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Some results concerning reconstruction conjecture

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SOME RESULTS CONCERNING RECONSTRUCTION CONJECTURE

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Abstract: B.Manvel first showed that, for every k, there exist two finite nonisomorphic graphs with the same collections of of k-point subgraphs. Here, we give some new results concerning Manvel's observation. We find the bounds of reconstructibility and nonreconstructibility of graphs from subgraphs for some classes of graphs /all graphs, all trees, all equivalences/.

0. Introduction

We consider finite undirected graphs without loops and multiple edges. More precisely: for a set X we denote $P_2(X)$ the set of all 2-point subsets of X; a graph is a couple $G = \langle V(G), E(G) \rangle$, where V(G) is a finite set and $E(G) \subseteq P_2(V(G))$.

A mapping $f: V(G) \longrightarrow V(H)$ is called the homomorphism from the graph G into the graph H if for every $Z \in V(G)$ $f(Z) \in V(H)$, and is called the isomorphism if f is a bijection and for every Z $f(Z) \in V(H)$ if and only if $Z \in V(G)$. We write $G \simeq H$ to indicate isomorphic graphs.

For every subset Y of the set V(G) of the graph G the induced graph $G/Y = \langle Y, V(G) \cap P_2(Y) \rangle$ is defined. The number of induced graphs of the graph G isomorhic to the graph H is called the frequency of H in G and denoted by frq(H,G).

We use homomorphisms of some special types. A homomorphism $f: G \longrightarrow H$ is called the monomorphism if $f: G \longrightarrow H/f(V(G))$ is an isomorphism and is called the semimonomorphism if for every component of connectivity C of the graph G $f: G/C \longrightarrow H/f(V(C))$ is an isomorphism. A homomorphism $f: G \longrightarrow H$ is said to be covering if $f(V(G)) = V(H)$. It is obvious that every covering monomorphism has to be an isomorphism.

The number of components of connectivity of the graph G will be denoted by cp(G). It is obvious that a semimonomorphism $f: G \longrightarrow H$ is a monomorphism if and only if $cp(G) = cp(H/f(V(G)))$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

We use some integral-valued functions:

card X ... denotes the number of elements of the set X,

$|G|$ ... = card(V(G)) for the graph G,

mono(G,H) ... denotes the number of monomorphisms from G into H,

semi(G,H) ... denotes the number of semimonomorphisms from G intoH

cov(G,H) ... denotes the number of covering semimonomorphisms from G into H,

aut(G) ... denotes the number of automorphisms of the graph G /we use the identity mono(G,H) = frq(G,H).aut G/.

1. The frequency and the similarity of graphs

Definition 1.1. Let G,H be two graphs such that $|G| = |H|$, let k be an integer. The graphs G,H are called k-similar /≤k-similar, ≤k-c-similar, respectively/ if for every graph R such that $|R| = k$ / $|R| \leq k$, $|R| \leq k$ & R is connected, respectively/ frq(R,G) = = frq(R,H) holds. We use the notation $G \overset{k}{\sim} H$ / $G \overset{\leq k}{\sim} H$, $G \overset{\leq k}{\underset{c}{\sim}} H$, respectively/.

Corollary 1.2. /Kelly's lemma/. For any two graphs G,H and any integer k, if $G \overset{k}{\sim} H$, then $G \overset{\leq k}{\sim} H$.

Proof. See [1] pp. 229-230.

Corollary 1.3. /Reconstruction conjecture/. It is conjectured that for any two graphs G,H such that $n = |G| = |H| \geq 2$ the implication " if $G \overset{n-1}{\sim} H$, then $G \simeq H$ " is true.

Now, we describe some "counting" rules for frequencies.

Lemma 1.4. If $G \overset{\leq k}{\underset{c}{\sim}} H$, then for every R such that $|R| \leq k$ semi(R,G) = semi(R,H).

Proof. The equality follows immediatelly from the observation that $\mathrm{semi}(R,G) = \prod_{m=1,\dots,cp(R)} \mathrm{mono}(C_m,G)$, where C_m are the components of R , from the identity $\mathrm{mono}(C_m,G) = \mathrm{frq}(C_m,G).\mathrm{aut}(C_m)$ and from their analogues for the graph H.

Lemma 1.5. Let $I = I_1 \cup I_2 \cup \dots \cup I_m \cup \dots$ be a set and let $\{R_i, i \in I\}$ be a collection of graphs such that:

1/ for every m, if $i \in I_m$, then $cp(R_i) = m$,

2/ for every graph R there is one and only one $i \in I$ such that $R_i \simeq R$.

Then for any two graphs R,G the identity $\mathrm{semi}(R,G) = \sum_{i \in I} \mathrm{cov}(R,R_i).\mathrm{frq}(R_i,G)$ holds.

Proof. Let $P = P_2(V(G))$. For $Z \in P$ let $\varphi(Z) = i$ so that $G/Z \simeq R_i$. Obviously $\mathrm{frq}(R_i,G) = \mathrm{card}(\varphi^{-1}(i))$. And now we can write $\mathrm{semi}(R,G) = \sum_{Z \in P} \mathrm{cov}(R,G/Z) = \sum_{Z \in P} \mathrm{cov}(R,R_{\varphi(Z)}) = \sum_{i \in I} \mathrm{cov}(R,R_i).\mathrm{frq}(R_i,G)$.

Lemma 1.6. If G,H are two graphs such that $G \overset{\leq k}{\underset{c}{\sim}} H$, then $G \overset{\leq k}{\sim} H$.

Proof. Let I, I_m, R_i be the same as in Lemma 1.5. We prove by induction that for every $j \leq k$ the proposition A(j): " if $q \in I_j$ a $|R_q| \leq k$, then $frq(R_q,G) = frq(R_q,H)$ " is true.

1/ A(1) is true because of assumption $G \overset{\leq k}{\underset{c}{\sim}} H$.

2/ A(1),A(2),...,A(j-1) are supposed to be true. We introduce $Q_G = \sum_{i \in I_1 \cup I_2 \cup \ldots \cup I_{j-1}} cov(R_q,R_i).frq(R_i,G)$, and analogically Q_H .

Using Lemma 1.5. we obtain $semi(R_q,G) = Q_G + aut(R_q).frq(R_q,G)$ and $semi\ R_q,H) = Q_H + aut(R_q).frq(R_q,H)$. But $Q_G = Q_H$ because for every $i \in I_1 \cup I_2 \cup \ldots \cup I_{j-1}$ $frq(R_i,G) = frq(R_i,H)$ /if we suppose $|R| \leq k$/. Moreover, $semi(R_q,G) = semi(R_q,H)$ according to Lemma 1.4. Thus, we have $frq(R_q,G) = \left[semi(R_q,G) - Q_H \right] / aut(R_q) = \left[semi(R_q,H) - Q_H \right] / aut(R_q) = frq(R_q,H)$.

Theorem 1.7. For any two graphs G,H and for any integer k,the following three properties are equivalent

/i/ $G \overset{k}{\sim} H$, /ii/ $G \overset{\leq k}{\sim} H$, /iii/ $G \overset{\leq k}{\underset{c}{\sim}} H$.

Proof. The theorem is the summary of Corollary 1.2. and Lemma 1.6.

Corollary 1.8. The reconstruction conjecture is true for disconnected graphs.

Proof. Let G,H be two disconnectedgraphs such that $n = |G| = |H| > 2$ and let $G \overset{n-1}{\sim} H$. Using Theorem 1.7. we get $G \overset{\leq(n-1)}{\underset{c}{\sim}} H$ and, since G,H are disconnected, even $G \overset{\leq n}{\underset{c}{\sim}} H$. Now, by Theorem 1.7., $G \overset{n}{\sim} H$, i.e. $G \simeq H$.

2. Bounds of reconstructibility and nonreconstructibility

Let N be the set of all natural numbers. For every subset M of N we define $\max M = +\infty$. Let us denote $N^{*} = N \cup \{+\infty\}$.

Definition 2.1. Let $\mathcal{F}$ be a subclass of the class of all finite graphs. We define the mapping $u_{\mathcal{F}}: N \longrightarrow N^{*}$ as $u_{\mathcal{F}}(n) = \max\{m; (\forall F_1,F_2 \in \mathcal{F})((|F_1| = |F_2| \leq m \,\&\, F_1 \overset{n}{\sim} F_2) \Rightarrow F_1 \simeq F_2)\}$.

Corollary 2.2. We denote $\mathcal{G}$ the class of all finite graphs. B. Manvel showed in [2] that for every $n \in N$ the unequality $u_{\mathcal{G}}(n) < +\infty$ holds. Further, the reconstruction conjecture can be written in the form $u_{\mathcal{G}}(n) \geq n+1$ for $n \geq 2$.

Proposition 2.3. Let $\mathcal{T}$ be the class of all finite trees. Then, for every $n > 1$, $n+1 \leq u_{\mathcal{T}}(n) < 2n$.

Proof. The first unequality expresses the fact that the reconstruction conjecture is true for the case of trees. The second one was proved in [5] where, for every $n > 1$, we constructed two nonisomorphic trees T_1, T_2 having 2n elements such that $T_1 \overset{n}{\sim} T_2$.

Propositon 2.4. If $\mathcal{G}$ is the class of all finite graphs, then, for evry $n > 1$, the unequality $u_{\mathcal{G}}(n) < \min(2n, 3n/2 + 15/2)$ holds.

Proof. To prove the unequality we use Proposition 2.3. and the construction from [5] where, for every $k \geq 2$, we constructed two nonisomorphic graphs G_1, G_2 having $3k + 6$ elements such that $G_1 \overset{2k}{\sim} G_2$.

Corollary 2.5. V.Müller in [3] showed that for every ρ, $1 < \rho < 2$, there exist a class $\mathcal{R}$ and a number n_ρ such that for every $n \in N$ $u_{\mathcal{R}}(n) > \rho . n$ and moreover, $\mathcal{R}$ contains asymptotically the most graphs on n elements.

Remark 2.6. It was proved in [4] that for every $n \in N$ in the class $\mathcal{E}$ of all finite equivalences the unequalities $n.(\ln n - 1) \leq u_{\mathcal{E}}(n) < (n+1).2^{n-1}$ hold /here ln denotes the logarithmus naturalis/.

Problem 2.7. Prove that, for every sufficiently "rich" class $\mathcal{F}$ of finite graphs, the unequality $u_{\mathcal{F}}(n) < +\infty$ holds for every $n \in N$.

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