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In: Zdeněk Frolík (ed.): Proceedings of the 12th Winter School on Abstract Analysis, Section of Topology. Circolo Matematico di Palermo, Palermo, 1984. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 6. pp. [215]--231.

Persistent URL: http://dml.cz/dmlcz/701841

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A SURVEY ON THE HAWKING - PENROSE THEORY

by Imre Major

ABSTRACT: Our aim is to give a brief summary of the concepts and proving-techniques of the theory of spacetime-singularities, and give a detailed proof for some theorems which have no completly exact or exhausting proof in the literature.

I. INTRODUCTION

DEFINITION 1: Light-cone at p: $C_p = \{X \in T_pM \mid g(X,X) \leq 0\}$.

DEFINITION 2: We say, that the spacetime /M,g/ is time-orientable, if the light-cones can be continuously split into two parts C_p^+ and C_p^- such that $C_p = C_p^+ \cup C_p^-$, $\dot{C}_p^+ \cap \dot{C}_p^- = 0$, $C_p^+ = -C_p^-$

We assume that /M,g/ is time-orientable.

DEFINITION 3: We call /M,g/ singular, if it cannot be isometrically imbedded into a geodesically complete spacetime. /There are three kinds of incompleteness, - namely timelike, null, and spacelike incompleteness -, so there are three kinds of singularities/.

DEFINITION 4: /Weak Energy condition, WEC/

$T(V,V) \geq 0$ for every timelike $V \in TM$

Physical meaning: the energy density is nonnegative /measured in any coordinate system/.

DEFINITION 5: /Null-convergence-condition, NCC/

$R(K,K) \geq 0$ for every $K \in TM$ nullvector. /R is the Ricci-tensor/

DEFINITION 6: The timelike convergence condition TCC is satisfied if $R(V,V) \geq 0$ for every timelike vector $V \in TM$.

PROPOSITION 1.

$$\text{WEC} \Rightarrow \text{NCC}$$

Proof: Write the vector K into the Einstein-equations.

PROPOSITION 2: In case of a T_{ab} stress-energy tensor of type

$$T_{ab} = \begin{pmatrix} \mu & & & 0 \\ & p_1 & & \\ & & p_2 & \\ 0 & & & p_3 \end{pmatrix}$$

/where μ is the energy-density, and p_i are the pressures/, the TCC follows from the condition $\mu + p_\alpha \geq 0$ /$\alpha = 1,\dots,3$/ and $\mu + \sum_\alpha p_\alpha \geq 0$.

/This condition is called the strong energy condition. /It falls only in case of negative energy-density, or large negative pressure.

DEFINITION 7: Let S be a spacelike 3-surface, $p \notin S$ and let γ be a timelike geodesic between p and S perpendicular to S at q, then we say that:

- The point p is conjugate to S /along γ/ if there is a Jacobi-field Z along γ for which $Z(p) = 0$, and $g(\nabla_V Z, X) = II(X,Z)$ for every $X \in T_q S$ where II is the second fundamental form of S, $V = \dot{\gamma}(q)$.
- We say that γ is maximal between p and S if the second variation of γ is negative semidefinite. /We consider such variations which are keeping p constant and moving the other endpoint of γ on the surface S./

PROPOSITION 3: If the TCC is satisfied and S, q, γ are as in the above definition, and if trace $II|_q = \vartheta_1 < 0$ then there is a point p' on γ conjugate to S within the distance $-\frac{3}{\vartheta_1}$ along γ /provided that γ can be extended to this parameter value/.

PROPOSITION 4: If TCC is satisfied and for $p = \gamma(s_1)$ there is a $Z \in T_p M$ for which $R(V,Z)(V) \neq 0$, /here γ is a timelike geodesic, $V = \dot{\gamma}(s_1)$/ then there are parameter values s_o, s_2 such that $\gamma(s_o)$ and $\gamma(s_2)$ are conjugate along γ /provided that γ can be extended to each parameter value/.

DEFINITION 8: We say, that the generic condition is satisfied /timelike or null, respectively/ if every timelike geodesic γ contains a point p where $R_{abcd}V^b V^d \neq 0$ /$V = \dot{\gamma}(p)$/ or if every null geodesic λ contains a point p where $K^c K^d K_{[a} R_{b]cd[e} K_{f]} \neq 0$ /$K = \lambda^{\cdot}(p)$/. The bracket $[\,,]$ stands for to anticommute the indices it contains.

Physical meaning: every particle has a moment in its life, when it feels the tidal forces of gravitation.

THEOREM 5: If γ is a timelike geodesic, then [γ is maximal between p and q $\Leftrightarrow$ there is no point conjugate to q between p and q]

THEOREM 6: If γ is a nonspacelike curve, which is not a null-geodesic, then γ can be varied to yield a timelike curve.

DEFINITION 9: For $S \subseteq M$ let

$J^+(S) = \{p\in M \mid \exists\ q\in S$ and $\gamma\subseteq M$ future-directed nonspacelike curve from q to p$\}$

$I^+(S) = \{p\in M \mid \exists\ q\in S$ and $\gamma\subseteq M$ future-directed timelike curve from q to p$\}$

$E^+(S) = J^+(S)\setminus I^+(S)$ is the horismos of S

We call $J^+(S)$ and $I^+(S)$ the causal and chronological future of S respectively.

LEMMA 7: $I^+(S)$ is open.

DEFINITION 10: We call a future-directed nonspacelike curve $\gamma\subseteq S\subseteq M$ /S is open/ future-in extendible in S, if it has no future endpoint in S.

DEFINITION 11: Let λ_n be a sequence of nonspacelike curves. Then $p\in M$ is a limit-point of λ_n, if for every $p\in V\subseteq M$ open set there is an n_o such that $\lambda_n\cap V \neq 0$ for every $n>n_o$.

DEFINITION 12: The nonspacelike curve λ is the limit-curve of the sequence of nonspacelike curves λ_n, if there is a subsequence λ_{n_k} of λ_n such that every $p\in\lambda$ is a limit point of λ_{n_k}.

THEOREM 8: If $S\subseteq M$ is open and $\lambda_n\subseteq S$ is a sequence of nonspacelike curves which are future inextendible in S, and $p\in S$ is a limit-point of λ_n, then there is a nonspacelike curve λ which is future-inextendible in S, $p\in\lambda$ and λ is a limit--curve of the sequence λ_n.

Moreover if we denote λ_{n_k} the subsequence converging to λ and if we have a point -sequence $q_k\in\lambda_{n_k}$ $q_k\to q$ such that there is a compact set K containing the section of λ_{n_k} between p and q_k, and $\lambda\not\subseteq K$, then $q\in\lambda$.

Proof: We call a continuous curve $\gamma: I\to M$ future-directed nonspacelike if for every $t_o\in I$ there is an open set $t_o\in G\subseteq I$ such that

$$\gamma(t)\in J^-(\gamma(t_o))\setminus\{\gamma(t_o)\} \qquad /t<t_o,\ t\in G/$$
$$\gamma(t)\in J^+(\gamma(t_o))\setminus\{\gamma(t_o)\} \qquad /t_o<t\in G/$$

Pre-lemma a./ If γ is a future-directed continuous nonspacelike curve, than $\gamma(t')\in J^+(\gamma(t))$ for every t, $t'\in I$, $t<t'$.

Proof: Suppose instead that there is a pair $t_1<t_1'\in I$ for which $\gamma(t_1')\notin J^+(\gamma(t_1))$ and let $p = \frac{t_1+t_1'}{2}$.

Then $\gamma(p)\notin J^+(\gamma(t_1))$ /in this case let $t_2 = t_1$, $t_2' = p$/ or $\gamma(t_1')\notin J^+(\gamma(p))$ /then let $t_2 = p$, $t_2' = t_1'$/.

Continuing the process we get a common limit-point q of the sequences t_i, t_i'.

Let $q\in G$ be an open set corresponding to the above definition then /for some n/ $t_n \le q \le t'_n$, $t_n, t'_n \in G$, so $\gamma(t_n) \in J^-(\gamma(q))$, $\gamma(t'_n) \in J^+(\gamma(q))$, thus $\gamma(t'_n) \in J(\gamma(t_n))$, contradiction.

Pre-lemma b./ If $U \subseteq M$ is a geodesically convex open set, and $p \in U$, then $J^+(p) \cap U$ is closed in U.

Note: The global version of the theorem is not true.

Proof: Let ω: $U \times U \to TM$, $\omega(a,b) = v \in T_aM$ for which $\exp|_a(v) = b$. We know from metric geometry, that ω is continuous. If $q \in U$ and $r_n \to q$ for a sequence $r_n \in J^+(p) \cap U$ then by convexity of U there is a nonspacelike geodesic joining p and r_n, so: $r_n = \exp|_p(v_n)$, where $v_n \in T_pM$ is nonspacelike i.e. $\omega(p,r_n) = v_n$. Now $v_n = \omega(p,r_n) \to \omega(p,q) = v$ by continuity of ω, so v is nonspacelike, thus $q \in J^+(p)$.

Pre-lemma c./ If p is the future endpoint of the future-directed nonspacelike curve γ then $\gamma \subseteq J^-(p)$.

Proof: Let U be a geodesically convex neighbourhood of p /$\bar{U}$ is compact/ and let $\gamma(t) \in U$ for $t \ge t_o$ and let $t_o \le t_n \to$ [the right-side endpoint of I]. By the compactness of U we can suppose that $\gamma(t_n) \to r$, and because M is a Hausdorff-manifold, we have $r = p$. Now $\gamma(t_n) \in J^+(\gamma(t_o))$ by pre-lemma a. and $p \in J^+(\gamma(t_o))$ by pre-lemma b., so $\gamma(t_o) \in J^-(p)$. Thus by pre-lemma a. we have the statement of the theorem.

Pre-lemma d./ There is a coordinate-pair /x,U/ around every $p \in M$, such that for every coordinate sphere S around p and for every pair of points $q,r \in J^+(p) \cap S$ there is no nonspacelike curve between q and r in U. /And the same is true for the surface $J^-(p) \cap S$./

Proof: Chose an open set $0 \in V \subseteq T_pM$ such that $\exp|_p$ is diffeomorphic on V and $\exp|_p(V)$ is geodesically convex, and choose an orthonormal base $E_o, E_1, \dots, E_3 \in T_pM$. Let y be the normal-coordinatesystem, defined by $E_o, \dots, E_3$ /i.e. $\exp^{-1}(q) = \sum_{a=0;3} y^a(q) E_a$ for $q \in \exp(V)$/ and let denote $f = (\exp|_p^{-1})^*$. Then there is an open set $p \in V' \subseteq V$ such that:

$$[2 f^o(Y)]^2 \ge \sum_{\alpha=1;3} [f^\alpha(Y)]^2 \quad (*)$$

for every nonspacelike vector $Y \in T_q(M)$, $q \in \exp(V')$. /Otherwise we would have a sequence $Y^{(n)}$ of nonspacelike vectors for which $Y^{(n)} \to Y \in T_pM, Y \ne 0$ and $Y^{(n)}$ would not satisfy * so by continuity of the metric g and function f we would have $[2 f^o(Y)]^2 \le \sum_{\alpha=1;3} [f^\alpha(Y)]^2 \le [f^o(Y)]^2$, contradiction./

Let $U = \exp(V')$ and $x^o = 4y^o$, $x^1 = y^1$, $x^2 = y^2$, $x^3 = y^3$ be a coordinatesystem on U and consider the coordinate-sphere $S = x^{-1}(S(o,r)) \subseteq U$. If the separation of $q,r \in S \cap J^+(p)$ would be nonspacelike, then there would be a nonspacelike geodesic η between q and r, $\eta \subseteq J^+(p)$ /because U is geodesically convex/. Let $t_o \in R$ be a point for which $(x \circ \eta)(t_o)$ is minimal, then $(x \circ \eta)^{\cdot}(t_o) = v$ is parallel with the tangent-hyperplane to $S(o,r)$ at the point:

$$z = r. \frac{x(\eta(t_o))}{|x(\eta(t_o))|}$$

We can choose $E_1,\ldots,E_3$ such that $v^2=v^3=0$, that is v is a tangent vector of the circle $(b \sin t,\ b \cos t, c, d)$ at the point $(b.\sin t_1,\ b.\cos t_1, c, d) = z$. The curve η is nonspacelike so by (*) we get

$$\frac{v_o^2}{4} \geq v_1^2 \text{ , that is } |\cos/t_1/| \geq 2|\sin/t_1/|.$$

But $\left|\frac{\sin t_1}{4}\right| \geq |\cos t_1|$ as $x^{-1}(z) \in J^+(p)$, a contradiction.

The proof of the theorem: Let /x,U/ be a coordinate-system around p as in the preceding pre-lemma d./ and let $B(p,r)$ be the closed coordinate-ball around p with radius r. We claim that every λ_n gets out of $B(p,r)$. Suppose, that there would be a $\lambda_n \subseteq B(p,r)$.

Let $t_i \to$[the right-side endpoint of the domain of λ_n] and $\lambda_n(t_i) \to q$, and denote $r_1 = |x(q)|$. As q is not the endpoint of λ_n, there is a sequence $t_i' \to$[the right-side endpoint of the domain of λ_n] such that $\lambda_n(t_i') \to q' \neq q$, and denote $r_1' = |x(q')|$. If $r_1 = r_1'$, then - by the continuity of ω - $q, q' \in J^+(p) \cap x^{-1}(S(o,r_1))$ would be nonspacelike-separated. If $r_1 \neq r_1'$, then there would be nonspacelike-separated points of $J^+(p) \cap x^{-1}(S(o,r'))$ for every $r_1 < r' < r_1'$.

So λ_n leaves $B(p,r)$.

Now let λ_{n_k} be a subsequence of λ_n for which $\lambda_{n_k} \cap \dot{B}(p,r) \to x_{11}$

Denoting $\lambda_{n_k} = \lambda(1,1)_n$ let $\lambda(i,0)_n = \lambda(i-1,i-1)_n$ and let $\lambda(i,j)_n$ be a subsequence of $\lambda(i,j-1)_n$ which converges to x_{ij} on $\dot{B}(p, \frac{j}{i} .r)$. We claim that the separation of x_{ij} and x_{nk} is nonspacelike because considering the sequence $a_n = \lambda(i,j)_n \cap \dot{B}(p, \frac{j}{i} r)$ and $b_m = \lambda(n,k)_m \cap \dot{B}(p, \frac{k}{n} r)$ one of the sequences $\lambda(i,j)_n$, $\lambda(n,k)_m$ will be the subsequence of the other /say $\lambda(n,k)_m$ is a subsequence of $\lambda(i,j)_n$/ so if we denote by a_m the corresponding subsequence of a_n we get that $a_m \to x_{ij}$, $b_m \to x_{nk}$ and the separation of a_m and b_m is nonspacelike, so by convexity, a_m and b_m can be joined by a nonspacelike geodesic /namely $\exp(t\omega(a_m,b_m))$.

So by continuity of ω the vector $\omega(x_{ij}, x_{nk})$ is nonspacelike, so the separation of x_{ij}, x_{nk} is nonspacelike.

Let $\gamma: Q\cap[0,1] \to M$, $\gamma(\frac{p}{q}) = x_{pq}$ and $t_n \in Q\cap[0,1]$, $t_n \to t \in [0,1]$. Then $\gamma(t_n)$ is a convergent sequence. /To see this we can write the sequences $\lambda(i,j)_n$ into an infinite matrix such that every row is a subsequence of its predecessor, and take the diagonal μ_n of this matrix. Then $\mu_n \cap \dot{B}(p,t)$ has a subsequence μ_{n_k} converging to the point $z \in \dot{B}(p,t)$. μ_{n_k} is a subsequence of each $\lambda(i,j)_n$, so we can see - as before - that the separation of $\gamma(t_n)$ and z is nonspacelike. If $\gamma(t_{n_j})$ is an arbitrary subsequence of $\gamma(t_n)$ for which $\gamma(t_{n_j}) \to q$ then: $q \in \dot{B}(p,t)$ and the separation of q and z is nonspacelike, so by the assumption on $S(p,r)$ we have $q = z$.

It follows easily that $\gamma(t_n) \to z$ and $\mu_n \cap \dot{B}(p,t) \to z$. It can be seen in the same manner that the curve γ is continuous.

Continuing the process with $p = x_{11}$ we get that the resulting curve λ is /future/ inextendible. /That is a limit-curve of λ_n is trivial./

Now suppose that λ_{n_k} is the sequence chosen above, $q_k \to q$ /$q_k \in \lambda_{n_k}$ is a point-sequence/, and there is a compact set K satisfying our conditions, so there is a point $z \in \lambda \backslash K$. Choose a finite sequence $B_1, \ldots, B_m$ of coordinate-balls from p to z around points of λ as in the proof of the theorem, such that $q \notin \bar{B}_1 \cup \ldots \cup \bar{B}_m$. Then there is a sequence $z_k \in \lambda_{n_k}$, $z_k \to z$ so there is a k_o such that $z_k \notin K$ /$k \geq k_o$/, that is q_k is between p and z_k on λ_{n_k} for $k \geq k_o$.

It can be seen by the construction in the proof, that there is a k_1 such that the section of λ_{n_k} betweem p and z_k is in the compact set $\bar{B}_1 \cup \ldots \cup \bar{B}_m$ for $k \geq k_1$, thus $q \in \bar{B}_1 \cup \ldots \cup \bar{B}_m$. contradiction.

NOTE: We can catch the deep difference between the Riemann and the Lorentz-metric by this theorem, because

$$c/t/ = \begin{cases} (t,\ t \sin(\frac{\pi}{1-t})) & 0 \leq t < 1 \\ (t,\ t \sin \frac{\pi}{1+t})) & -1 < t \leq 0 \end{cases}$$

is an inextendible curve in $\mathbb{R}^2$, so $\lambda_n = \frac{1}{n} \cdot c$ is a sequence of such curves, and there is no curve λ for which $\lambda_n \to \lambda$.

DEFINITION 13: $S \subset M$ is achronal if $I^+(S) \cap S = \emptyset$.

DEFINITION 14: $S \subseteq M$ is an achronal boundary, if it is a closed, achronal imbedded 3-submanifold /of C^{1-} class/

PROPOSITION 9: If $I^+(S) \subseteq S$ for $S \subseteq M$ i.e. S is a future set then ∂S is an achronal boundary.

PROPOSITION 10: If K is a closed set, then $\dot{J}^+(K)$ is the union of null-geodesic, which can have past endpoints only on K.

DEFINITION 15: M is causally simple if $\dot{J}^+(K) = E^+(K)$, $\dot{J}^-(K) = E^-(K)$ for every compact K. /i.e. $J^+(K)$ and $J^-(K)$ are closed for compact K/

DEFINITION 16: We say, that M is strongly causal at p, if every neighbourhood V of p contains a neighbourhood W of p such that $\gamma \cap W$ is connected for every nonspacelike curve γ.

LEMMA 11: If M is not strongly causal at p, then there is a geodesically convex open set U and there is a sequence of open sets $\ldots \subseteq V_n \subseteq \ldots \subseteq V_1 \subseteq U$, shrinking to p and a sequence of future-directed nonspacelike curves $\lambda_1, \ldots, \lambda_n, \ldots$ such that the fore-part of λ_i is in V_i, then λ_i leaves U, after then λ_i returns to V_i.

Proof: Let $E_0, \ldots, E_3$ be an orthonormal base at p and y be the normal-coordinate-system defined by this base, and let $(\exp|p^{-1})_*(Z) = \sum_{a=0;3} f^a(Z) E_a$ for $Z \in TM$, $\pi/Z/\in U'$, /as in pre-lemma d./

Because of the conditions there is an open set $p \in W \subseteq M$ such that for every open set $p \in W' \subseteq W$ there is a future-directed nonspacelike curve that leaves W' and returns to it.

As in the proof of pre-lemma d we can say that: there is ar $r>0$ such that $3.\,[f^0(Z)]^2 \geq \sum_{\alpha=1;3} [f^\alpha(Z)]^2$ for every nonspacelike $Z \in TM$ for which $\pi(Z) \in y^{-1}(B(0,r))$ /i.e. the angle between $(\exp|p^{-1})_*(Z)$ and the hyperplane spanned by E_1, E_2, E_3 is greater than $\frac{\pi}{6}$/.

Let $K = \{x \in R^4 \,|\, x_1^2 + x_2^2 + x_3^2 < \frac{r^2}{4}\,,\ |x_0| < \sqrt{r^2 - x_1^2 - x_2^2 - x_3^2} - \frac{\sqrt{3}r}{2}\}$ and $V_n = y^{-1}(\frac{1}{n} . K)$. Then $V_n \subseteq W$ for $n \geq n_0$. Let λ_n be a future-directed nonspacelike curve leaving V_n and returning to it, and let $\gamma_n = (\exp|p)^{-1} \lambda_n$. It is easy to see, that γ_n comes out of the set $\frac{1}{n}$ K above the hyperplane S spanned by E_1, E_2, E_3 and re-enters under it.

If γ_n would not leave the ball $B(0,r)$, then the tangent vector at its farthest point from S would be parallel to S, so the corresponding tangent vector to λ_n would be spacelike.

THEOREM 12: If the following conditions are satisfied on M

1. NCC
2. Null-genericity
3. There is no closed timelike curve
4. Null-geodesic completeness

then M is strongly causal.

Proof: Suppose that there is a point $p\in M$ violating the strong causality and let U, V_i, λ_i /$i\in N$/ be as in the preceding lemma. Let us extend the curves λ_i in future direction to get a sequence of future-inextendible curves and let λ be the limit-curve of this sequence, guaranted by theorem 8. /then λ is a future-directed /future/ inextendible nonspacelike curve/. Extending the λ_i-s in the other direction we get the limit-curve λ'. If $\lambda\cap\lambda'\neq 0$ or either of it meets the point p once more, then a closed nonspacelike curve will be, which can be only a closed null-geodesic /by condition 3 and theorem 6/. But then this closed null-geodesic would contain a pair of conjugate-points /by conditions 1.2.4 and proposition 4/ so it would have a variation which would yield a closed timelike curve: contradiction.

Let $r_i\in\lambda_i\cap V_i$ be a point of λ_i after the point on λ_i not belonging to U. We have the following two cases:

case 1: There is a subsequence λ_{i_k} of λ_i, and there is $q_{i_k}\in\lambda_{i_k}$, $q\in\lambda$, $q_{i_k}\to q$ such that r_{i_k} precedes q_{i_k} on λ_{i_k}. Then covering the section of λ between p and q with open sets of finite number used in the proof of theorem 8 we get that λ will meet the point p once more: contradiction. /As the curve λ_{i_k} contains a point not belonging to U between p and r_{i_k} the curve λ will contain a point not belonging to U between "p and p"./

case 2: In case of $q_i\in\lambda_i$, $q\in\lambda$, $q_i\to q$ the point q_i precedes r_i for $i>i_o$. Then any pair of distinct points of λ can not have timelike separation, because otherwise we would have $q\in\lambda$, $q\in I^+(p)$ for some q, so we could choose open sets $p\in W$, $q\in W'$ such that $b\in I^+(a)$ for every $a\in W$, $b\in W'$.

So choosing a sequence $q_i\in\lambda_i$, $q_i\to q$ there is an i_o such that q_i precedes r_i, $r_i\in W$, $q_i\in W'$ for $i>i_o$. Then $q_i\in I^+(r_i)$ and $r_i\in J^+(q_i)$ which is a contradiction.

So we can say that λ /and λ'/ is a null-geodesic. Suppose there are points $q'\in\lambda'$, $q\in\lambda$ with timelike separation. Then let

$q\in K$, $q'\in K'$ be open sets such that $b\in I^+(a)$ for every $a\in K'$, $b\in K$. Let $q_i\in\lambda_i$, $q_i\to q$, $q_i'\in\lambda_i'$, $q_i'\to q'$ be sequences and let $q_i\in K$, $q_i'\in K'$, q_i precedes q_i' on λ_i for $i>i_o$. So $q_i\in I^+(q_i')$ and $q_i'\in J^+(q_i)$, contradiction. So $\lambda\cup\lambda'$ is a complete null-geodesic /by theorem 6/. But it contains a pair of conjugate points by condition 1.2. and proposition 4 so it would have pairs of points with timelike separation by theorem 5.

NOTE: The proof of this theorem in [1] - in our opinion - is rather scamped /for example it does not distinguish the above mentioned two cases, though this distinction is necessary by all means/.

DEFINITION 17: If $K\subseteq M$ is a compact set, then the curve γ: $I\to M$ is totally imprisoned in K, if there is a $b\in I$ for which $\gamma((b,+\infty)\cap I)\subseteq K$. We say, that γ is partially imprisoned in K, if it is not totally imprisoned in K but $\gamma^{-1}(K)$ has no upper bound in I.

LEMMA 13: If $K\subseteq M$ is a strongly causal compact set, then there is no future-directed inextendible nonspacelike curve totally or partially future-imprisoned in K.

DEFINITION 18: For $S\subseteq M$:

$$\tilde{D}^+(S) = \left\{p\in M \mid \begin{array}{l}\text{every past-directed inextendible timelike}\\ \text{curve from } p \text{ intersects } S.\end{array}\right\}$$

$$D^+(S) = \left\{p\in M \mid \begin{array}{l}\text{every past-directed inextendible nonspace-}\\ \text{like curve from } p \text{ intersects } S\end{array}\right\}$$

DEFINITION 19: If S is an achronal set, then:

$$\text{edge}(S) = \left\{p\in\bar{S} \mid \begin{array}{l}\text{for every open set } p\in U\subseteq M \text{ there are}\\ \text{points } q\in I^-(p)\cap U,\ r\in I^+(p)\cap U \text{ such}\\ \text{that } r \text{ and } q \text{ can be joined in } U \text{ by a}\\ \text{timelike curve which does not inter-}\\ \text{sect } S.\end{array}\right\}$$

PROPOSITION 14: $\overline{\tilde{D}^+(S)} = \overline{D^+(S)}$ for every closed set $S\subseteq M$.

PROPOSITION 15: If S is a closed achronal set, then:

$$\text{edge}(S) = \text{edge}(H^+(S)) = \text{edge}(H^-(S))$$

DEFINITION 20: Let $N\subseteq M$ be open. Then we call N globally hyperbolic if:

1. The strong causality condition holds at every point of N,
2. $J^+(p)\cap J^-(q)\subseteq N$ is compact for every $p,q\in N$.

DEFINITION 21:

$$C(p,q) = \left\{ \gamma \mid \begin{array}{l} \gamma \text{ is a nonspacelike curve connecting } p \\ \text{and } q, \text{ and } \gamma = \gamma', \text{ if } \gamma' \text{ can be got from} \\ \gamma \text{ by reparametrizing it.} \end{array} \right\}$$

Let $\gamma \in C(p,q)$ and $\gamma \subseteq U \subseteq M$ be an open set.
Denote $U(\gamma) = \{\lambda \in C(p,q) \mid \lambda \subseteq U\}$, and denote $C^{O}(p,q)$ the set $C(p,q)$ with the topology defined by the sets $K(\gamma) = \{U(\gamma) \mid U \subseteq M \text{ is open}\}$ a neighbourhood base at γ.

PROPOSITION 16: If N is a strongly causal open set, for which $N = J^{-}(N) \cap J^{+}(N)$, then:
[N is globally hyperbolic $\Leftrightarrow$ $C(p,q)$ is compact for all $p,q \in N$]

THEOREM 17: If S is a closed achronal set, then $\operatorname{int}(D(S))$ is globally hyperbolic.

Proof:

a./ If $p \in \operatorname{int}(D(S))$ and - say - $p \in D^{+}(S)$, then every past-inextendible nonspacelike curve $\lambda:[a,b) \to M$ $\lambda(a) = p$ intersects $I^{-}(S)$.
Let $s \in \lambda \cap S$, $t_i \to b$ monotone increasing sequence, $r \in I^{+}(p) \cap D^{+}(S)$.
Let γ_1 be a timelike curve from r to $\lambda(t_1)$ and let q_1 be an interior point of γ_1, further let γ_2 be a timelike curve from q_1 to $\lambda(t_2)$ and q_2 be an interior point of γ_2, and so on.
Choose the sequence q_i such that the distance between q_i and $\lambda(t_i)$ converges to zero for some positive definite metric on M.
Then we have a curve $\gamma = \cup\{\text{segment of } \gamma_i \text{ between } q_{i-1} \text{ and } q_i\}$.
If $\gamma \cap S \neq \emptyset$, then $\lambda \cap I^{-}(S) \neq \emptyset$. If $\gamma \cap S = \emptyset$, then γ must have a past endpoint q, thus $q_i \to q$ and $\lambda(t_i) \to q$. There is a past-inextendible timelike curve through q, which has to intersect S at a point u. So there is an index i for which $s \in J^{+}(\lambda(t_i))$ and $\lambda(t_i) \in I^{+}(u)$: contradiction because S is achronal. So every /both future and past/ inextendible, nonspacelike curve through a point $p' \in \operatorname{int}D(S)$ intersects both $I^{+}(S)$ and $I^{-}(S)$.

b,/ Suppose, that M is not strongly causal at the point $p \in \operatorname{int}D(S)$, so let $p \in U \subseteq M$ be an open set, $U \supseteq V_1 \supseteq \ldots \supseteq V_n \supseteq \ldots$ be a sequence of open sets $\lambda_1, \ldots, \lambda_n, \ldots$ be a sequence of nonspacelike curves as in Lemma 11. Extending the curves λ_i in the future direction, we get the future-inextendible limit-curve λ, and extending them in the past direction we get the past-inextendible limit-curve λ' /see the proof of the theorem 12/. If $\lambda \cap \lambda' \neq \emptyset$, or either of them meets the point p once more, then we get a closed nonspacelike curve through p, which intersects both $I^+(S)$, $I^-(S)$: contradiction. So, as in the proof of theorem 12 the second case is satisfied.

Let $q\in\lambda\cap I^+(S)$, $q\in K\subseteq I^+(S)$, $q'\in\lambda'\cap I^-(S)$, $q'\in K'\subseteq I^-(S)$, K, K' are open, $q_i\in\lambda_i\cap K$, $q_i'\in\lambda_i'\cap K$, $q_i\to q$, $q_i'\to q'$, $q_i\in J^-(q_i')$ for $i\geq i_o$. But $q_i\in K\subseteq I^+(S)$, $q_i'\in K'\subseteq I^-(S)$.

c./ $J^-(p)\cap J^+(q)\subseteq \mathrm{int}(D(S))$ for $p,q\in \mathrm{int}(D(S))$

Let first $p\in D^+(S)$, $q\in D^-(S)$ and λ be a nonspacelike curve joining p and q. Let $r\in[M\setminus(\mathrm{int}\, D(S)]\cap\lambda$. Let $p'\in D^+(S)\cap I^+(p)$. $q'\in D^-(S)\cap I^-(q)$, $r\in K\subseteq I^-(p')\cap I^+(q')$ be an open set, $r'\in K\setminus D(S)$, and $r'\in\gamma$ be a timelike curve joining p' and q'. If $r'\in\lambda'$ is a nonspacelike curve, both future and past-inextendible, and $\lambda'\cap S=\emptyset$, then $\gamma\cap S$ has at least two points: contradiction.

Now let $p\in I^-(S)$, $q\in I^-(S)$, $r\in J^-(p)\cap J^+(q)$, $q'\in I^-(q)\cap D^-(S)$, $r\in K\subseteq I^-(S)\cap I^+(q')$ be open, $r'\in K\setminus D(S)$, γ be a timelike curve joining q' and r', and λ be a future inextendible nonspacelike curve from r', $\lambda\cap S=\emptyset$. But then $\gamma\cap S\neq 0$ /because $q'\in D^-(S)$/. This is a contradiction, because $r'\in I^-(S)$.

d./ Now we are going to show, that $C^o(p,q)$ is compact for $p,q\in\mathrm{int}(D(S))$.

Let first $p,q\in I^-(S)$, and $\lambda_n\in C^o(p,q)$ be a sequence. Then the sequence λ_n has a future-inextendible nonspacelike limit-curve λ in $M\setminus\{p\}$. If p is not the end point of λ in M, then $\lambda\cap I^+(S)\neq\emptyset$, by a., so $\lambda_n\cap I^+(S)\neq\emptyset$ for some n. But $\lambda_n\subseteq J^-(p)\subseteq I^-(S)$: contradiction. So $\lambda\cup\{p\}\in C(p,q)$. /We denote the curve $\lambda\cup\{p\}$ by λ too./

Let $\lambda\subseteq U\subseteq M$ be open, and let $r\in V_r\subseteq U$ /$r\in\lambda$/ be such an open set, that there is no nonspacelike curve starting from and then returning to it.

Choose a finite system $\lambda\subseteq V_1\cup\ldots\cup V_m$ of $\{V_r|r\in\lambda\}$. Let $W_o=V_1$, $W_1=V_1\cap V_2,\ldots,W_{m-1}=V_{m-1}\cap V_m$, $W_m=V_m$ and let λ_{i_k} be a sequence of λ_i, converging to λ. Then $\lambda_{i_k}\cap(\bigcap_1^{m-1}W_j)\neq\emptyset$ from some $k>k_o$. Let $v\in\lambda_{i_k}\setminus(\bigcup_1^m V_m)$ be a point between W_j,W_{j+1} /$j=0,\ldots,m-1$/. Then λ_{i_k} would leave V_{j+1} then return to it. Thus $\lambda_{i_k}\subseteq V_1\cup\ldots\cup V_m\subseteq U$ for $k>k_o$, that is $C^o(p,q)$ is compact.

Now let us prove that $J^+(q)\cap J^-(S)$ is compact for $q\in\mathrm{int}(D(S))$.

Let $a_n \in J^+(q) \cap J^-(S)$ be a sequence, $a_n \in \lambda_n$ be a sequence of future-inextendible nonspacelike curves starting from q, and λ be the limit-curve of the sequence λ_n. Then there exists $r \in \lambda \cap I^+(S)$. Let λ_{i_k} be a subsequence of λ_i such that every point of λ is a limit point of λ_{i_k}. Covering the section $[q,r]$ of λ with coordinate-balls $B_1, \dots B_n$, as in the proof of theorem 8 we can see that there is a k_0 such that $a_{i_k} \in \bigcup_{j=1}^{n} B_j$ /$k>k_0$/. So a_{i_k} has a subsequence $a_j' \to a \in \lambda$.

If a_j' has a subsequence in S, then $a \in S$.
If $a_j' \in J^+(q) \cap [J^-(S) \setminus S]$, then there is no point of S on λ_j' between q and a_j', because if r would be such a point, then let $q' \in I^-(q) \cap D^-(S)$, $r' \in \lambda_j' \setminus S$ be a point between r and a_j', further let γ be a timelike curve joining q' and r', and γ' be a future-inextendible timelike curve from r'. Then:
$\gamma \cap S = \emptyset$ because $a_j' \in J^-(S)$ and $\gamma' \cap S = \emptyset$ because $r \in S$: contradicting to the fact that $q \in D^-(S)$.
Let b_n be the first point of $\lambda_n \cap S$. So a_n is between q and b_n and $b_{n_k} \in \bigcup_{j=1}^{n} B_j$ for $k \geq k_0$, so there is a subsequence $b_j' \to b \in \lambda \cap S$ of b_{j_k}, so $a \in J^-(b)$, that is $J^+(q) \cap J^-(S)$ is compact.
Now let $p \in D^+(S)$, $q \in D^-(S)$, $a_n \in J^-(p) \cap J^+(q)$, $a_n \in \lambda_n$ be a nonspacelike curve joining p and q. Then $\lambda_n \cap S \neq \emptyset$. If a subsequènce a_{n_k} of a_n is in S, then we can suppose, that $a_{n_k} \to a \in J^+(q) \cap S$, and $a \in J^-(p)$ is also satisfied.
If $a_n \notin S$, then, as in the preceding proof, there will be a first point b_n of S on λ_n /as measured from q/, so we can suppose, that $a_n \to a \in J^+(q) \cap J^-(S)$ and $b_n \to b \in J^-(p) \cap J^+(S)$, and $a \in J^-(b)$. So $a \in J^-(p) \cap J^+(q)$.

<u>THEOREM 18</u>: If S is a closed achronal set, for which $J^+(S) \cap J^-(S)$ is strongly causal, and acausal or compact, then D(S) is globally hyperbolic.

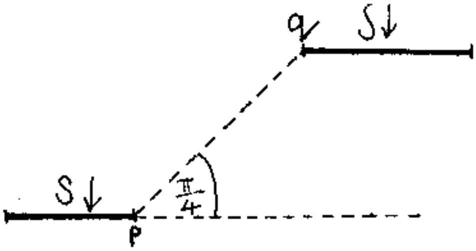
<u>Proof</u>:

a./ D(S) is strongly causal.

It is sufficient to examine the case $p \in D^+(S) \cap I^+(S)$. Let $U \subseteq I^+(S)$, $p \in V_i \subseteq U$ be open sets, λ_i is a sequence of nonspacelike curves as in Lemma 11.
Extend λ_i in the past direction to get a past-inextendible

nonspacelike curve λ_i' and let λ' be the limit-curve of the sequence λ_i'. λ' can not meet the point p once more because of the achronality of S, so we have the second case of theorem 12. If $q\in\lambda'$, $q\in K\subseteq M$ is an arbitrary open set, $q'\in I^-(q)\cap K$, $p\in V_i\subseteq I^+(q')$ /$i>i_0$/ γ_i is nonspacelike curve, which joins q' with the first point of the /future-directed/ λ_i, $q_i\in\lambda_i'$, $q_i\to q$, then /with finite exception/ $q_i\in\lambda_i$, so M is not strongly causal at q. But $\lambda'\cap S\neq\emptyset$: contradiction.

b./ If $p,q\in D(S)$, then $J^+(p)\cap J^-(q)\subseteq D(S)$.
This is trivial, if $J^+(S)\cap J^-(S)$ is acausal, and not true, if $J^+(S)\cap J^-(S)$ is compact. /Consider for example the following set in Minkowski-spacetime:



But disregarding this, the further claims of the theorem are true.

c./ $J^+(p)\cap J^-(q)$ is compact /for $p,q\in D(S)$/.
First we prove the compactness of $J^+(p)\cap J^-(S)$. Let $a_n\in J^+(p)\cap J^-(S)$, $a_n\in\lambda_n$ be a future-inextendible nonspacelike curve, starting from p, and let $a_n\in J^-(s_n)$, $s_n\in\lambda_n\cap S$, and let λ be the limit-curve of the sequence λ_n. As $\lambda\cap S\neq\emptyset$, λ leaves $J^-(S)$, if $J^+(S)\cap J^-(S)$ is acausal. If $J^+(S)\cap J^-(S)$ is compact, then λ leaves S too. /by Lemma 13/
Let $x\in\lambda\setminus J^-(S)$, $c_n\in\lambda_n$, $c_n\to x$. If there is a subsequence λ_{n_k} for which c_{n_k} precedes s_{n_k}, then $x\in D^-(S)$, so $x\in J^-(S)$.
So covering the section $[p,x]$ of the curve λ with coordinate-balls of finite number, we get, that $a_n\to a\in\lambda$, $s_n\to s\in\lambda$ and $a\in J^-(S)$. From this it is easy to prove the compactness of $J^+(p)\cap J^-(q)$ in the case of $q\in D^+(S)$, $p\in D^-(S)$.
If $p,q\in I^-(S)$, then consider a sequence $\lambda_n\in C(p,q)$. Thus $\lambda_n\to\lambda$ in $M\setminus\{q\}$.
If q is an endpoint of λ, then $\lambda\subseteq D^-(S)$, and we can see - as in the preceding theorem - that $\lambda_n\to\lambda$ in the topology of $C^0(p,q)$.
If q is not the endpoint of λ, then $\lambda\cap S\neq\emptyset$, so there exists

a point $x\in\lambda\setminus J^-(S)$. But $\lambda_n \subseteq D^-(S)$, so $\lambda \subseteq \tilde{D}^-(S) \subseteq J^-(S)$, contradiction.

THEOREM 19: If N is a globally hyperbolic set, $p,q\in N$, and $q\in J^+(p)$, then there is a nonspacelike geodesic joining p and q in N which has maximal length among all the nonspacelike curves joining p and q.

THEOREM 20: If S is a closed achronal set, for which $J^+(S)$ is strongly causal, and $E^+(S)$ is compact, then there is a future-inextendible timelike curve $\gamma \subseteq D^+(E^+(S))$, such that $\gamma \cap E^+(S) \neq \emptyset$.

Proof: See in [1].

PROPOSITION 21: Every globally hyperbolic set is causally simple.

DEFINITION 22: We call a compact, closed /without boundary/ spacelike two-surface closed trapped surface, if the two null-component of its second fundamental form is negative semidefinite. /We use then a base $E_1, E_2, K, L, g(E_i,E_j) = \delta_{ij}$, $g(E_i,K) = g(E_i,L) = g(K,K) = g(L,L) = 0$, $g(K,L) = -1$.

DEFINITION 23: We call a set S Cauchy-surface, if

1./ It is acausal $/J^+(p)\cap S = \{p\}$, $/p\in S//$

2./ Edge $S = 0$

3./ $D(S) = D^+(S)\cup D^-(S) = M$.

THEOREM 22: /PENROSE, 1965/

If M contains a closed trapped surface and a noncompact Cauchy-surface, and if the NCC is satisfied, then M is not null-geodesically complete.

Proof: The idea of the proof is to suppose null-geodesical completeness, and then to construct a homeomorphism between a compact and a noncompact set, to get a contradiction in such a way.

a./ If T is a closed trapped surface, then $\dot{J}^+(T)$ is compact.

By theorem 17. M is globally hyperbolic, thus causally simple, so $\dot{J}^+(T) = E^+(T)$.

So if $f: T\times[0,+\infty)\times\{1,2\}\to M$, $f(p,t,1) = \exp|_p(tK)$, $f(p,t,2) = \exp|_p(tL)$, then $\dot{J}^+(T) \subseteq f(T\times[0,+\infty)\times\{1,2\})$

Let $c = \min(\inf_T |tr/II^1/|, \inf_T |tr/II^2/|)$

$/II^1, II^2$ are the two components of the second fundamental form in the base E_1E_2,K,L, where E_1,E_2 is chosen such that $E_{1p},E_{2p} = T_pT$./

From the null-version of proposition 3 and theorem 5 we get that a null-geodesic starting from $p \in T$ and orthogonal to T will enter $I^+(T)$ at latest as the parameter t reaches the value $\frac{2}{c}$. So $\dot{J}^+(T) \subseteq f(T \times [0, \frac{2}{c}] \times \{1,2\}$, $\dot{J}^+(T)$ is compact.

Now project $\dot{J}^+(T)$ to the noncompact Cauchy-surface C via the integral curves of a timelike vector-field. The projection is a compact 3-submanifold of C, without boundary /because $\dot{J}^+(T)$ is such a set, too/.

And now here is our main theorem:

THEOREM 23: /Hawking - Penrose 1970/

If:

1. TCC is satisfied.
2. The timelike and null-generic condition is satisfied
3. There is no closed timelike curve
4. There is at least one of the following sets:
   a. S compact achronal set, edge $(S) = 0$
   b. Closed trapped surface
   c. $p \in M$ point, for which $\vartheta(p) < 0$ along every null-geodesic, starting from p.

then M cannot be both timelike and null-geodesically complete.

Proof: /an outline/

We prove instead that

In a timelike and null-geodesically complete spacetime the following five conditions cannot be satisfied at the same time:

a. Every inextendible nonspacelike geodesic contains a pair of conjugate points
b. There are no closed timelike curves
c. There is a closed achronal set S for which $E^+(S)(E^-(S))$ is compact.
d. NCC
e. Null-generic condition

If these conditions are satisfied, then M is strongly causal by theorem 12, so there is a timelike curve $\gamma \subseteq D^+(E^+(S))$, as in the theorem 20. Then the set $F = \overline{E^+(S) \cap J^-(\gamma)}$ is compact and $E^-(F) \subseteq F \cup \dot{J}^-/\gamma/$.

It can be seen from a. - by corollary 10 - that every null-geodesic which has a section on $\dot{J}^-(\gamma)$, will enter to $I^-(\gamma)$.

The distance of these "points of entering" from F is a continuous function of the tangent vectors of these geodesics at the point, where they intersect F. So $E^-(F)$ is compact. There is a curve $\lambda \subseteq D^-(E^-(F))$ as in theorem 20. Then $\lambda \cup \gamma \subseteq \mathrm{int}(D(E^-(F)))$.

Let $a_n \in \lambda$, $b_n \in \gamma$, $a_{n+1} \in I^-(a_n)$, $b_{n+1} \in I^+(b_n)$, $b_1 \in I^+(a_1)$. The sequence a_n /and b_n/ leaves every compact section of λ /and γ, respectively/. Because of global hyperbolicity of $\mathrm{int}(D(E^-(F)))$ /theorem 17/, there is a maximal timelike geodesic μ_n between a_n and b_n. We can suppose /by compactness of F/ that $\mu_n \cap F \to x$ $(x \in F)$ and $\mu_n'(\mu_n \cap F) \to V \in T_x M$. Consider the geodesic μ for which $\mu(0) = x$, $\mu'(0) = V$. It will contain a pair of conjugate points u,v by a. Let $u' \in J^-(u) \cap \mu$, $v' \in J^+(v) \cap \mu$. Varying the curve μ between u' and v' to get a longer curve α, we can prove by the aid of this curve, that there is a μ_n, which is not maximal: contradiction.

<u>THEOREM 24</u>: If:

1. TCC is satisfied
2. There is a compact acausal 3-surface S such that edge S = 0, and tr(II) is negative definite on S.

Then the spacetime M is not timelike geodesically complete.

<u>Proof</u>: The idea of the proof is to show, that $H^+(S)$ is compact, and then find a contradiction with the fact, that null-geodesics laying in $H^+(S)$ can have past endpoints only on edge (S). See e.g. [1] for the details of the proof.

<u>NOTE</u>: This theorem describes a closed contracting Universe. As Friedmann exhibited in 1922, that every homogenous and isotropic cosmological model is contracting or expanding and as we know, that the real Universe is expanding /contracting in the reversed time direction/, we can say, that our Universe contains a singularity in its past, if it is closed. However this singularity is not necessarily the "beginning point of the Universe".

Though there are incomplete geodesics, it is not necessarily true, that <u>every</u> past directed geodesic is incomplete. /At least it is by no means the consequence of the above results./

<u>EPILOGUE</u>: We tried to summarize the main results of S.W.Hawking and R.Penrose. We gave a full-exhausting proof of the statement of theorem 8. The second part of the statement is important, though not mentioned /only tacitly used/ in the literature /see e.g. [1], [2], [3]/. The same is true for our pre-lemma d. and Lemma 11.

It is necessary to distinguish the two cases in theorem 12, though e.g. in [1] these two cases are collected into one sentence.

Our proof of Theorem 17 is slightly and that of Theorem 18 is rather different, than usual proofs.

So we feel, that it is succeeded to fill some of the gaps of inexactness in the theory of singularities.

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