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SOME OLD AND NEW APPLICATIONS OF CHOQUET THEORY IN POTENTIAL THEORY

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Potential theory has a long and interesting history. It is well known that it influenced many mathematical disciplines but there is also a kind of a feedback. Many branches of mathematical analysis and other fields brought an essential contribution to the development of modern potential theory.

One can easily follow this idea considering the interplay between the Choquet theory and potential theory. The origins of the Choquet theory may be traced to the theory of capacities. Also the latest results show its important role in modern potential theory.

Apart from some minor changes, this is a text of a lecture delivered at the Winterschool Srní, 1984.

1. Preliminaries. Studying concave functions on a convex compact subset of a locally convex space, G. Choquet in (1956) started his celebrated theory. Among list of papers of that period we mention here at least G. Choquet (1960), H. Bauer (1958, 1960, 1961), E. Bishop and K. de Leeuw (1959), D. A. Edwards (1963), D. Sibony (1968), N. Boboc and A. Cornea (1965, 1967).

Let $\mathcal{S}$ be a convex cone of (lower finite) lower semicontinuous functions on a compact Hausdorff space Y satisfying the following two conditions:

- (S_1) $\mathcal{S}$ is linearly separating (i.e. for each couple $x, y \in Y$, $x \neq y$ and for each $\lambda \geq 0$ there is $f \in \mathcal{S}$ such that $f(x) \neq \lambda f(y)$),
- (S_2) the set $\mathcal{S}^+$ of all nonnegative elements of $\mathcal{S}$ contains a finite and positive function.

(A more general approach considering only locally compact spaces or non-convex cones can be also treated.)

An ordering on the set $\mathcal{M}^+(Y)$ of all positive Radon measu-

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This paper is in final form and no version of it will be submitted for publication elsewhere.

res on Y is introduced in the following way:

$$\mu \prec \nu \text{ if } \mu s \leqq \nu s \text{ for every } s \in \mathcal{S}.$$

The measures of $\mathcal{M}_x(\mathcal{S}) := \{\mu \in \mathcal{M}^+(Y): \mu \prec \varepsilon_x\}$ (ε_x is the Dirac measure at x) are called representing measures for x. Define the Choquet boundary of $\mathcal{S}$ as the set of those $x \in Y$ for which the Dirac measure is the only representing measure for x.

By Zorn's lemma, there is always a minimal element of $\mathcal{M}_x(\mathcal{S})$. We say that $\mathcal{S}$ is simplicial if for each $x \in Y$ there is unique minimal representing measure.

2. Regular points and the simpliciality of $S(U)$. In what follows, we shall consider a $\mathfrak{P}$-harmonic space X with a countable base in the sense of the axiomatics of C. Constantinescu and A. Cornea. For undefined terminology we refer the reader to their monograph (1972).

Let U be a (relatively compact) open subset of X. If $S = S(U)$ denotes the set of all continuous functions on $\overline{U}$ which are superharmonic on U, then S is a convex cone on $\overline{U}$ and it is almost obvious that $Ch_S(\overline{U}) \subset U_{reg}$ where U_{reg} stands for the set of all regular points of U.

In the case of classical harmonic functions given by Laplace equation, $U_{reg} \subset Ch_S(\overline{U})$ in view of the Keldysh lemma (see M. Brelot (1960), H. Bauer (1961)). The same inclusion holds in the axiomatic potential theory if the additional domination axiom D is supposed (N. Boboc and A. Cornea (1965), (1967)). In (1967), J. Köhn and M. Sieveking gave a simple example of an open set U in a harmonic space corresponding to the heat equation for which $Ch_S(\overline{U}) \neq U_{reg}$. Finally, J. Bliedtner and W. Hansen (1974), (1975) showed that $Ch_S(\overline{U}) = U_{reg}$ provided the set of all irregular points of U is negligible (in the sense that it is of harmonic measure zero at each point of U).

To derive this result, J. Bliedtner and W. Hansen use the simpliciality of the cone $S(U)$ in an essential way. The history of discovery of the simpliciality which seems to be one of the most important results of the potential theory in the last decade is also interesting. If the domination axiom D holds in a harmonic space, then $S(U)$ is simplicial (N. Boboc and A. Cornea (1967); using the method of affine dilations E. G. Effros and J. L. Kazdan (1970)). Remark that in this case the harmonic measure is always a minimal representing measure for $S(U)$. The case of the heat equation was settled by E. G. Effros and J. L. Kazdan (1971) for

U with "smooth boundary" and by P. D. Taylor (1972) for other examples of domains U. Finally, J. Bliedtner and W. Hansen (1975) established the simpliciality in the general case. They use the so-called essential base also for the characterization of the Choquet boundary of $S(U)$. Another idea of the proof consisting in the solution of the weak Dirichlet problem for compact subsets of the Choquet boundary has recently appeared in J. Bliedtner and W. Hansen (*).

3. Semiregular points and the simpliciality of $H^*(U)$. The notion of a semiregular set goes back to H. Bauer (1962): A set U is termed semiregular if the Perron solution can be extended continuously to $\bar{U}$ for any continuous boundary function $f \in \mathcal{C}(\partial U)$. It is not difficult to realize that the set of all irregular points of a semiregular set U must be open and negligible in its boundary ∂U. J. Král (1972) raised essentially the question of whether it is possible to characterize semiregular sets by this property. The answer is affirmative as was shown by I. Netuka (1973) in the presence of the axiom of polarity, and by J. Lukeš (1975) using simpliciality of $S(U)$ and a characterization of $Ch_S(\bar{U})$, or by J. Bliedtner and W. Hansen (1975).

Denote by $H^*(U)$ the set of all lower semicontinuous functions v on $\bar{U}$ hyperharmonic on U and fulfilling

$\liminf_{U \ni x \to z} v(x) = v(z)$ for every $z \in \partial U$. J. Bliedtner and W. Hansen (1976) have shown that $H^*(U)$ is simplicial and they characterized the Choquet boundary of $H^*(U)$.

The local version of the above introduced notion of a semiregular set leads to the following definition: An irregular point $z \in \partial U$ is termed semiregular if $\lim_{U \ni x \to z} H_f^U(x)$ exists for every continuous function f on ∂U. This notion was introduced by J. Lukeš and J. Malý (1981) where the next key theorem was proved.

THEOREM A. A point $z \in \partial U$ is not semiregular if and only if there is a sequence $x_n \in U$ such that $f(z) = \lim H_f^U(x_n)$ for each $f \in \mathcal{C}(\partial U)$.

Some conclusions from the above theorem can be derived. We mention the following one.

THEOREM B. The Choquet boundary $Ch_{H^*(U)}(\overline{U})$ consists of all non-semiregular points of U.

4. Fine potential theory. In the forties, H. Cartan defined the fine topology as a topological interpretation of the notion of thinness introduced by M. Brelot. This one is the coarsest topology making all superharmonic functions continuous. Since that time various nice problems concerning the fine topology have been studied by many authors. It was M. Brelot himself who used methods of the fine topology for the study of Dirichlet problem. Nevertheless, fundamental contributions to the fine potential theory are due to B. Fuglede whose theory in the framework of harmonic spaces fulfilling the domination axiom D is collected in (1972). There is a number of recent papers investigating and applying fine topology methods. Remark at least the theory of finely holomorphic functions, examination of probabilistic aspects of the fine topology, or a large part of the whole theory in the framework of standard H-cones of N. Boboc, Gh. Bucur and A. Cornea (1981).

We now focus on the notion of fine hyperharmonicity which, perhaps, plays a central role in the fine potential theory. If f is a (lower finite) lower semicontinuous function on an open subset U of a harmonic space, then the following assertions are equivalent:

(A_1) if $x \in V \subset \overline{V} \subset U$ (V open), then
(*) $$f(x) \geq \int f \, d\varepsilon_x^{CV}$$
(ε_x^{CV} stands for the balayaged measure),

(A_2) there is a base $\mathcal{B}$ of the topology on U such that (*) holds for every $x \in V \in \mathcal{B}$, $\overline{V} \subset U$,

(A_3) for each $x \in U$ there is a local base $\mathcal{V}_x$ of neighborhoods of x such that (*) holds for every $V \in \mathcal{V}_x$, $\overline{V} \subset U$.

This equivalence is no longer true for the case of fine topology. Guided by (A_1), (A_2) and (A_3), we consider its analogue in a setting of the fine topology. By replacing all topological notions by the fine ones with the exception of " $\overline{V} \subset U$ " and by adding a condition of certain "lower boundedness" of f, we define the families $\mathcal{H}^*(U)$, $\mathcal{H}_F^*(U)$ and $\mathcal{H}_{loc}^*(U)$ of "finely hyperharmonic functions" as those functions satisfying (A_1), (A_2) and (A_3), respectively (compare with J. Lukeš and J. Malý (1982)). Obviously, $\mathcal{H}_{loc}^*(U) \supset \mathcal{H}_F^*(U) \supset \mathcal{H}^*(U)$. With these preliminaries we may state now the main theorems.

THEOREM C (B. Fuglede (1972)). $\mathcal{H}_F^*(U) = \mathcal{H}^*(U)$ provided the domination axiom D holds.

THEOREM D (T. Lyons (1982)). $\mathcal{H}_{loc}^*(U) = \mathcal{H}_F^*(U)$ provided the domination axiom D holds.

Proof of this assertion uses the Choquet theory and it is based on the Bliedtner and Hansen's simpliciality theorem together with the following Brelot's result (1967): If f is a finely lower semicontinuous function on a finely open set U, then for each $x \in U$ there is its compact fine neighborhood $K \subset U$ such that f/K is lower semicontinuous (axiom D is supposed).

THEOREM E (J. Lukeš and J. Malý (1982)). In any harmonic space, $\mathcal{H}_{loc}^*(U) = \mathcal{H}_F^*(U)$.

THEOREM F (J. Malý (1980), L. Stoica (1983)). If $f \in \mathcal{H}_{loc}^*(U)$ is lower semicontinuous, then $f \in \mathcal{H}^*(U)$.

Proof (J. Malý (1980), unpublished). We merely outline the main steps and invite the reader to fill in the details: Given $x \in V \subset \overline{V} \subset U$, V finely open, there is $W(x,V)$ such that $x \in W(x,V)$ and $f(x) \geq \int^* f \, d\varepsilon_x^{CW(x,V)}$. Fix a relatively compact finely open set $T \subset \overline{T} \subset U$. Denote

$$\mathcal{W} := \{s: \ s \text{ is lower semicontinuous on } \overline{T} \text{ and } s(x) \geq \int s \, d\varepsilon_x^{CV} \text{ provided } x \in V \subset \overline{V} \subset T\}.$$

Then $Ch_{\mathcal{W}}(\overline{T}) \subset \overline{T} \setminus T$. Using Bauer's minimum principle ($s \in \mathcal{W}$, $s \geq 0$ on $Ch_{\mathcal{W}}(\overline{T}) \Rightarrow s \geq 0$) and a characterization of the balayaged measures due to J. Bliedtner and W. Hansen (1978) we obtain

$$\int f \, d\varepsilon_x^{CT} \leq \sup \{t(x): \ t \in -(\mathcal{W} \cap \mathcal{H}^*(T)), \ t \leq f \text{ on } \overline{T} \setminus T\} \leq f(x).$$

Remark. Another proof of Theorem F based on a minimum principle for functions of $\mathcal{H}_{loc}^*(U)$ appeared in a preprint to lectures at Erlangen and Eichstätt Universities in April 1982. Its enlarged version will appear in a forthcoming publication J. Lukeš, J. Malý and L. Zajíček, Fine topology methods in real analysis and potential theory.

To have a good theory of finely hyperharmonic and, of course, finely harmonic functions we need the "fine minimum principle" for those functions. Assuming the domination axiom D, the quasi-Lindelöf property of the fine topology yields this principle (B. Fuglede (1972)). Another method of the proof of the fine minimum principle is based in a general context of abstract harmonic spaces on the Luzin-Menshov property of the fine topology (cf. J. Lukeš (1977)). In the above mentioned publication, a fine potential theory without the axiom D is developed making an essential use of this property.

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