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ON THE COMPUTATION OF SOME QUANTITIES IN THE THEORY OF FREDHOLM OPERATORS

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0. Introduction

Although the theory of Fredholm operators is usually approached with Banach algebra techniques (e.g. [6],[4]) there has been some interest in "measures of non-compactness" (e.g. [19],[28]). Roughly speaking, they may be used as a substitute for the operator norm if one is interested in Fredholm inverses and the essential spectrum instead of the usual notion of invertibility and spectrum. For a bounded linear operator T in a Banach space X they are defined in terms of X and T (not via quotient algebras like the Calkin algebra) but it is often not easy to calculate them and this is the problem we address in this report.

We will concentrate on the following variante of these quantities (introduced by M. Schechter in [30]):

(*) $$\Delta(T) = \sup_{M} \inf_{N\subset M} \|T_{|N}\|$$

where the 'sup' and 'inf' is taken over all infinite dimensional subspaces of X. At first glance it may look hopeless to try to calculate this quantity for concrete operators, especially since one has to consider 'arbitrary' subspaces N of X and the restriction $T_{|N}$ will in general distroy any concrete representation of T that might be useful to estimate the norm (it is well known that every contraction in a Hilbert-space is similar to a restriction of the shift-operator, [27]).

But we will see that for operators in the classical Banach spaces $l_p, L_p(\mu), C(K)$ one has to deal in (*) only with very special subspaces M and N for which the form of $T_{|N}$ is in general still very close to the original operator T. Using modern Banach space theory one can show that usually it is enough to consider 'bands' in these spaces (i.e. subspaces of functions that vanish outside a given measurable set). In $L_p(\mu)$ this is of special interest for

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operators of 'local type' according to Simonenko ([31]). As an illustration we calculate Δ for certain classes of Toeplitz-, Hankel- singular integral- and pseudo-differential operators.

We will also show that Δ is nicely related to other interesting quantities:

- For a large class of operators Δ(T) equals the distance of T to the ideal of strictly singular operator. This ideal, introduced by Kato is the maximal ideal of admissable Fredholm perturbations in $L_p(\mu)$ and C(K) (see [22],[34]).
- For operators in l_p and operators of local type many of the usual measures of non compactness are just a multiple of Δ.

We detail these results in the following sections:

1. General properties of Schechter's Δ-characteristic.
2. Operators in l_p and in Hilbert spaces.
3. Singular integral operators and Toeplitz operators.
4. Operators in $L_p(\mu)$, $1< p < \infty$.
5. Pseudo-differential operators.
6. Operators in $L_1(\mu)$ and C(K).
7. Comparison with measures of non-compactness.

1. General properties of Schechter's Δ-characteristic.

Let X and Y be Banach spaces and B(X,Y) the space of all bounded linear operators. Recall that T ∈ B(X,Y) is called a ϕ_+-operator (a ϕ_--operator) if T has closed range and dim(Ker T) < ∞ (codim T(X) < ∞). These properties can be expressed in terms of the following quantities.

(1) $$\Gamma(T) = \inf_M \|T_M\| \quad , \quad \Gamma^*(T) = \inf_M \|\phi_M T\|$$

where the inf is taken over all closed subspaces with dim M = codim M = ∞ and ϕ_M denotes the quotient map $\phi_M : X \to X/M$. Γ was introduced by Gramsch in [11] and Γ* in [33]. It is not difficult to see that

1.1 Proposition: T is a ϕ_+-operator ⇔ Γ(T) > 0([11])
T is a ϕ_--operator ⇔ Γ*(T)> 0([33])

In particular, T is a Fredholm operator if and only if Γ(T) > 0 and Γ*(T) > 0. In the subsequent sections we shall see that many natural conditions that guarantee that certain concrete operators (e.g. singular integral operators) are Fredholm operators, can be expressed in terms of Γ and Γ^*.

Γ and Γ^* also appear in the perturbation theory of (Semi-) Fredholm operators. In order to obtain a more general ideal of admissible perturbation of ϕ_+-operators than compact operators Kato introduced in [17] the class of strictly singular operators. These are operators such that for all infinite dimensional subspaces M of X the restriction $T_{|M}$ is not an isomorphism into. The dual notion of strictly cosingular operators ($\phi_M T$ is never surjective if codim $M = \infty$) was considered by Pelczynski ([23]) and it was observed in [32] that these operators are admissable perturbations of ϕ_--operators. The following quantity Δ introduced by Schechter ([30]) generalizes measures of non-compactness in the same way strictly singular operators generalize compact operators

(2) $$\Delta(T) = \sup_M \Gamma(T_{|M})$$

where the sup is taken over all closed subspaces M with $\dim M = \operatorname{codim} M = \infty$. The dual quantity

$$\Delta^*(T) = \sup_M \Gamma^*(\phi_M T)$$

was considered in [33]. Indeed we have

1.2. Proposition: Δ and Δ^* are continuous and multiplicative seminorms on $B(X)$ and

$$\Delta(T) = 0 \Leftrightarrow T \text{ is strictly singular ([30])}$$

$$\Delta^*(T) = 0 \Leftrightarrow T \text{ is strictly cosingular ([33])}$$

Using Δ and Γ Schechter formulated the following perturbation theorem which generalizes Kato's theorem on strictly singular operators and the well known fact that the set of ϕ_+-operators is open in $B(X,Y)$:

1.3. Theorem: a) ([30]) If $\Delta(T) < \Gamma(S)$ then $T+S$ is a ϕ_+-operator and $\operatorname{ind}(S) = \operatorname{ind}(S+T)$.

b) ([33]) If $\Delta^*(T) < \Gamma^*(S)$ then $T+S$ is a ϕ_--operator and $\operatorname{ind}(S) = \operatorname{ind}(S+T)$.

Furthermore, the quantities (1) and (2) can be used to give asymptotic formulas for the radius $r_e(T)$ of the essential spectrum of $T \in B(X)$ (see [6] for its properties) and for the so called semi-Fredholm radii.

(3) $$S_+(T) = \sup\{\varepsilon \geq 0 : T - \lambda I \text{ is a } \phi_+\text{-operator for } |\lambda| < \varepsilon\}$$

$$S_-(T) = \sup\{\varepsilon \geq 0 : T - \lambda I \text{ is a } \phi_-\text{-operator for } |\lambda| < \varepsilon\}$$

1.4. Theorem: For $T \in B(X)$ we have

a) ([30],[33]) $r_e(T) = \lim_n \Delta(T^n)^{1/n} = \lim_n \Delta^*(T^n)^{1/n}$

b) ([39]) $s_+(T) = \lim_n \Gamma(T^n)^{1/n}$, $s_-(T) = \lim_n \Gamma^*(T^n)^{1/n}$

1.1. to 1.4. suggest that if we deal with perturbations of (Semi-) Fredholm operators and the essential spectrum instead of perturbations of invertible operators and the usual spectrum the role of the operator norm will be taken over by Δ, Δ^*, and Γ, Γ^* may be considered as substitutes for the minimum modulus of the operator T

(4) $$m(T) = \inf\{\|Tx\| : \|x\| = 1\}$$

and the so called surjection modulus

(5) $$q(T) = \sup\{\varepsilon \geq 0 : TU_X \supset \varepsilon U_Y\}$$

where U_X, U_Y are the unit balls of X and Y. It is one of the main points of the next sections to show that the 'abstract' quantities $\Delta, \Gamma, \ldots$ for operators in classical Banach spaces can be expressed in terms of the more intuitive quantities (4) and (5). This will allow to calculate $\Delta, \Delta^*, \Gamma, \Gamma^*$ and to apply theorem 1.3. and 1.4. in some concrete situations.

The 'duality' between Δ, Γ and Δ^*, Γ^* takes a simple form in reflexive spaces:

1.5. Proposition: If X and Y are reflexive then

$$\Delta(T^*) = \Delta^*(T) \quad , \quad \Gamma(T^*) = \Gamma^*(T)$$

But in general Banach spaces, it might happen that e.g. $\Delta(T^*) > \Delta^*(T) = 0$ (see [23],[8],[35]). We shall also need the following useful relations which hold for general $T \in B(X,Y)$ (see [24] B. 3.8)

(6) $$m(T^*) = q(T) \quad , \quad q(T^*) = m(T).$$

2. Operators in l_p and in Hilbert spaces.

For multiplication operators it is rather easy to calculate Δ and Γ but this special case is useful in determing Δ and Γ for more general classes of operators (see e.g. 2.2., 2.4.).

2.1. Proposition: Let $m \in L_\infty(\Omega,\Sigma,\mu)$. For the multiplication operator $T: L_p(\Omega,\Sigma,\mu) \to L_p(\Omega,\Sigma,\mu)$ with $Tf(w) = m(w)\cdot f(w)$, $w \in \Omega$, we have

$$\Delta(T) = \Delta^*(T) = \overline{\lim}\, R(m)$$
$$\Gamma(T) = \Gamma^*(T) = \underline{\lim}\, R(m)$$

Here R(m) denotes the essential range of the function m, i.e.

$$R(m) = \{\lambda \in \mathbb{C}: \quad \mu(|m-\lambda| < \varepsilon) > 0 \quad \text{for all } \varepsilon > 0\}$$

If the measure space is purely atomic then T is a diagonal operator

$T(f_n) = (m_n f_n)$ in a sequence space l_n and

$$\Delta(T) = \overline{\lim_n}|m_n| \quad , \quad \Gamma(T) = \underline{\lim_n}|m_n|$$

while $\|T\| = \sup|m_n| \quad , \quad \alpha(T) = q(T) = \inf_n |m_n|$.
If (Ω,Σ,μ) has no atoms then

$$\Delta(T) = \|T\| = \sup R(m), \ \Gamma(T) = \alpha(T) = \inf R(m)$$

(this observation is also true for multiplication operators in Banach function lattices and in spaces with an unconditional basis, see e.g. [5],[20] for definitions).

Sketch of proof: As subspaces on which Δ and Γ are (almost) attained we may take the span of χ_{A_n}, $n \in \mathbb{N}$, where the $A_n \in \Sigma$ are pairwise disjoint and chosen in such a way that the values of $|m|\chi_{A_n}$ are very close to $\overline{\lim}\, R(m)$ or $\underline{\lim}\, R(m)$.

Using results on basic sequences from Banach space theory (see e.g. [20]) we can reduce the calculation of Δ and Γ for an operator T in l_p to certain diagonal operators. More precisely, if $x_n \in lp$ satisfies

$$\|x_n\| = 1, \quad x_n \xrightarrow{w} 0, \quad \|Tx_n\| \longrightarrow d$$

then there is a subsequence x_{n_k}, such that x_{n_k} and

$y_k = T(x_{n_k}) \cdot \|T(x_{n_k})\|^{-1}$ are equivalent to the unit vector basis (e_k)

of l_p. Then the diagramm

$$\begin{array}{ccc} [x_{n_k}] & \xrightarrow{T} & [Tx_{n_k}] \\ U\uparrow & & \uparrow V \\ lp & \xrightarrow[D]{} & lp \end{array}$$

commutes, where $U(e_k) = x_{n_k}$, $V(e_k) = y_k$ are (almost) isometries and

D is the diagonal operator $D(\alpha_n) = (\|Tx_n\|\alpha_n)$. In particular

$$d = \Delta(D) = \Gamma(D) \approx \Delta(T|_{[x_{n_k}]}) \approx \Gamma(T|_{[x_{n_k}]})$$

This is the essential step in the proof of the following theorem:

2.2. Theorem: For a bounded linear operator
$T : l_p \longrightarrow l_p$, $1 \le p < \infty$ or $T : c_o \longrightarrow c_o$ we have

$$\Delta(T) = \Delta^*(T) = \overline{\lim_{n,m\to\infty}} \|P_nTP_m\|^{1)}$$

$$= \sup\{\overline{\lim_n}\|Tf_n\| : f_n \text{ pairwise disjoint}, \|f_n\| = 1\}^{2)}$$

$$= \inf\{\|T-K\| : K \text{ compact}\}$$

$$\Gamma(T) = \underline{\lim_{n,m\to\infty}} m(P_nTP_m)$$

$$= \inf\{\underline{\lim}\|Tf_n\| : f_n \text{ pairwise disjoint}, \|f_n\| = 1\}$$

$$= \sup\{m(T-K) : K \text{ compact}\}$$

$$\Gamma^*(T) = \underline{\lim_{n,m\to\infty}} q(P_nTP_m)$$

$$= \sup\{q(T-K) : K \text{ compact}\}$$

In sections 4 and 6 we shall extend these formulas to operators in $L_p(\mu)$ and $C(K)$. They improve the original definition of Δ and Γ in several ways:

- Instead of arbitrary infinite dimensional subspaces M of l_p we only have to consider the 'special' subspaces $P_n(l_p) = [e_K, K \geq n]$ on which T still may have a form similar to the whole operator. E.g. if $T : l_p \to l_p$ is given with respect to the unit vector basis by an infinite matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots \\ a_{21} & a_{22} & \cdots\cdots \\ \vdots & \vdots & \end{pmatrix}$$

then P_nTP_m is given by

$$\left(\begin{array}{c|cc} & \multicolumn{2}{c}{0} \\ \hline 0 & a_{nm} & a_{n,m+1} \cdots \\ & a_{n+1,m} & a_{n+1,m+1} \cdots \\ & \vdots & \vdots \end{array}\right)$$

And on the subspaces $P_n(l_p)$, i.e. for these truncated matrices, we only have to estimate the 'traditional' quantities $\| \ \|$, $m(\cdot)$, and $q(\cdot)$. As an illustration for this, we shall consider Toeplitz- and Hankel matrices in Sec. 3.

1) If (e_K) is the unit vector basis of l_p we denote by P_n the coordinate projection $P_n(\sum_{K=1}^{\infty} \alpha_K e_K) = \sum_{K=n}^{\infty} \alpha_K e_K$.

2) In this line let $p > 1$.

- In characterizations of semi-Fredholm operators sometimes called singular sequences (i.e. $\|x_n\| = 1$, $x_n \longrightarrow 0$) are used (see e.g. [19], [17]). Our formulas imply that it is enough to consider very special singular sequences formed by pairwise disjoint vectors. This corresponds to the use of 'Weyl-sequences' (which are orthogonal) in Hilbert space theory.
- Since the Fredholm property of $T : l_p \to l_p$ corresponds to the invertibility of the equivalence class $\hat{T} \in B(l_p)/K(l_p)$ one may formulate properties of Fredholm operators or the essential spectrum in terms of the quotient norm $\|\hat{T}\|$. From this point of view Δ provides a manageable 'lifting' of $\|\hat{T}\|$ to the original space.

2.1. and 2.2. contain some results on Hilbert space operators which are essentially known, although not stated in terms of Δ and Γ. Since a separable Hilbert space is isometric to l_2 we obtain from 2.2.

2.3. Corollary ([14] §3, Theorem 1): For a bounded linear operator $T : H \to H$ in a Hilbert space H we have

$$\Delta(T) = \Delta^*(T) = \sup\{\overline{\lim_n}\|Tx_n\| : \|x_n\| = 1,\ (x_n) \text{ orthogonal}\}$$

$$= \inf\{\|T-K\| : K \text{ compact}\}$$

Of course analoguous formulas can be derived for Γ and Γ^*. Since self-adjoint operators are similar to multiplication operators the next statements follows from 2.1. but they also follow from results in [19], p. 61/62; [2] and [40].

2.4. Corollary: Let T be a bounded linear operator in a Hilbert space and T* its Hilbert space adjoint. Then

$$\Delta(T) = \Delta^*(T) = \sup \sigma_e([T^*T]^{1/2}) = \sup \sigma_e([TT^*]^{1/2})$$
$$\Gamma(T) = \inf \sigma_e([T^*T]^{1/2})$$
$$\Gamma^*(T) = \inf \sigma_e([TT^*]^{1/2})$$

Proof: By the polar decomposition we have

$$T = U|T| \qquad T^* = |T^*|U^*$$

where $|T| = [TT^*]^{1/2}$ and U and U* are partial isometries. Hence $\Delta(T) = \Delta(|T|) = \Delta(|T^*|)$ and we may restrict ourselves to self adjoint operators. But by the spectral theorem for self adjoint operators T there is a space $L_2(\Sigma,\mu)$, an isometry $U : H \longrightarrow L_2(\Sigma,\mu)$ and a multiplication operator $Mf(y) = m(y)f(y)$ in $L_2(\Sigma,\mu)$ such that $T = U^{-1}JU$. By 2.1. we have

$$\Delta(T) = \Delta(M) = \overline{\lim}\, R(m) = \sup \sigma_e(M) = \sup \sigma_e(T)$$
$$\Gamma(T) = \Gamma(M) = \underline{\lim}\, R(m) = \inf \sigma_e(M) = \inf \sigma_e(T)$$

the formula for $\Gamma^*(T)$ follows by duality.

We mention one more consequence of 2.2. which is helpful in calculating Γ and Γ^* for Fredholm operators.

2.5. Corollary: If T and S are bounded linear operators in l_p such that $T\cdot S - I$ and $ST - I$ are compact then

$$\Delta(T) = \Gamma(S)^{-1} = \Gamma^*(S)^{-1}$$

This 'generalizes' the well known fact that for a invertible operator T we have

$$\|T\| = m(T^{-1})^{-1} = q(T^{-1})^{-1}.$$

Using the corollaries 2.3, 2.4 and 2.5 one can sometimes deduce $\Gamma(T)$ and $\Delta(T)$ easily from known results. For examples, see Sec. 3 and 5.

3. Singular integral operators, Toeplitzoperators and Hankel operators in $L_2(T)$.

Let T be the unit circle in $\mathbb{C}$ and μ be Lebesgue's measure on T. Here we are interested in operators in $L_2(T,\mu)$ of the form

(1) $$M_c + M_d S \qquad c,d \in L_\infty(T)$$

where M_c denotes the multiplication operator by c, i.e.

$$M_c(f)(x) = c(x)f(x)$$

and S denotes the operator of singular integration

$$Sf(x) = \frac{1}{\pi i}\int_T \frac{f(y)}{y-x}\,dy \qquad \text{(Cauchy-principal-value)}.$$

Let us recall the connection of the singular integral operator (1) with Toeplitz- and Hankel-operators: $f_n(e^{it}) = e^{int}$, $n \in \mathbb{Z}$, form a orthonormal basis of $L_2(T)$ and

$$P\Big(\sum_{n\in\mathbb{Z}} \alpha_n f_n\Big) = \sum_{n\geq o} \alpha_n f_n, \quad Q\Big(\sum_{n\in\mathbb{Z}} \alpha_n f_n\Big) = \sum_{n<o} \alpha_n f_n$$

are orthogonal projections onto the Hardy space $H_2(T)$ and its complement $H_2^-(T)$, respectively. Since $P = \frac{1}{2}(Id + S)$, $Q = \frac{1}{2}(Id - S)$ we can write (1) in the form

(2) $$M_c + M_d S = M_a P + M_b Q \quad \text{where } a = c + d,\ b = c - d\ .$$

With respect to the decomposition $L_2(T) = H_2(T) + H_2^-(T)$ the singular integral operator (1) has therefore the 'matrix'-representation

(3) $$\begin{pmatrix} PM_a|_{H_2} & PM_b|_{H_2^-} \\ QM_a|_{H_2} & QM_b|_{H_2^-} \end{pmatrix}$$

where the 'diagonal' operators are similar to Toeplitz operators and the 'off-diagonal' operators are essentially Hankel operators.

More precisely, a Toeplitz-operator is of the form

(4) $$PM_a|_{H_2} : H_2 \longrightarrow H_2 \ , \ a \in L_\infty(T)$$

If $J : L_2(T) \longrightarrow L_2(T)$ is given by $J(f_n) = f_{-n}$, then $P - JQJ$ and $JPJ - Q$ is the orthogonal projection onto (1) and we have that

(5) $$QM_bQ - J^{-1}[PM_{J(b)}P]J = QM_bQ - JPJM_bJPJ$$

is a finite dimensional operator. Hence the diagonal operators of (3) are essentially similar to Toeplitz-operators. A Hankel-operator is given by

(6) $$PJM_a|_{H_2} : H_2(T) \longrightarrow H_2(T), \ a \in L_\infty(T)$$

Then the following operators are one-dimensional

(7) $$QM_aP - J[PJM_aP]$$
$$PM_bQ - [PJM_{\bar{b}}P]^*J = PM_bQ - PM_bJPJ$$

i.e. the 'off-diagonal' operators in (3) can be expressed in terms of Hankel-operators.

Furthermore, we want to recall the connections of definitions (4) and (5) with Toeplitz- and Hankel-matrices. The Fourier-transform of the Toeplitz-operator $U = PM_a|_{H_2}$ is an operator $\hat{U} : l_2(\mathbb{Z}^+) \to l_2(\mathbb{Z}^+)$ given by an infinite matrix of the form

(8) $$(d_{j-k})_{j,k\in\mathbb{Z}^+} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_{-1} & a_0 & a_1 & \ddots \\ a_{-2} & a_{-1} & a_0 & a_1 \ \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \text{ where } a_j = \hat{a}(j), j \in \mathbb{Z}^t$$

Such a matrix is called a Toeplitz-matrix.

The Fouriertransform $\hat{V} : l_2(\mathbb{Z}^+) \to \ell_2(\mathbb{Z}^+)$ of a Hankel-operator $V = PJM_a|_{H_2} : H_2 \longrightarrow H_2$ is given by the matrix

(9) $$(a_{K+j})_{K,j\in\mathbb{Z}^+} = \begin{pmatrix} a_0 & a_1 & a_2 & \cdot \\ a_1 & a_2 & \cdot & \\ a_2 & \cdot & & \end{pmatrix} \text{ where } a_j = \hat{a}(-j), j \in \mathbb{Z}^+$$

which one calls a Hankel-matrix.

As indicated by (3),(5) and (7) one can reduce the calculation of Δ and Γ for singular integral operators to Toeplitz- and Hankel-operators; we look therefore first at these operators.

3.1. Satz: For a Hankel-operator $V = PJM_a|_{H_2}, a \in L_\infty(T)$, we have:

$$\Gamma^*(T) = \Gamma(T) = 0$$
$$\Delta(T) = \inf\{\|a-\zeta\|_\infty : \zeta \in H_\infty(T) + C(T)\}$$

where $H_\infty(T) = L_\infty(T) \cap H_2(T)$.

Proof: Since the Fourier transform : $H_2(T) \longrightarrow l_2(\mathbb{Z}^+)$ is an isometry we have $\Gamma(V) = \Gamma(\hat{V})$. From (9) we obtain for the unit vectors $e_k \in l_2(\mathbb{Z}^+)$ that

$$\|\hat{V}e_K\|^2 = \sum_{j=K}^{\infty} |a_i|^2 \longrightarrow 0 \quad \text{for } K \to \infty$$

Now 2.3. implies that $\Gamma(\hat{V}) = 0$. $\Gamma^*(\hat{V}) = 0$ follows by duality since $\hat{V}^*$ is given by the Hankel matrix $(\bar{a}_{i+j})_{i,j\in\mathbb{Z}^+}$.

The second statement follows directly from 2.3. and the well known fact, that

$$\inf\{\|T-K\| : K \text{ compact}\} = \inf\{\|a-\zeta\|_\infty : \zeta \in H_\infty + C\}$$

(See Hartmann's theorem, e.g. in [25] §1). But in 3.2. and 3.3. we will only use "≤" and this inequality is easy to prove: If $\zeta = \zeta_\infty + \zeta_0$ with $\zeta_\infty \in H_\infty(T)$ and $\zeta_0 \in C(T)$ and $\varepsilon > 0$ choose a trigonometric polynomial $\zeta_1 = \sum \beta_n e^{in}$ such that $\|\zeta_0 - \zeta_1\| \leq \varepsilon$. Then $U_{\zeta_1} = PJM_{\zeta_1}|H_2$ is a finite dimensional operator and

$$\Delta(U_a) = \Delta(U_a - U_{\zeta_1}) \leq \Delta(U_{a-\zeta_\infty-\zeta_1}) \leq \|a-\zeta_\infty \ \zeta_1\|_\infty$$

$$\leq \|a-\zeta\| + \varepsilon .$$

3.1. gives a criterion for when the 'off-diagonal' operators in (3) are negligeable in calculating Γ and Δ. If $a, \bar{b} \in H_\infty(T) + C(T)$ then by (7) and (9) the operators $PM_{b|H_2^-}$ and $QM_{a|H_2}$ are compact. This motivates the following definition:

A function $a \in L_\infty(T)$ is called <u>quasi-continuous</u> if $a \in H_\infty(T) + C(T)$ and $\bar{a} \in H_\infty(T) + C(T)$ (see also [29]).

<u>3.2. Satz:</u> For a Toeplitz operator $U = PJM_{a|H_2}$ $a \in L_\infty(T)$, we always have

$$\Delta(U) = \Delta^*(U) = \|U\| = \sup|a|$$

$$\inf \overline{co}R(a) \leq \Gamma(U) \leq \inf|a|$$

If a is quasi-continuous we have in addition

$$\Gamma(U) = \Gamma^*(U) = \inf|a| .$$

Remark: Even for a piecewise monotone function a we may have $\Gamma(U) = 0 < \inf|a|$, see [6] 7.20.

<u>Proof:</u>The first statements follow from

$$r_e(U) \leq \Delta(U) \leq \|U\| \leq \|M_a\| = \sup|a|$$

$$\Gamma(U) \leq S_+(U) \leq \inf\{|\lambda| : \lambda \in \sigma_e(U)\}$$

(see theorem 1.4) and the well known fact that $R(a) \subset \sigma_e(U)$ (see [6] Corollary 7.7).

Choose $\lambda \in \overline{\mathrm{co}\,(a)}$ such that $\mu := \inf|\overline{\mathrm{co}R(a)}| = |\lambda|$. Put $\lambda_o = \lambda|\lambda|^{-1}$ and fix some $\mu_1 < \mu$. Then, for r large enough $\overline{\mathrm{co}(R(a))}$ is contained in the ball $B(r\lambda_o, r-\mu_1)$ with center $r\lambda_o$ and radius $r-\mu_1$. From the general inequality

$$\Gamma(U+V) \leq \Gamma(U) + \Delta(V)$$

we obtain then

$$\begin{aligned}\Gamma(T) &\geq \Gamma(r\lambda_o Id) - \Delta(r\lambda_o Id-T)\\ &\geq r|\lambda_o| - (r-\mu_1) = \mu_1\end{aligned}$$

by our first estimate. Since $\mu_1 < \mu$ was arbitrary the second inequality holds.

If a is quasi-continuous, it follows from 3.1. and (5) that QM_aP and PM_aQ are compact and therefore we obtain from 2.1 and lemma 3.3 that

$$\begin{aligned}\inf|a| &= \Gamma(M_a) = \Gamma(PM_aP+QM_aP+PM_aQ+QM_aQ)\\ &= \Gamma(PM_aP + QM_aQ) = \min(\Gamma(PM_{a|H_2}), \Gamma(QM_{a|H_2^-}))\\ &\leq \Gamma(U)\ .\end{aligned}$$

The same argument works for $\Gamma^*(U)$.

<u>3.3. Lemma:</u> If $W: L_2(T) \rightarrow L_2(T)$ has $H_2(T)$ and $H_2^-(T)$ as invariant subspaces then

$$\Delta(W) = \max(\Delta(W_{|H_2}),\ \Delta(W_{|H_2^-}))$$

$$\Gamma(W) = \min(\Gamma(W_{|H_2}),\ \Gamma(W_{|H_2^-}))$$

$$\Gamma^*(W) = \min(\Gamma^*(W_{H_2}), \Gamma^*(W_{|H_2^-}))$$

This can be shown using 2.3 and that $L_2(T) = H_2(T) + H_2^-(T)$ is an orthogonal decomposition.

<u>3.4. Theorem:</u> For a singular integral operator $U = M_c + M_dS$ with quasi-continuous c and d we have

$$\Delta(U) = \max(\sup|c+d|,\ \sup|c-d|)$$

$$\Gamma(U) = \Gamma^*(U) = \min(\inf|c+d|,\ \inf|c-d|)\ .$$

<u>Proof:</u> By (2) $U = M_aP + M_bQ$ where $a = c+d$ and $b = c-d$ are quasi-continuous. By (7) and 3.1 the operators $PM_{b|H_2^-}$ and $QM_{a|H_2}$ are compact and we obtain from (3) and 3.3:

$$\Delta(W) = \max(\Delta(PM_{a|H_2}),\ \Delta(QM_{b|H_2^-}))$$

$$\Gamma(W) = \min(\Gamma(PM_{a|H_2}),\ \Gamma(QM_{b|H_2^-}))$$

By (5) we have $\Delta(QM_{b|H_2^-}) = \Delta(PM_{J(b)|H_2})$ and $\Gamma(QM_{b|H_2^-}) = \Gamma(PM_{J(b)|H_2})$. Now the result follows from 3.2. The same argument works for Γ^*. □

3.2. and 3.3. contain well-known characterizations of the Fredholm property for Toeplitz and singular integral operators: e.g. M_c+M_dS with quasi-continuous c and d defines a Fredholm operator if and only if $\inf|c+d|\cdot|c-d| = \inf|c^2-d^2| > 0$ (see e.g. [10] Kap. III §7, [6] Theorem 7.26,[29]
These results can be generalized using localization techniques (see section 4).
Let us come back to Hankel-operators for a moment. Besides the symbol a and the Hankel-matrix (a_{K+j}) there is a third discription of $V = PJM_{a|H_2}$. If V is a positive operator (in the Hilbert space sense) then there is a positive measure λ on [-1,1] such that $a_n = \hat{a}(-n)$ is the n^{th} moment of λ:

(10) $$a_n = \hat{a}(-n) = \int_{-1}^{1} t^n d\mu(t)$$

Now $\Delta(V)$ can be estimated as follows (compare Th. 1.6. in [25])
3.5. Corollary: If V is positive in $H_2(T)$ then

$$\tfrac{1}{\pi}\Delta(V) \leq \overline{\lim_n}\, n\cdot|a_n| \leq \lim_{t\to 1} \frac{\mu([-t,t]^c)}{1-t} \leq 4\Delta(T)$$

Proof: For $\sum|f_i|^2 \leq 1$, $\sum|g_i|^2 \leq 1$ we have

$$\Big|\sum_{j,K\geq n} a_{K+j} f_K g_j\Big| \leq \sup_{K,j\geq n} (|a_{K+j}|(1+K+j))\Big(\sum_{K,j} \frac{1}{K+j+1}|f_K|\cdot|g_j|\Big)$$

Since the Hilbert matrix $(\frac{1}{K+j+1})$ has norm π and $P_n\hat{V}P_n \triangleq (a_{K+j})_{K,j\geq n}$ it follows that

$$\Delta(V) = \lim_n |P_n\hat{V}P_n| \leq \pi\, \overline{\lim_n}\, n\cdot|a_n|.$$

The remaining inequalities also follow along the lines of the proof of theorem 1.6 in [25] . □

3.6. Example: Let v be a positive and finite measure on $[0,\omega)$ and

$k(x) = \int_0^\infty e^{-xy} dv(y)$ its Laplace transform. It is shown in the proof

of Theorem 2.5 in [25] that a singular integral operator of Carleman type

(11) $$S_k : L_2(0,\infty) \longrightarrow L_2(0,\infty),\quad S_k f(x) = \int_0^\infty k(x+y)f(y)dy$$

is unitary equivalent to the Hankel matrix $(a_{K+j})_{K,j\in\mathbb{Z}^+}$ on $l_2(\mathbb{Z}^+)$ given by the measure

(12) $$\lambda(E) = \frac{1}{4\pi}\int_E (1+t)^2 d\nu\circ\tau^{-1}, \quad \tau(t) = \frac{1-t}{1+t}$$

i.e. $$a_n = \int t^n d\lambda(t)$$

Then $\Delta(S_k) = \Delta((a_{K+j})_{K,j\in\mathbb{Z}^+})$ and by 3.5 and an estimate connecting the vanishing conditions for the measures λ and ν there is a constant $c > 0$ such that

$$\frac{1}{c}\Delta(S_k) \le \overline{\lim_{\substack{x\to 0\\ x\to\infty}}} \frac{\nu([0,x])}{x} \le c\Delta(S_k)$$

4. Operators in $L_p(\mu)$.

In trying to generalize our results from operators in sequence spaces l_p to function spaces $L_p(X,\mu)$ it is natural to look out for forms as of the kind

(1) $$\Delta(T) = \overline{\lim_{\mu(A),\mu(B)\to 0}} \|\chi_A T\chi_B\| ,$$

i.e. one tries to replace the bands $l_p(M)$, $M \subset \mathbb{N}$, by the bands $L_p(A)$, $A \subset X$. But (1) does not hold for arbitrary operators in $L_p(\mu)$, e.g. there are convolution operators $Tf = f*\mu$, μ a 'biased coin' measure, with

$$\Delta(T) > 0 , \quad \overline{\lim_{\mu(A)\to 0}} \|\chi_A T\|_{L_2} = 0 .$$

On the other hand, we will discuss in this section a class of operators for which (1) holds and which is large enough to contain many interesting operators from analysis.

Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be separable measure spaces and μ_o, ν_o are finite measure equivalent with μ and ν respectively. $S,T,\ldots$ denote bounded linear operators from $L_p(X,\mu)$ to $L_p(Y,\nu)$, $1 < p < \infty$.

<u>4.1. Definition:</u> i) $S: L_p(X,\mu) \to L_p(Y,\nu)$ is called <u>almost compact</u>, if for all $\varepsilon > 0$ there are $A \in \mathcal{A}$, $B \in \mathcal{B}$ with $\mu_o(A) \le \varepsilon$, $\nu_o(B) \le \varepsilon$ and such that $T - \chi_A T\chi_B$ is compact.

ii) $T : L_p(X,\mu) \to L_p(Y,\nu)$ is of <u>diagonal type</u> if for all $\varepsilon > 0$ there are $A_1,\ldots,A_n \in \mathcal{A}$, $B_1,\ldots,B_n \in \mathcal{B}$

(2) $$\begin{aligned} &0 < \mu(A_i) < \varepsilon, \ A_i \cap A_j = \emptyset \text{ for } i \ne j, \ \cup A_i = X \\ &0 < \mu(B_i) < \varepsilon, \ B_i \cap B_j = \emptyset \text{ for } i \ne j, \ \cup B_i = Y \end{aligned}$$

such that $T - \sum_{i=1}^{n} \chi_{B_i} T \chi_{A_i}$ is almost compact.

4.2. Examples: a) It was shown in [36] that an integral operator

$$Tf(y) = \int k(y,x)\ f(x)d\mu(x)$$

with an absolute kernel (i.e. $|k(y,x)|$ also defines a L_p-bounded integral operator) is almost compact. So is the singular integral operator $S : L_2(0,\infty) \to L_2(0,\infty)$

$$Sf(y) = \int_0^\infty \frac{1}{x+y}\ f(x)\ dx$$

since $S\chi_{[a,\infty)}$, $\chi_{[a,\infty)}S$, $a > 0$, are compact. On the other hand, S is unitarily equivalent to multiplication by $\pi\ (\cosh(\frac{x\pi}{2}))^{-1}$ on $L_2(\mathbb{R})$. So one should not take the word 'almost' too liberally.

b) T is of diagonal type if

$$Tf(g) = a(y)f(\delta(y)) + \int k(y,x)f(x)d\ (x)$$

where the measurable functions $a : Y \to \mathbb{C}$, $\delta : Y \to X$ and $|k| : Y \times X \to \mathbb{R}^+$ are such that the above operators are L_p-bounded.

c) T is of diagonal type if

$$T = S + \int k(y,x)f(x)d\mu(x)$$

where S is 'local type', i.e. $X = Y$ is a metric space and for disjoint compact subsets $A,B \subset X$ we have that $\chi_B S\chi_A$ is compact. Typical examples of local type are singular integral operators and pseudo-differential operators with a smooth symbol (see e.g. Sec.5)

Proof: a) See [36] and [25], Theorem 2.6.

b) Since for $B = \delta^{-1}(A)$, $A \in \mathcal{A}$, we have $\nu(B) = \int \chi_{\delta^{-1}(A)} d\nu = \int \chi_A (\frac{d\nu\circ\delta^{-1}}{d\mu}) d\mu$ it is possible to choose $A_n \subset X$ and $B_n = \delta^{-1}(A_n)$ such that (2) is satisfied.

c) For $A_1,\ldots,A_n \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$ we have

$$S = \sum_{i=1}^{n} \chi_{A_i} S\chi_{A_i} + \sum_{\substack{i,j=1 \\ i\ j}}^{n} \chi_{A_i} S\chi_{A_j} .$$

The second sum is an almost compact operator since S is of local type and μ_o, ν_o are regular measures.

4.3. Theorem: Let $T : L_p(X,\mu) \to L_p(Y,\nu)$ be of diagonal type. Then we have that

$$\Delta(T) = \Delta^*(T) = \overline{\lim_{\substack{\mu_o(A)\to o \\ \nu_o(B)\to o}}} \|\chi_B T\chi_A\|$$

$$= \sup\{\overline{\lim}\|Tf_n\| : \|f_n\| = 1 \text{ and } f_n \text{ pairwise disjoint}\}$$

$$= \inf\{\|T-S\| : S \text{ compact}\}.$$

$$\Gamma(T) = \lim_{\mu_o(A)\to o} m(T\chi_A) = \sup\{m(T-S) : S \text{ compact}\}$$

$$\Gamma^*(T) = \lim_{\nu_o(A)\to o} q(\chi_A T) = \sup\{q(T-S) : S \text{ compact}\}$$

Using some information about subspaces and basic sequences of L_p one can reduce this result to 2.2. Instead of going into the details of the proof we discuss some examples which illustrate our ealier remark, that formulas as given in 4.3. reduce the estimation of Δ to norm estimates for operators which have a form similar to the original T.

4.4. Examples: For a composition operator

$$Tf(y) = a(y)f(\delta(y))$$

we define $s(x) = E(|a|^p|\delta)(x)\cdot \frac{d\nu\cdot\delta^{-1}}{d\mu}(x)$ where $E(|a|^p|\delta)$ is the conditional expectation of $|a|^p$ given δ and $\frac{d\nu\cdot\delta^{-1}}{d\mu}$ denotes the Radon Nikodym derivatives of the image measure $\nu\circ\delta^{-1}$ with respect to μ. Then

$$\Delta(T) = \|T\| = \sup|s|$$
$$\Gamma(T) = \alpha(T) = \inf|s|$$

Proof: For $f \in L\ (X,\mu)$ we have

$$\|Tf\|^p = \int|a(y)|^p|f(\delta(y)|^p d\nu(y)$$

$$= \int E(\ a|^p \mid \delta)(x)|f(x)|^p \frac{d\nu\circ\delta^{-1}}{d\mu}(x)d\mu(x)$$

Now the estimates are the same as for multiplication operators.

4.5. Example. For $f \in L_p(0,1)$ define

(3) $$Tf(y) = u(y)\int_0^y v(x)f(x)dx\ ,\ y \in (0,1)$$

Here we assume that for all $C \in (0,1)$, $\frac{1}{p}+\frac{1}{q} = 1$, we have

(4) $$u \in L_p(C,1),\ \mu(\mathrm{supp}(u)\cap(C,1]) > 0$$
$$v \in L_q(0,C),\ \mu(\mathrm{supp}(v)\cap(0,C)) > 0$$

Put $K(a,b) = \sup_{a<y<b} (\int_a^y|v(x)|^q dx)^{1/q}\cdot(\int_y^b|u(x)|^p dx)^{1/p}$

Tomaselli has shown, that T is a bounded operator in $L_p(0,1)$ if and only if $K(0,1)$ is finite and

(5) $$K(0,1) \le \|T\| \le p^{1/p}\cdot q^{1/q}\cdot K(0,1)\ .$$

In particular, if $u \in L_p(0,1)$ and $v \in L_q(0,1)$ then

$$\|\chi_B T \chi_A\| \le D \cdot (\int_A |v(x)|^q dx)^{1/q} (\int_B |u(x)|^p dx)^{1/p}$$

$$\longrightarrow 0 \quad \text{for } \mu(A),\ \mu(B) \longrightarrow 0$$

i.e. T is compact in this case. In order to estimate $\Delta(T)$ we conclude from the special form of T that for all $0 < a < b < 1$

$$T = \chi_{[a,b]^c} T \chi_{[a,b]^c} + \chi_{[b,1]} T \chi_{[a,b]} + \chi_{[a,b]} T \chi_{[0,b]}$$

The last two operators are again of the form (3) but with u and v replaced by restrictions of these functions that belong to $L_p(0,1)$ and $L_q(0,1)$ respectively. Therefore these operators are compact and we have

$$\Delta(T) = \Delta(\chi_{[a,b]^c} T \chi_{[a,b]^c}) \le \| \chi_{[a,b]^c} T \chi_{[a,b]^d} \|$$

$$\le p^{1/p} \cdot q^{1/q} \max(K(0,a), K(b,1))$$

by (5). Hence we obtain from 5.3. and (5)

$$K \le \Delta(T) \le p^{1/p} q^{1/q} \cdot K$$

where $$K = \max\{\lim_{a\to 0} K(0,a),\ \lim_{b\to 1} K(b,1)\} .$$

In [15] Juberg and Stuart have estimated measures of non-compactness for integral operators of form (3). It will follow from section 7 that (6) is an improvement of their results. □

For an operator $U : L_p(T) \to L_p(T)$ of local type Δ and Γ can be expressed by the 'localized' quantities

$$\delta(U,x) = \inf\{\|U\chi_{K(x,\varepsilon)}\| : \varepsilon > 0\}$$
$$\gamma(U,x) = \sup\{\alpha(U\chi_{K(x,\varepsilon)}) : \varepsilon > 0\}$$

where $K(x,\varepsilon) = \{y \in T : |x-y| < \varepsilon\}$. Such quantities were studied in [1]; up to a factor 2 the following result was already given in [31].

<u>4.6. Corollary:</u> For an operator $U : L_p(T) \to L_p(T)$ of local type we have

$$\Delta(U) = \sup_{x\in T} \delta(U,x)$$

$$\Gamma(U) = \inf_{x\in T} \gamma(U,x) \quad .$$

This local approach may be used to refine the results of section 3:

4.7. Example: Consider a singular integral operator

$$U = aP + Q : L_2(T) \to L_2(T)$$

as defined in Section 3. If a is piecewise continuous, then

$$\Gamma(U) = \inf|a^{\#}|$$

where $a^{\#} : T \times [0,1] \to \mathbb{C}$ is defined by

$$a^{\#}(t,\mu) = a(t-0)\mu + a(t+0)(1-\mu)$$

i.e. in the jumping points of a we add vertical lines to the graph of a.

Proof: For a continuity point x of a one can show easily

$$\gamma(U,x) = |a(x)| = |a^{\#}(x,\mu)| \quad \text{for all } 0 \le \mu \le 1 .$$

If $a(x-0) \neq a(x+0)$ define $V = bP + Q$ by a function b which equals a on $K(x,\varepsilon)$ and is continuous outside $K(x,\varepsilon)$ such that

$$\overline{co}R(b)) \subset \overline{co}(\{a(y) : y \in \mathring{K}(x,\varepsilon)\}) =: D$$

Applying 3.2. to V we obtain

$$\Gamma(U\chi_{K(x,\varepsilon)}) \ge \inf\{|\lambda| : \lambda \in D\}$$

$$\longrightarrow \inf\{ a^{\#}(x,\mu) : 0 \le \mu \le 1\} \text{ for } \varepsilon \to 0 .$$

On the other hand, one always has by [10] IX 2.5

$$\Gamma(U) \le \inf\{|\lambda| : \lambda \in \sigma_e(U)\} = \inf|a^{\#}| .$$

One can estimate $\Delta(T)$ and $\Gamma(T)$ also for more general classes of singular integral operators in L_p-spaces. But these cases are more complicated and they will appear elsewhere together with the proofs of 4.3. and 4.6.

5. Pseudo-Differential operators.

First we will discuss an algebra of pseudo-differential operators introduced by Herman and Cordes in [3]; later on we consider operators of the Hörmander class $S^m_{\rho,g}$ on $\mathbb{R}^n$.

5.1. The Laplace Comparison algebra (cf. [3],[4]). Denote by A the C^*-subalgebra of $B(L_2(\mathbb{R}^n))$ generated by all operators of the form $a(M)$ and $b(D)$. Here $a(M)$ denotes the multiplication operator

$$a(M)g(x) = a(x)g(x) \quad , \; g \in L_2(\mathbb{R}^n)$$

where a is a bounded, continuous function on $\mathbb{R}^n$ whose 'local oscillation' vanishes at ∞, i.e.

(1) $$\max\{|a(x+h) - a(x)| : h \in \mathbb{R}^n, |h| \le 1\} \to 0$$

for $|x| \to \infty$. On the other hand, $b(D)$ stands for the formal Fourier multiplier

$$b(D)g = F^{-1}(b\cdot F(g)) \quad , \; g \in L_2(\mathbb{R}^n)$$

where b is a bounded, continuous function on $\mathbb{R}^n$ such that the limits $\lim_{\rho\to\infty} b(\rho z)$ exist uniformly for $z \in \mathbb{R}^n$, $|z| = 1$.

The algebra A contains for example the operator

$$(I-\Delta)^{-1}$$

where Δ is the Laplace operator on $\mathbb{R}^n$, and also the ideal of compact operators on $L_2(\mathbb{R}^n)$. Furthermore, the quotient A/K is a commutative C*-algebra with a maximal ideal space M. Now the symbol $\sigma(T)$ of an operator $T \in A$ is defined as the Gelfand transform of the equivalence class $\hat{T} \in A/K$. In particular, $\sigma(T) \in C(M)$ and the symbol is invariant under compact perturbations. It follows directly from this definition of the symbol, some standard properties of the Gelfand homomorphism and Corollary 2.5. that for every $T \in A$ we have

$$\Delta(T) = \Delta^*(T) = \max|\sigma|$$
$$\Gamma(T) = \Gamma^*(T) = \min|\sigma| .$$

There is a rather concrete discription of the maximal ideal space M in [4], Chap. IV, Theorem 1.6, from which it follows for example that for an operator T of the form

$$T = \sum_{i=1}^{n} a_i(M)b_i(D) \in A$$

we have

$$\Delta(T) = \overline{\lim_{|x|+|\zeta|\to\infty}} \sum_1^n a_i(x)b_i(\zeta)$$

(2)
$$\Gamma(T) = \lim_{|x|+|\zeta|\to\infty} \sum_1^n a_i(x)b_i(\zeta) .$$

Operators in the Sobolev spaces H_s,

(3) $$H_s = \{u \in S' : \|u\|_s^2 = \int |\hat{u}(\zeta)|^2(1+|\zeta|^2)d\zeta < \infty\}$$

may be treated in the same way using the isometries

(4) $$\Lambda^t = b(D) : H_s \to H_{s+2t} , \; b(\zeta) = (1+|\zeta|^2)^{-t}$$

(see [4] Sec. 8,9). We illustrate this by an example.

<u>5.2. Example:</u> Let K be the partial differential operator

$$K = \sum_{|\alpha|\le N} a_\alpha D^\alpha$$

with coefficients a_α that satisfy (1). $K : H_s \to H_{s-N}$ is bounded and by (4) we have

$$\Delta(K : H_s \to H_{s-N}) = \Delta(T : H_o \to H_o)$$
$$\Gamma(K : H_s \to H_{s-N}) = \Gamma(T : H_o \to H_o)$$
where $T = \Lambda^{-\frac{s-N}{2}} K \Lambda^{s/2}$.

But T belongs to A and has the form

$$T = \sum_{|\alpha|\le N} a_\alpha(M)b_\alpha(D) \quad , \quad b_\alpha = (1+|\zeta|^2)^{-N/2}\zeta^\alpha .$$

Since $|b_\alpha(\zeta)| \to 0$ for $|\alpha|<N$ and $|\zeta|\to\infty$ we get from (2):

$$\Delta(T) = \max\{ \sup_{x\in\mathbb{R}^n, \zeta\in S^{n-1}} |\sum_{|\alpha|=N} a_\alpha(x)\zeta^\alpha|, \sup_{\zeta\in\mathbb{R}^n} \overline{\lim_{|x|\to\infty}} |\sum_{|\alpha|\le N} a_\alpha(x)b_\alpha(\zeta)|\}$$

$$\Gamma(T) = \min\{ \inf_{x\in\mathbb{R}^n, \zeta\in S^{n-1}} |\sum_{|\alpha|=N} a_\alpha(x)\zeta^\alpha|, \inf_{\zeta\in\mathbb{R}^n} \underline{\lim_{|x|\to\infty}} |\sum_{|\alpha|\le N} a_\alpha(x)b_\alpha(\zeta)|\}.$$

This gives a Fredholm criterion for K which corresponds to theorem 33 in [3].

Now we turn to classes of pseudo-differential operators defined by Fourier integrals.

<u>5.3. Symbols of class</u> $S^m_{\rho,\delta}$. We say that a C^∞-function $p(x,\zeta)$ defined on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\zeta$ is a symbol of class $S^m_{\rho,\delta}$, $-\infty < m < \infty$, $0 \le \delta < \rho \le 1$, if for any multi-index α,β there is a bounded function $C_{\alpha,\beta}(x)$ such that

(5) $$|\partial^\alpha_\zeta \partial x^\beta p(x,\zeta)| \le C_{\alpha,\beta}(x)(1+|\zeta|)^{m+\delta|\beta|-\rho|\alpha|}$$

for all $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^n$. For a rapidly decreasing function $u \in S$ we define

(6) $$Pu(x) = \iint e^{i(x-x')} p(x,\zeta)u(x')dx'd\zeta .$$

P extends to bounded linear operator $P : H_{s+m} \to H_s$ (see e.g. [18] Chap. 3, Theorem 3.7). There are some estimates for the distance of T to the class of compact operators from H_{s+m} to H_s due to Hörmander [13] and some sufficient conditions for the Fredholm property of T due to Grushin ([12]) and Kumano-go ([18]). Their methods (e.g. Garding's inequality and special parametrices) lead to the following estimates:

<u>5.4. Proposition.</u> Let p be a symbol of class $S^m_{\rho,\delta}$ and assume that its derivatives are slowly varying in the sense that in addition to (5) we have:

(7) $$|C_{\alpha,\beta}(x)| \longrightarrow 0 \text{ if } |x| \to \infty \quad \text{for all } \alpha \text{ and } \beta \ne 0 .$$

Then for $P : H_{s+m} \to H_s$ defined by (6) we have

$$\Delta(P) \ge \overline{\lim_{|x|+|\zeta|\to\infty}} |p(x,\zeta)(1+|\zeta|^2)^{-m/2}| .$$

If $p(x,\zeta) = 0$ for all x outside some compact set, then we have equality.

§6 Operators in $L_1(\mu)$ and $C(K)$.

In the situations we considered so far the class of strictly (co-) singular operators coincides with the ideal of compact operators. In $L_1(\mu)$ and $C(K)$ they form a larger class and - this is more important for us - they interact better with the Banach space structure of $L_1(\mu)$ and $C(K)$ than compact operators do. Pelczynski [23] has shown for example that an operator in $L_1(\mu)$ is strictly singular if and only if it is weakly compact, i.e.

(1) $$\Delta(T) = 0 \quad \text{iff} \quad \overline{\lim_{\nu_0(A)\to 0}} \|\chi_A T\| = 0$$ [1)]

The latter conditions means that $T(U_{L_1})$ is equi-integrable. As a generalization of (1) we get the following formula for Δ.

6.1. Theorem ([38]): For an arbitrary bounded linear operator $T : L_1(X,\mu) \to L_1(Y,\nu)$ we have

$$\Delta(T) = \Delta^*(T) = \overline{\lim_{\nu_0(A)\to 0}} \|\chi_A T\|$$

$$= \inf\{\|T-S\| : S \text{ weakly compact}\} .$$

This formula has again the useful properties we discussed after 2.2. (just replace the bands $l_p(M), M \subset \mathbb{N}$, by $L_1(B), B \subset Y$, and compact operators by weakly compact operators.) E.g. we obtain a formula for the essential spectral radius which applies to a perturbation argument in transport theory (see [37]). In [37] we also discribe a connection between the formula $\Delta(T^n)^{1/n} \longrightarrow r_e(T)$ and the so called Doeblin-condition for Markov processes.

6.2. Examples ([38]): a) Let $U : L_1(X,\mu) \longrightarrow L_1(Y,\nu)$

(2) $$U(f)(y) = \int k(y,x) f(x) d\mu(x)$$

be an integral operator defined by a measurable kernel k. Then

$$\Delta(U) = \inf_n \sup_x \int_{|k|>n} |k(y,x)| d\nu(y)$$

b) Let $V : L_1(X,\mu) \to L_1(Y,\nu)$ be defined by

(3) $$Vf(y) = \int f d\mu_y \qquad \nu\text{-a.e.}$$

where $(\mu_y)_{y\in Y}$ is a stochastic kernel of μ-singular measures on (X,A). Then

$$\Delta(V) = \|V\| = \sup_y \|\mu_y\|$$

where $\|\mu_y\|$ is the total variation of μ_y.

---

1) We continue with the notation of section 4.

Since an arbitrary operator $V : L_1(X,\mu) \longrightarrow L_1(Y,\nu)$ can be written as a sum $T = U + V$ where U and V have the form (2) and (3) resp. (see[37]), the above examples already cover the general case.

For operators in C(X) we get results which are essentially dual to our results in the L_1-case.

6.3. Satz: For compact spaces X and Y and a bounded linear operator $T : C(X) \longrightarrow C(Y)$ we have:

$$\Delta(T) = \sup\{\overline{\lim_n} \|T\chi_{A_n}\| : A_n \subset X \text{ open and } A_n \cap A_m = \emptyset \text{ for } n \neq m\} .$$

If X is extremely disconnected or Y metric then

$$\Delta^*(T) = \Delta(T)$$

and if Y is extremely disconnected we also have

$$\Delta(T) = \inf\{\|T-S\| : S \text{ weakly compact}\}$$

One can also 'dualize' the formulas given in 6.2. But it can happen that $\Delta^*(T) = 0 < \Delta(T)$ (e.q. choose $C(X) \approx c_0$, $C(Y) \approx l_\infty$ and T as an isomorphic embedding. Then $\Delta(T^*) = 0$ since by a theorem of Rosenthal [26] every separable quotient of l^∞ is reflexive).

6.4. Remark. So far we can estimate $\Gamma(T)$ and $\Gamma^*(T)$ only for special operators in L_1 and C(X). For example, if T has the form

$$Tf(y) = \int k(y,x)f(x)d\mu(x) + \sum_{n=1}^{\infty} a_n(y)f(\sigma_n(y))$$

then for $T : L_1(X,\mu) \rightarrow L_1(Y,\nu)$ one can show that

$$\Gamma(T) = \underline{\lim}_{\mu(A)\to 0} m(T\chi_A)$$

and for $T : C(X) \longrightarrow C(Y)$ we have

$$\Gamma^*(T) = \inf\{\underline{\lim}_n \alpha(\chi_{A_n}T) : A_n \subset X \text{ offen, } A_n \cap A_m \neq \emptyset \text{ for } n \neq n\}$$

I don't know at this point if these formulas hold for general bounded linear operators in $L_1(\mu)$ or C(X).

7. Comparison with measures of non-compactness.

There is an abundance of measures of non-compactness and quantities similar to Δ and Γ in the literature (see [19],[28],[30],[39] and their references. We have concentrated here on Δ and Γ since these formulas seem to be well suited for the application of Banach space methods also in $L_1(\mu)$- and C(K)-spaces. But our methods also give that in many situations of interest in applications Δ and Γ actually coincide with various other notions. Here are some examples.

a) In [30] and [39] it was shown that the quantities

$$\tau(T) = \sup\{m(T_{|M}) : \dim M = \infty\}$$
$$\tau^*(T) = \sup\{q(\Omega_M T) : \operatorname{codim} M = \infty\}$$

have properties similar to Δ and Δ^*. This is no coincidence. In a joint work with O. Beucher we shall show that

$$\Delta(T) = \tau(T) \quad , \quad \Delta^*(T) = \tau^*(T)$$

for arbitrary bounded linear operators in a large class of Banach-spaces including all $L_p(\mu)$-spaces, $1 \le p < \infty$, Lorentz function spaces, certain Orlicz function spaces, Hardy-spaces and Sobolev-spaces.

b) In all situations considered in sections 2,3,4 and 5 the popular measures of non-compactness coincide with Δ and Γ. More precisely, let us call <u>'measure of non-compactness'</u> any function

$$\alpha : C \longrightarrow \mathbb{R}^+ ,$$

where C is the class of absolutely convex, bounded subsets of the underlying Banach-space X, with the following properties

i) If $A,B \in C$ and A is relatively compact then $\alpha(A+B) = \alpha(B)$

ii) If $A,B \in C$ and $A \subset B$ then $\alpha(A) \le \alpha(B)$.

iii) If $A \in C$ and $r \in \mathbb{R}^+$ then $\alpha(rA) = r\alpha(A)$.

iv) There is a constant C such that $\alpha(U_M) = C$ for the unit ball U_M of every infinite dimensional subspace M of X.

For example the well-known Kuratowski and Hausdorff-measures of non-compactness have these properties, see [28] §1.2.

For X and $T \in B(X)$ as in section 2 to 5 we then have

$$\alpha(TU_X) = C\Delta(T) .$$

Indeed it is not difficult to see that

$$C\cdot\tau(T) \le \alpha(TU_X) \le C \inf\{\|T-K\| : K \text{ comp.}\}$$

and then equality follows directly from our earlier results.

This observation does not extend to $X = L_1(\mu)$ or $X = C(K)$. There are integral operators T in C[0,1] and $L_1[0,1]$ which are weakly compact but not compact and therefore satisfy

$0 = \Delta(T) < q(TU_X)$

c) Different kinds of measures of non-compactness were studied in [30] and [39]:

$$c(T) = \inf\{\|T|_M\| : \operatorname{codim} M < \infty\}$$
$$k(T) = \inf\{\|\Omega_M T\| : \dim M = \infty\}$$

Since $\Delta(T) \le c(T) \le \|T+K\|$, $\Delta^*(T) \le k(T) \le \|T+K\|$ for all compact K, we have

$$\Delta(T) = c(T) = k(T) = \Delta^*(T)$$

for all situation considered in sections 2 to 5 but not for

$X = L_1(\mu)$ or $X = C(K)$.

In [30] and [39], some relatives of Γ and Γ^* were considered:

$$B(T) = \sup\{m(T|_M) : \text{codim } M < \infty\}$$
$$M(T) = \sup\{q(Q_M T) : \dim M = \infty\}$$

It is shown in these papers that

$$m_\infty(T) := \sup\{m(T-K) : K \text{ finite dim.}\} \le B(T) \le \Gamma(T)$$
$$q_\infty(T) := \sup\{q(T-K) : K \text{ finite dim.}\} \le M(T) \le \Gamma^*(T) .$$

Since compact operators in $L_p(\mu)$ can be approximated by finite dimensional operators we have equality in these inequalities in all situations considered in sections 2 to 5. A similar remark applies to the characteristic $\gamma_\infty(T)$ considered in [39] .

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