$$\sum_{j=1}^{m} e_j\left(\frac{\partial}{\partial r_j} + \frac{k_j+1}{r_j}\right) f_{k_1,\ldots,-k_j-1,\ldots,k_m} = 0,$$

which may be splitted into systems, parametrised by $(k_1,\ldots,k_m)\in\mathbb{N}^m$ of 2^m equations.
Notice that special solutions to these systems are given by

$$f_{k_1,\ldots,k_m} = r_1^{k_1}\ldots r_m^{k_m},$$

which gives rise to holomorphic functions of several complex variables.

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