STIEFEL-WHITNEY CHARACTERISTIC CLASSES AND PARALLELIZABILITY OF GRASSMANN MANIFOLDS

Vojtěch Bartík and Július Korbaš

Introduction

The solution to the parallelizability of real Grassmann manifolds was given in 1975 by Toshio Yoshida [14]. Another proof is outlined by Hiller and Stong in [3] (see Observation, p. 367).

The purpose of our paper is to present an independent and more elementary proof, showing directly that for any real Grassmann manifold, which is not diffeomorphic to projective space, its first or the second or the fourth Stiefel-Whitney characteristic class does not vanish.

The Stiefel-Whitney characteristic classes from the first to the ninth are computed here in full generality. As a product of our method we obtain also some upper bounds for the span of grassmannians, and we prove one non-embedding theorem.

1. Preliminaries and statement of results

We recall that the span of a smooth, closed, connected manifold M, abbreviated span M, is defined to be the maximal number of linearly independent tangent vector fields on M (see Thomas [11]). If span $M = n$, the n-manifold M is called parallellizable. Obviously, M is parallelizable if and only if its tangent bundle, $T(M)$, is trivial.

Let $G_{n,r}$ denote the Grassmann manifold of all linear r-subspaces of the real Euclidean n-space R^n (we shall suppose $r > 0$), and $\gamma_{n,r}$ the canonical vector bundle over $G_{n,r}$ (its total space consists of pairs (D,v) where D is an element of $G_{n,r}$ and v is a vector in D).

Writing as usual $w(\varepsilon) = 1 + w_1(\varepsilon) + \ldots + w_m(\varepsilon)$ for the total Stiefel-Whitney class of an m-dimensional vector bundle ε and putting $w(G_{n,r}) = w(T(G_{n,r}))$ for the total Stiefel-Whitney

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class of $G_{n,r}$ we prove

<u>1.1. Theorem.</u> Let w_i abbreviate $w_i(\gamma_{n,r}) \in H^i(G_{n,r};Z_2)$, and $n = \sum_{i \geq 0} n_i 2^i$, $r = \sum_{i \geq 0} r_i 2^i$ be the dyadic expansions of n,r respectively. Then:

$$w_1(G_{n,r}) = n_o w_1 ;$$
$$w_2(G_{n,r}) = (1+n_1+r_o)w_1^2 + n_o w_2 ;$$
$$w_3(G_{n,r}) = n_o(1+n_1+r_o)w_1^3 + n_o w_3 ;$$
$$\begin{aligned} w_4(G_{n,r}) = {} & (n_1(1+r_o)+n_2+r_1)w_1^4 + n_o(1+n_1+r_o)w_1^2 w_2 + (n_1+r_o)w_2^2 + \\ & + n_o w_4 ; \end{aligned}$$
$$\begin{aligned} w_5(G_{n,r}) = {} & n_o(n_1(1+r_o)+n_2+r_1)w_1^5 + n_o(1+n_1+r_o)w_1^2 w_3 + \\ & + n_o(n_1+r_o)w_1 w_2^2 + n_o w_5 ; \end{aligned}$$
$$\begin{aligned} w_6(G_{n,r}) = {} & (n_2+r_1)(1+n_1+r_o)w_1^6 + n_o(n_1(1+r_o)+n_2+r_1)w_1^4 w_2 + \\ & + (n_1+r_o)w_1^2 w_2^2 + n_o(1+n_1+r_o)w_1^2 w_4 + n_o(n_1+r_o)w_2^3 + \\ & + (n_1+r_o)w_3^2 + n_o w_6 ; \end{aligned}$$
$$\begin{aligned} w_7(G_{n,r}) = {} & n_o(n_2+r_1)(1+n_1+r_o)w_1^7 + n_o(n_1(1+r_o)+n_2+r_1)w_1^4 w_3 + \\ & + n_o(n_1+r_o)w_1^3 w_2^2 + n_o(1+n_1+r_o)w_1^2 w_5 + n_o(n_1+r_o)w_2^2 w_3 + \\ & + n_o(n_1+r_o)w_1 w_3^2 + n_o w_7 ; \end{aligned}$$
$$\begin{aligned} w_8(G_{n,r}) = {} & (n_1(1+r_o)(n_2+r_1)+(1+n_2)r_1+n_3+r_2)w_1^8 + \\ & + n_o(n_2+r_1)(1+n_1+r_o)w_1^6 w_2 + (n_1+r_o)(n_2+r_o+r_1)w_1^4 w_2^2 + \\ & + n_o(n_1(1+r_o)+n_2+r_1)w_1^4 w_4 + n_o(n_1+r_o)w_1^2 w_2^3 + \\ & + (1+n_1+r_o)w_1^2 w_3^2 + n_o(1+n_1+r_o)w_1^2 w_6 + (1+n_2+(1+n_1)r_o+r_1)w_2^4 + \\ & + n_o(n_1+r_o)w_2^2 w_4 + n_o(n_1+r_o)w_2 w_3^2 + (n_1+r_o)w_4^2 + n_o w_8 ; \end{aligned}$$
$$\begin{aligned} w_9(G_{n,r}) = {} & n_o(n_1(n_2+r_1)(1+r_o)+(1+n_2)r_1+r_2+n_3)w_1^9 + \\ & + n_o(n_2+r_1)(1+n_1+r_o)w_1^6 w_3 + n_o(n_1+r_o)(n_2+r_o+r_1)w_1^5 w_2^2 + \\ & + n_o(n_1+r_o)w_1^2 w_2^2 w_3 + n_o(1+n_1+r_o)w_1^3 w_3^2 + \\ & + n_o(n_1(1+r_o)+n_2+r_1)w_1^4 w_5 + n_o(1+n_1+r_o)w_1^2 w_7 + \\ & + n_o(1+n_2+(1+n_1)r_o+r_1)w_1 w_2^4 + n_o(n_1+r_o)w_2^2 w_5 + \\ & + n_o(n_1+r_o)w_3^3 + n_o(n_1+r_o)w_1 w_4^2 + n_o w_9 . \end{aligned}$$

Moreover, if n is even, then $w_i(G_{n,r}) = 0$ for any odd i.

As a consequence we obtain very quickly Yoshida's result on the parallelizability of Grassmann manifolds.

1.2. Theorem. The only parallelizable Grassmann manifolds are $G_{2,1}$, $G_{4,1}$, $G_{4,3}$, $G_{8,1}$ and $G_{8,7}$.

Theorem 1.1 yields also the following estimations of span $G_{n,r}$:

1.3. Theorem. If n is even, $\min\{r,n-r\}\geq 3$ is odd and $\max\{r,n-r\}\geq 9$, then

$$\text{span } G_{n,r} \leq r(n-r) - 8 .$$

Moreover, span $G_{6,3}\leq 7$, span $G_{8,3} = 7$, span $G_{10,3} \leq 13$, span $G_{10,5} \leq 17$, span $G_{12,5} \leq 27$ and span $G_{14,7} \leq 41$.

Finally we prove

1.4. Theorem. If n is odd, then $G_{n,r}$ does not embed in $R^{r(n-r)+\min\{r,n-r\}}$.

This result is in infinite many cases weaker than that given by Oproiu ([9], Theorem 1). However, it is also in infinite many cases equally strong or stronger (in some of them considerably).

2. Proof of Theorem 1.1

We shall need some lemmas.

2.1. Lemma. Let η be an r-dimensional $(r>0)$ vector bundle over a paracompact space. Then the i-th Stiefel-Whitney class $w_i(\eta\otimes\eta)$ of the tensor square $\eta\otimes\eta$ vanishes for any odd i.

Moreover, abbreviating $w_i(\eta) = w_i$, and writing $r = \sum_{i\geq 0} r_i 2^i$ in the dyadic expansion, we have:

$$w_2(\eta\otimes\eta) = (1+r_0)w_1^2 ;$$
$$w_4(\eta\otimes\eta) = (1+r_0+r_1)w_1^4 + r_0 w_2^2 ;$$
$$w_6(\eta\otimes\eta) = (1+r_0)(1+r_1)w_1^6 + r_0 w_1^2 w_2^2 + r_0 w_3^2 ;$$
$$w_8(\eta\otimes\eta) = ((1+r_0)(1+r_1)+r_2)w_1^8 + (1+r_0)w_1^2 w_3^2 + (1+r_1)w_2^4 + r_0(1+r_1)w_1^4 w_2^2 + r_0 w_4^2 .$$

Proof. Let $\sigma_1,\dots,\sigma_r$ denote the elementary symmetric functions in variables $x_1,\dots,x_r$. Then (see Borel and Hirzebruch [2] or Thomas [12]) there is a unique element Φ_r in the ring $Z_2[x_1,\dots,x_r]$ of polynomials over the integers modulo 2, having the property

$$\Phi_r(\sigma_1,\ldots,\sigma_r) = \prod_{i,j=1}^{r}(1+x_i+x_j)\ , \tag{2.1.1}$$
$$w(\eta\otimes\eta) = \Phi_r(w_1,\ldots,w_r)\ .$$

Essentially it remains to study the polynomials Φ_r .

First we note that $\prod_{i,j=1}^{r}(1+x_i+x_j) = \prod_{i<j}(1+x_i+x_j)^2$ in the ring $Z_2[x_1,\ldots,x_r]$. Further,

$$\prod_{i<j}(1+x_i+x_j)^2 = (1+\bar{\sigma}_1+\ldots+\bar{\sigma}_{(r,2)})^2 = \tag{2.1.2}$$
$$= 1+\bar{\sigma}_1^2+\ldots+\bar{\sigma}_{(r,2)}^2\ ,$$

$\bar{\sigma}_s$ being the s-th elementary symmetric function in variables x_i+x_j , $i<j$ ((p,k) will always denote the binomial coefficient $p!/(p-k)!k!$).

Observing that each $\bar{\sigma}_i$ is a homogeneous symmetric polynomial of the i-th degree in $x_1,\ldots,x_r$, we conclude that each $\bar{\sigma}_i$ can be expressed in a unique way as a polynomial in $\sigma_1,\ldots,\sigma_i$. For our purposes it would be sufficient to determine the coefficients mod 2 of this expression for $i=1,2,3,4$.

Hence, for practical reasons, let us consider the biindexed variables $u_{(i;j)} = x_i+x_j$, $i<j$, with the biindices $(i;j)$ ordered lexicographically, and write down the induced "list":

$$x_1+x_2,x_1+x_3,\ldots,x_1+x_r,x_2+x_3,\ldots,x_2+x_r,\ldots,x_{r-1}+x_r \tag{2.1.3}$$

Since $\bar{\sigma}_1 = \sum_{i<j}x_i+x_j = a_1\sum_{i=1}^{r}x_i$, it is clear, that the coefficient $a_1\in Z_2$ equals to the number (mod 2) of entries x_1 in the sum $\sum_{i<j}x_i+x_j$. Looking at the list (2.1.3) we read $a_1 = r-1 = r+1$ mod 2.

Similarly, $\bar{\sigma}_2$ must be of the form $b_1\sigma_1^2+b_2\sigma_2$, i.e. $\bar{\sigma}_2 = b_1(\sum_{i=1}^{r}x_i)^2+b_2(\sum_{i<j}x_ix_j)$. Therefore we find b_1 and b_2 as numbers (mod 2) of entries x_1^2 and x_1x_2 respectively in $\sum_{(i;j)<(m;n)}(x_i+x_j)(x_m+x_n)$. Again, an easy calculation using (2.1.3) gives us $b_1 = (r-1,2)$ mod 2 and $b_2 = r-2+(r-1)(r-2) = r$ (mod 2) (note that the real number of entries x_1x_2 is $(r-2)r-2(r-1,2)$).

Following this pattern (but taking some more care of relations among coefficients) we find:

$$\bar{\sigma}_3 = (r-1,3)\sigma_1^3 + (r(r-1,2)+(r-2,2)+(r-1,3))\sigma_1\sigma_2 + $$
$$+ (r(r-1,2)+(r-2,2)+(r-1,3))\sigma_3 \bmod 2 ;$$

$$\bar{\sigma}_4 = (r-1,4)\sigma_1^4 + (r(r-2,2)+r(r-3,2)+(r+1)(r-1,2)+(r-3,2) +$$
$$+ (r-2,3))\sigma_1\sigma_3 + ((r-1,2)(r-2,2)+r(r-2,2))\sigma_2^2 +$$
$$+ (r-2,3)\sigma_1^2\sigma_2 + r\sigma_4 \bmod 2 .$$

It is a well-known property of binomial coefficients, that

(2.1.4) $$(p,k) \equiv \prod_{i\geq 0}(p_i,k_i) \bmod 2 ,$$

where $p = \sum_{i\geq 0} p_i 2^i$ resp. $k = \sum_{i\geq 0} k_i 2^i$ are the dyadic expansions of p resp. k .

Similarly, there is no difficulty in deriving the following identities mod 2:

(2.1.5) $$(p-1,k) \equiv \sum_{i=0}^{k}(p,i) , \quad p\geq 1 .$$

(2.1.6) $$(p-2,2s) \equiv \sum_{i=0}^{s}(p,2i) , \quad p\geq 2 .$$

(2.1.7) $$(p-2,2s+1) \equiv \sum_{i=0}^{s}(p,2i+1) , \quad p\geq 2 .$$

(2.1.8) $$(p-3,k) \equiv (p,k)+(p,k-1)+(p,k-4)+(p,k-5)+\ldots, \quad p\geq 3 .$$

Using this one calculates, that

$$\Phi_r(\sigma_1,\ldots,\sigma_r) = 1 + (1+r_0)\sigma_1^2 + (1+r_0+r_1)\sigma_1^4 + r_0\sigma_2^2 +$$
$$+ (1+r_0)(1+r_1)\sigma_1^6 + r_0\sigma_1^2\sigma_2^2 + r_0\sigma_3^2 +$$
$$+ ((1+r_0)(1+r_1)+r_2)\sigma_1^8 + (1+r_0)\sigma_1^2\sigma_3^2 +$$
$$+ (1+r_1)\sigma_2^4 + r_0(1+r_1)\sigma_1^4\sigma_2^2 + r_0\sigma_4^2 +$$
$$+ f_{10,r}(\sigma_1,\ldots,\sigma_r) ,$$

$f_{10,r}$ being a polynomial over Z_2 , which consists of nomomials having weights at least ten. This together with (2.1.1) and (2.1.2) proves the lemma.

We remark that the method above enables us theoretically to compute step by step the sequence $w_1(\eta\otimes\eta),\ldots,w_k(\eta\otimes\eta)$ for any k .

<u>2.2. Lemma.</u> Let η and w_i be the same as in 2.1, $n\eta$ denote the n-fold Whitney sum $\eta\oplus\ldots\oplus\eta$, and

$n = \sum_{i \geq 0} n_i 2^i$ be the dyadic expansion of n. Then:

$w_1(n\eta) = n_o w_1$;

$w_2(n\eta) = n_o w_2 + n_1 w_1^2$;

$w_3(n\eta) = n_o w_3 + n_o n_1 w_1^3$;

$w_4(n\eta) = n_o w_4 + n_1 w_2^2 + n_o n_1 w_1^2 w_2 + n_2 w_1^4$;

$w_5(n\eta) = n_o w_5 + n_o n_1 w_1^2 w_3 + n_o n_1 w_1 w_2^2 + n_o n_2 w_1^5$;

$w_6(n\eta) = n_o w_6 + n_o n_1 w_1^2 w_4 + n_1 w_3^2 + n_o n_1 w_2^3 + n_o n_2 w_1^4 w_2 + n_1 n_2 w_1^6$;

$w_7(n\eta) = n_o w_7 + n_o n_1 w_1^2 w_5 + n_o n_1 w_2^2 w_3 + n_o n_2 w_1^4 w_3 + n_o n_1 w_1 w_3^2 +$
$+ n_o n_1 n_2 w_2 w_1^7$;

$w_8(n\eta) = n_o w_8 + n_o n_1 w_1^2 w_6 + n_1 w_4^2 + n_o n_1 w_2^2 w_4 + n_o n_2 w_1^4 w_4 + n_o n_1 w_2 w_3^2 +$
$+ n_2 w_2^4 + n_1 n_2 w_1^4 w_2^2 + n_o n_1 n_2 w_1^6 w_2 + n_3 w_1^8$;

$w_9(n\eta) = n_o w_9 + n_o n_1 w_1^2 w_7 + n_o n_1 w_2^2 w_5 + n_o n_2 w_1^4 w_5 + n_o n_1 w_3^3 +$
$+ n_o n_1 n_2 w_1^6 w_3 + n_o n_1 w_1 w_4^2 + n_o n_2 w_1 w_2^4 + n_o n_1 n_2 w_1^5 w_2^2 + n_o n_3 w_1^9$.

Moreover, if n is even, $w_i(n\eta)$ vanishes for any odd i.

Proof. The first part: since $w(n\eta) = (1+w_1+\ldots+w_r)^n$, it holds

$$w_k(n\eta) = \sum (n,i_o)(n-i_o,i_1)\ldots \ldots(n-i_o-i_1-\ldots-i_{k-1},i_k)w_1^{i_1}\ldots w_k^{i_k}, \tag{2.2.1}$$

where the sum runs throughout the set of (k+1)-tuples $(i_o,i_1,\ldots,i_k)$ with $i_o+i_1+\ldots+i_k = n$ and $i_1+2i_2+\ldots+ki_k = k$. This and a little calculation using (2.1.4) gives the assertion.

The second part: since n is now even, we can write $w(n\eta) = (w(\eta)^2)^{n/2}$, which yields the result immediately.

2.3. Proof of Theorem 1.1. Since $G_{n,r}$ is compact, the canonical bundle $\gamma_{n,r}$ may be identified with its dual $\gamma^*_{n,r}$, and

$$T(G_{n,r}) \oplus \gamma_{n,r} \otimes \gamma_{n,r} \approx n\gamma_{n,r} \tag{2.3.1}$$

(see Hsiang and Szczarba [4]). The first part of 1.1 follows, using product formula for Stiefel-Whitney classes, from 2.1 and 2.2.

In the case n is even we know (see 2.1 and 2.2) that the odd - dimensional Stiefel-Whitney classes of $\gamma_{n,r} \otimes \gamma_{n,r}$ resp. $n\gamma_{n,r}$ vanish. By (2.3.1) we obtain

$$w_s(G_{n,r}) = \sum_{\substack{k+p=s\\ p>0}} w_k(G_{n,r}) w_p(\gamma \otimes \gamma) + w_s(n\gamma) .$$

This with an obvious induction completes the proof.

3. Proofs of the remaining theorems

Let $G_{\infty,r}$ denote the infinite Grassmann manifold of all linear r-subspaces of R^{∞} .

3.1. Lemma. The restriction morphism

$$i^* : H^p(G_{\infty,r};Z_2) \longrightarrow H^p(G_{n,r};Z_2)$$

is an isomorphism for $p \leq n-r$. Hence, there is no polynomial relation among $w_1,\dots,w_r$ in $H^p(G_{n,r};Z_2)$ for $p \leq n-r$.

Proof. The cell decomposition of Grassmann manifolds (Milnor [7]) implies that the (n-r)-skeletons $(G_{n,r})_{(n-r)}$ and $(G_{\infty,r})_{(n-r)}$ may be identified. The commutative diagram

$$\begin{array}{ccc} (G_{n,r})_{(n-r)} & \xrightarrow{\ id\ } & (G_{\infty,r})_{(n-r)} \\ \Big\downarrow j & & \Big\downarrow k \\ G_{n,r} & \xrightarrow{\ i\ } & G_{\infty,r} \end{array}$$

i,j,k being inclusions, induces - in cohomology with arbitrary coefficients - the commutative diagram

$$\begin{array}{ccc} H^p(G_{\infty,r}) & \xrightarrow{\ k^*\ } & H^p((G_{\infty,r})_{(n-r)}) \\ \Big\downarrow i^* & & \Big\downarrow id \\ H^p(G_{n,r}) & \xrightarrow{\ j^*\ } & H^p((G_{n,r})_{(n-r)}) \end{array} .$$

Cohomology properties of CW-complexes imply that k^* and j^* are isomorphisms for $p < n-r$ and monomorphisms for $p = n-r$. Therefore, i^* is an isomorphism for $p < n-r$ and a monomorphism for $p = n-r$.

Moreover, it is known that the mod 2 cohomology ring $H^*(G_{\infty,r};Z_2)$ is the polynomial ring $Z_2[w_1',\dots,w_r']$, $w_i' = w_i(\gamma_r)$ being the i-th Stiefel-Whitney class of the canonical vector bundle γ_r over $G_{n,r}$ (see Milnor [7]).

On the other hand, the ring $H^*(G_{n,r};Z_2)$ is generated multiplicatively by the Stiefel-Whitney classes $w_1,\dots,w_r$ of the canonical bundle $\gamma_{n,r}$ over $G_{n,r}$ (see Borel [1]). Since, by "universality" of γ_r and naturality of Stiefel-Whitney classes, $w_i(\gamma_{n,r}) = i^*(w_i(\gamma_r))$, the ring morphism $i^* : H^*(G_{\infty,r};Z_2) \to H^*(G_{n,r};Z_2)$ is an epimorphism, and 3.1 is proved.

3.2. Proof of Theorem 1.2. The map $h : G_{n,r} \to G_{n,n-r}$ sending each linear r-subspace of R^n into its orthogonal complement in R^n is a diffeomorphism. Therefore we shall suppose throughout this section $n \geq 2r$.

It follows from 1.1 and 3.1 that

(i) $w_2(G_{n,r}) = w_1^2 \neq 0$ for $n \equiv 2 \bmod 4$, $r \geq 3$ and odd,

(ii) $w_4(G_{n,r}) = w_2^2 + \dots \neq 0$ for $n \equiv 0 \bmod 4$, $r \geq 3$ and odd,

(iii) $w_1(G_{n,r}) = w_1 \neq 0$ for n odd,

(iv) $w_2(G_{n,r}) = w_1^2 \neq 0$ for $n \equiv 0 \bmod 4$, r even,

(v) $w_4(G_{n,r}) = w_2^2 + \dots \neq 0$ for $n \equiv 2 \bmod 4$, r even,

which clearly implies the non-parallelizability of $G_{n,r}$ in each of the cases (i) - (v).

This, however, covers all $G_{n,r}$'s not diffeomorphic to projective spaces. To make the proof complete, it is sufficient to observe that - since the only parallelizable spheres are S^1, S^3 and S^7 (Kervaire [5], Milnor [8]) - the only parallelizable projective spaces are $G_{2,1}$, $G_{4,1}$ and $G_{8,1}$.

3.3. Remark. The non-parallelizability of even dimensional $G_{n,r}$'s (and even more, the non-existence of any vector field without zeros on them) could also be proved in other, but less intrinsic in relation to the present context, way using the fact that their Euler characteristic does not vanish. For the Grassmann manifolds $\tilde{G}_{n,r}$ of oriented linear r-subspaces of R^n this follows, for instance, from the theory of symmetric spaces (Wolf [13]), and because $\tilde{G}_{n,r}$ is a double covering of $G_{n,r}$, we have also

$\chi(G_{n,r}) = \frac{1}{2}\tilde{G}_{n,r} \neq 0$.

3.4. Remark. From 1.1 and 3.1 one obtains immediately that $G_{n,r}$ is a spin-manifold iff $n \equiv 0 \bmod 4$, r is odd or $n \equiv 2 \bmod 4$, r is even.

3.5. Proof of Theorem 1.3. Clearly we may suppose $n \geq 2r$. We have

$w_8(G_{n,r}) = w_4^2 + \ldots \neq 0$ for $n \equiv 0 \bmod 4$, $r \geq 5$,

$w_8(G_{n,r}) = w_2^4 + \ldots \neq 0$ for $n \equiv 0 \bmod 8$, $r = 3$,

$w_8(G_{n,r}) = w_1^4 w_2^2 + \ldots \neq 0$ for $n \equiv 4 \bmod 8$, $r = 3$,

$w_8(G_{n,r}) = w_1^2 w_3^2 + \ldots \neq 0$ for $n \equiv 2 \bmod 4$,

which proves the first part of the theorem.

For $G_{6,3}$ we obtain $w_2(G_{6,3}) = w_1^2 \neq 0$, $w_4(G_{6,3}) = w_6(G_{6,3}) = 0$ and $w_8(G_{6,3}) = w_1^2 w_3^2 + w_2^4$, which, however, vanishes. To see this it is enough to analyse the relations among w_1, w_2, w_3 in $H^8(G_{6,3};Z_2)$ (recall, that all relations among $w_1,\ldots,w_r$ in $H^*(G_{n,r};Z_2)$ are determined by $(1+w_1+\ldots+w_r).(1+\bar{w}_1+\ldots+\bar{w}_{n-r}) = 1$, $\bar{w}_i = \bar{w}_i(\gamma_{n,r})$ being the dual Stiefel-Whitney class (see Borel [1])).

On the other hand, a similar analysis shows, that the 8-th Stiefel-Whitney classes of $G_{8,3}$, $G_{10,3}$, $G_{10,5}$, $G_{12,5}$ and $G_{14,7}$ does not vanish.

Moreover, when n is even and r is odd, span $G_{n,r} \geq$ span S^{n-1} (see Leite and Miatello [6]). Hence, we get span $G_{8,3} = 7$, the remaining estimations being clear. This completes the proof.

<u>3.6. Proof of Theorem 1.4.</u> If an n-dimensional manifold M can be embedded in R^{n+k} , then $\bar{w}_p(M) = 0$ for $p \geq k$, $\bar{w}_p(M)$ being the p-th dual Stiefel-Whitney class of M (see for instance Switzer [10]). Hence, for to prove our theorem we show that $\bar{w}_{n-r}(G_{n,r}) \neq 0$ if n is odd and $n \leq 2r$.

It is clear from (2.2.1) that $w_p(n\gamma)$ is of the form

(3.6.1) $$w_p(n\gamma) = nw_p(\gamma) + \text{terms without } w_p(\gamma)$$

for any real vector bundle γ over a paracompact space.

For $p \leq n-r \leq r$, n odd we can write in a unique way

(3.6.2) $$w_p(G_{n,r}) = w_p(\gamma_{n,r}) + \text{terms without } w_p(\gamma_{n,r}) .$$

Namely, from (2.3.1) we obtain

(3.6.3) $$w_p(G_{n,r}) = w_{p-2}(G_{n,r})w_2(\gamma \otimes \gamma) + \ldots + w_1(G_{n,r})w_{p-1}(\gamma \otimes \gamma) + w_p(n\gamma) ,$$

if p is odd, and

(3.6.4) $$w_p(G_{n,r}) = w_{p-2}(G_{n,r})w_2(\gamma \otimes \gamma) + \ldots + w_2(G_{n,r})w_{p-2}(\gamma \otimes \gamma) + w_p(\gamma \otimes \gamma) + w_p(n\gamma) ,$$

if p is even.

Then one gets (3.6.2) from (3.6.1) and (3.6.3) resp. (3.6.4) observing that $w_p(\gamma \otimes \gamma)$ does not contain $w_p(\gamma)$ (see the proof of 2.1) and keeping in mind that $p \leq n-r$ ensures the absence of polynomial relations among $w_1(\gamma_{n,r}),\dots,w_r(\gamma_{n,r})$.

Since $\overline{w} = (1+w_1+w_2+\dots)^{-1}$, it is clear that

$$\overline{w}_{n-r}(G_{n,r}) = w_{n-r}(G_{n,r}) + \text{monomials without } w_{n-r}(G_{n,r}) .$$

Thus, expressing the right-hand side in terms of $w_i(\gamma_{n,r})$ we obtain (see (3.6.2)):

$$\overline{w}_{n-r}(G_{n,r}) = w_{n-r}(\gamma_{n,r}) + \text{terms without } w_{n-r}(\gamma_{n,r}) ,$$

which does not vanish, and the proof is complete.

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MATEMATICKÝ ÚSTAV ČSAV
ŽITNÁ 25
115 67 PRAHA 1
CZECHOSLOVAKIA

KATEDRA MATEMATIKY VŠDS
MARXA-ENGELSA 25
010 88 ŽILINA
CZECHOSLOVAKIA