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The basic notion of this paper is the notion of a twisted sum. A topological vector space (tvs) X is called a <u>twisted sum</u> for a pair of tvs Y, Z if X contains an isomorphic copy X_1 of Y such that $X/X_1 \simeq Z$. In terms of diagrams it may be expressed equivalently (and more precisely) as follows: diagram of tvs and continuous relatively open linear mappings (homomorphisms) of the form

$$(*) \qquad 0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0$$

is said to be a <u>twisted sum</u> of Y and Z if it is a short exact sequence (i.e., the image of every map is equal to the kernel of the next map).

There are two main problems in the theory of twisted sums which we study in the present paper. The first one is the <u>three space problem</u> for local convexity, i.e., the question for what pairs of tvs all of their twisted sums are locally convex. It is known (and proved independently in [7], [14] and [15]) that there is a nonlocally convex twisted sum of the one-dimensional space K and the Banach space l_1.

We say that a locally convex space (lcs) Z is a <u>TSC-space</u> (Twisted Sum Convex, comp. [4]) if, for every lcs Y, every twisted sum of Y and Z is locally convex. We study this class of lcs in our paper in detail (characterizations, permanence properties). This class is closely related to the second main problem which we examine: the <u>problem of splitting</u>.

We say that the twisted sum (*) <u>splits</u> if the space j(Y) is complemented in X, in fact then (*) is a direct sum of Y and Z. A tvs Z belongs to the <u>class S(Y)</u> if every twisted sum of Y and Z splits.

In Section 2 we characterize TSC-spaces as lcs Z such that every twisted sum of $l_\infty(A)$ and Z splits (i.e., $Z \in S(l_\infty(A))$) for every set A.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In Section 4 we consider the open problem whether every locally convex $\mathcal{K}$-space Z (i.e., such that $Z \in S(K)$, $\dim K = 1$, see [9]) is necessarily a TSC-space (the converse is trivial). We give some partial answers; for example, the answer is positive for arbitrary products of countable direct sums of metrizable lcs. The first partial answer was given by S. Dierolf [3], Theorem 2.4.1 and by N. J. Kalton [7], Theorem 4.10.

In Section 3 we study permanence properties of the classes of all TSC-spaces, $\mathcal{K}$-spaces, and $S(Y)$. The table given there contains all such properties known to the author.

Section 1 is of technical nature. It contains a study of the so-called quasilinear technique - a general method of constructing all twisted sums for the given pair of tvs Y and Z. This method is used throughout the paper. The first general description of the technique was given in [4], Sections 2 and 3, but the present description includes some new facts and improvements, including some better proofs. Those proofs which have not been changed, are omitted. The applications of this technique given in this paper and in [4] fully motivate an extensive study of it.

The quasilinear technique for constructing twisted sums of two locally bounded tvs was used for the first time in [7] and [14]. The first splitting condition for the locally bounded case was given in [7]. In [10] it was shown that all twisted sums of locally bounded spaces can be obtained by using quasilinear homogeneous maps. A similar results for the pairs consisting of the one-dimensional space and a Fréchet space was given in [8] and for the pairs consisting of two nuclear Fréchet spaces, the second one with a basis, in [12]. This latter result is even stronger, all twisted sums of such pairs of tvs can be obtained with use of linear maps instead of quasilinear. It can be generalized to pairs of arbitrary nuclear spaces or even pairs consisting of an arbitrary lcs and a TSC-space. The generalization as well as some other applications of the quasilinear technique will be given in the author's next paper "Twisted sums of nuclear and Banach spaces." Some other considerations and results related to the two problems mentioned above are contained in [4].

We finish the introduction with some auxiliary notions and facts.

In this paper we consider only tvs over the field of real or complex numbers which is denoted by K. The same notation is used for the one-dimensional tvs.

A balanced subset U of a tvs X is called pseudoconvex if there is $c > 0$ such that $U + U \subset cU$ and it is called r-convex, where $0 < r \leqslant 1$, if $aU + bU \subset U$ holds for all $a, b \geqslant 0$ with $a^r + b^r = 1$. The tvs X is said to be locally pseudoconvex (r-convex) iff it has a 0-neighbourhood base consisting of pseudoconvex (r-convex) sets.

By Theorem 6.8.3 [6] and its proof (the so-called Aoki-Rolewicz Theorem), a tvs X is locally pseudoconvex iff it has a 0-neighbourhood base $(U_a)_{a \in A}$ such that each U_a is r_a-convex for some $0 < r_a \leqslant 1$. It is obvious that every r-convex set is pseudoconvex and that the gauge functionals of r-convex sets are r-seminorms.

A metrizable complete tvs is called an F-space, and a locally convex F-space is called a Fréchet space. By $\tilde{X}$ we denote the completion of the tvs X (i.e., the direct sum of the closure of zero and the completion of its Hausdorff associated tvs). If j is a continuous linear mapping, then $\tilde{j}$ denotes its (unique) extension to the completion of its domain. In diagrams, the arrow $\hookrightarrow$ is used to indicate topological linear embeddings. For other notions we refer to [6] and [16] in general.

1. Quasilinear technique. First, let us recall some definitions from [4], Section 3. Let Y, Z be tvs. Two twisted sums of Y and Z,

$$(1.1) \qquad 0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0$$

and

$$0 \longrightarrow Y \xrightarrow{j_1} X_1 \xrightarrow{q_1} Z \longrightarrow 0$$

are equivalent if there is a topological isomorphism (which establishes the equivalence) $T: X \longrightarrow X_1$ such that the following diagram commutes:

$$(1.2) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & Y & \xrightarrow{j_1} & X_1 & \xrightarrow{q_1} & Z & \longrightarrow & 0. \end{array}$$

Let $F: Z \longrightarrow Y$ be an arbitrary map, and let us define two other maps $A_F: Z \times Z \longrightarrow Y$, $J_F: K \times Z \longrightarrow Y$ by

$$A_F(z_1, z_2) = F(z_1 + z_2) - F(z_1) - F(z_2),$$
$$J_F(a, z) = F(az) - aF(z).$$

We say that the map F is quasilinear if A_F, J_F are continuous at zero and $F(0) = 0$.

For $i = 0,1,2,3,4$ we define the sets $Q^i(Z,Y)$ of maps $F: Z \longrightarrow Y$ as follows:

$$Q^0(Z,Y) = \{F\colon F \text{ is quasilinear}\},$$
$$Q^1(Z,Y) = \{F\in Q^0(Z,Y)\colon F \text{ is homogeneous}\},$$
$$Q^2(Z,Y) = \{F\in Q^0(Z,Y)\colon A_F \text{ and } J_F \text{ are continuous}\},$$
$$Q^3(Z,Y) = \{F\in Q^1(Z,Y)\colon A_F \text{ is continuous}\},$$
$$Q^4(Z,Y) = \{F\colon F \text{ is linear}\}.$$

Now assume that Y is a subspace of another tvs Y_1, and let $q_0\colon Y_1 \longrightarrow Y_1/Y$ be the natural quotient map. Then we will denote by $Q^i(Z,Y,Y_1)$ the set of all $F\in Q^i(Z,Y_1)$ such that $q_0\circ F\colon Z \longrightarrow Y_1/Y$ is continuous at zero. We will say that a map between two tvs belongs to $\mathfrak{M}^i$ $(i = 0,1,2,3,4)$ if it is, respectively,

- continuous at zero $(i = 0)$,
- continuous at zero and homogeneous $(i = 1)$,
- continuous $(i = 2)$,
- continuous and homogeneous $(i = 3)$,
- continuous and linear $(i = 4)$.

The superscript $i = 0$ is often omitted.

Using quasilinear maps we can construct twisted sums. Let $F\in Q(Z,Y,Y_1)$; then $Y\oplus_F Z$ will denote the product linear space $Y\times Z$ endowed with the topology generated by the 0-neighbourhood base consisting of the sets of the form:

$$W_F^Y(U,V) = \{(y,z)\in Y\times Z\colon y - F(z)\in U,\ z\in V\},$$

where U, V are arbitrary 0-neighbourhoods in Y_1, Z, respectively. It is proved in [4], Proposition 3.1 that the following diagram

$$0 \longrightarrow Y \xrightarrow{\ j\ } Y\oplus_F Z \xrightarrow{\ q\ } Z \longrightarrow 0$$

forms a twisted sum, where j and q are the natural embedding and projection, respectively.

R e m a r k 1.1. If $i\colon Y \hookrightarrow Y_1$ is the natural embedding, then the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\ j\ } & Y\oplus_F Z & \xrightarrow{\ q\ } & Z & \longrightarrow & 0\\ & & \downarrow i & & \downarrow i\times \mathrm{id}_Z & & \downarrow \mathrm{id}_Z & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{\ j_1\ } & Y_1\oplus_F Z & \xrightarrow{\ q_1\ } & Z & \longrightarrow & 0, \end{array}$$

where j_1 and q_1 are again the natural embedding and projection, respectively. Moreover, $i\times \mathrm{id}_Z$ is a topological embedding. In fact, the condition "$q_0\circ F\colon Z \longrightarrow Y_1/Y$ is continuous at zero" in the definition of $Q(Z,Y,Y_1)$ is equivalent to the fact that $q = q_1\circ(i\times \mathrm{id}_Z)$ is open or, equivalently, that the "restriction" of the twisted sum $Y_1\oplus_F Z$ to $Y\oplus_F Z$ is a twisted sum too.

We will say that a map s is a <u>section</u> for the twisted sum (1.1) iff $s\colon Z \longrightarrow X$, $s(0) = 0$ and $q\circ s = \mathrm{id}_Z$. Further, a map $p\colon X \longrightarrow Y$ is said to be <u>projection</u> for the twisted sum (1.1) if $p(0) = 0$ and $p(x + j(y)) = p(x)+y$ for every $x \in X$, $y \in Y$. The latter condition is fulfilled by every additive map p with $p|_{j(Y)} = j^{-1}$.

R e m a r k 1.2. Let us observe that if s is a section for (1.1) then $p = j^{-1}\circ(id_X - s\circ q)$ is a projection for (1.1). On the other hand, if p is a projection for (1.1), then we can define a section s for (1.1) by $s(q(x)) = x - j\circ p(x)$ for every $x \in X$. Therefore there is a bijective correspondence between projections and sections.

LEMMA 1.1. For an arbitrary twisted sum (1.1) of tvs Y, Z the following conditions are equivalent ($i = 0,1,2,3,4$):

(a) For every algebraic isomorphism $T: X \longrightarrow Y \times Z$ for which (1.2) with $X_1 = Y \times Z$, commutes, there is a map $F \in Q^i(Z,Y)$ such that (1.1) is equivalent to the twisted sum

$$0 \longrightarrow Y \longrightarrow Y \oplus_F Z \longrightarrow Z \longrightarrow 0,$$

and the equivalence is established by T (recall that $Y \times Z = Y \oplus_F Z$ algebraically).

(b) The twisted sum (1.1) has a section $s \in M^i$.

(c) The twisted sum (1.1) has a projection $p \in M^i$.

P r o o f: (a)$\Longleftrightarrow$(b) is proved implicitly in [4], Lemma 3.2.

(b)$\Longrightarrow$(c): The projection p defined as in Remark 1.2 belongs to M^i iff $s \in M^i$.

(c)$\Longrightarrow$(b): If p is continuous at zero, then for every 0-neighbourhood $V \subset Y$ there is a 0-neighbourhood $U \subset X$ such that $p(U) \subset V$. Hence for $z \in q(U)$, $s(z) \in U - j(V)$ whenever s is defined as in Remark 1.2. Therefore s is continuous at zero too. The other cases are similar.

LEMMA 1.2. (Comp. [4], Lemma 2.1) Let

$$(1.3)\qquad 0 \longrightarrow (Y,\lambda) \xrightarrow{j} (X,\tau_1) \xrightarrow{q} (Z,\gamma_1) \longrightarrow 0$$

and

$$(1.4)\qquad 0 \longrightarrow (Y,\lambda) \xrightarrow{j} (X,\tau_2) \xrightarrow{q} (Z,\gamma_2) \longrightarrow 0$$

be twisted sums, where $\tau_1 \geqslant \tau_2$.

Let $s: (Z,\gamma_2) \longrightarrow (X,\tau_2)$ be a section for (1.4) belonging to the class M^i; then $s: (Z,\gamma_1) \longrightarrow (X,\tau_1)$ is a section for (1.3) belonging to the class M^i, too.

P r o o f. Using Lemma 1.1 and the remark before it, we may consider respective projections for the twisted sums (1.3) and (1.4). This makes Lemma 1.2 obvious.

PROPOSITION 1.1. Let $(Y_a)_{a\in A}$ be a family of tvs and let Z be a tvs. If for every $a\in A$ and for every twisted sum of Y_a and Z there is a projection $p\in \mathcal{M}^i$, then for every twisted sum of $Y = \prod_{a\in A} Y_a$ and Z there is a projection $p\in \mathcal{M}^i$ $(i = 0,1,2,3,4)$.

P r o o f. Let us consider the following twisted sum

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0.$$

For every $a\in A$, let $r_a\colon Y\longrightarrow Y_a$ be the natural projection. Then, for the natural quotient map $T_a\colon X \longrightarrow X/j(\prod_{b\in A\setminus\{a\}} Y_a) = X_a$, there are maps $j_a\colon Y_a\longrightarrow X_a$ and $q_a\colon X_a\longrightarrow Z$ such that the following diagram commutes and the second row forms a twisted sum:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \downarrow r_a & & \downarrow T_a & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & Y_a & \xrightarrow{j_a} & X_a & \xrightarrow{q_a} & Z & \longrightarrow & 0 . \end{array}$$

Let $p_a\colon X_a\longrightarrow Y_a$ be a projection for the twisted sum X_a.

The desired projection $p\colon X\longrightarrow Y$ is then given by the formula

$$p(x) = (p_a\circ T_a(x))_{a\in A},$$

Indeed, let $x\in X$, $y\in Y$, $a\in A$, then

$$p_a\circ T_a(x + j(y)) = p_a(T_a(x) + T_a\circ j(y)) = p_a(T_a(x) + j_a\circ r_a(y))$$
$$= p_a\circ T_a(x) + r_a(y).$$

Finally,

$$p(x + j(y)) = (p_a\circ T_a(x))_{a\in A} + (r_a(y))_{a\in A} = p(x) + y.$$

Clearly $p\in \mathcal{M}^i$ whenever p_a belongs to the class $\mathcal{M}^i$ for every $a\in A$.

PROPOSITION 1.2. Let Y be a Fréchet space and let Z be an F-space.

(a) (E. Michael, [13]; [2], Theorem II.7.1) Every twisted sum of Y and Z admits a continuous section.

(b) If Y is a Banach space, then every twisted sum of Y and Z admits a continuous section s such that $s(az) = as(z)$ for every $z\in Z$, $a\in K$, $|a| = 1$.

(c) If Y is a Banach space and Z is locally bounded, then every twisted sum of Y and Z admits a continuous homogeneous section.

P r o o f. (a): It is proved in [2], Theorem II.7.1.

(b): Let us denote by T the multiplicative group $\{a\in K\colon |a| = 1\}$.

Let us consider an arbitrary twisted sum of Y and Z. By part (a), it possesses a continuous section s. For every $x \in q^{-1}(z)$, $z \in Z$, we have $(1/a)s(az) - x \in j(Y)$, $a \in K$, where j, q are "associated" to the given twisted sum embedding of Y and quotient map onto Z, respectively. Hence the function $f: K \times Z \longrightarrow Y$ such that

$$f(a,z) = (1/a)s(az) - s(z), \quad a \in K, \ z \in Z,$$

is continuous. Thus we can define:

$$s_1(z) = s(z) + \int_T f(a,z)\, da ,$$

where the integral is the Bochner integral taken with respect to the normalized Haar measure on the group T.

In the real case we have simply $s_1(z) = (s(z) + s(-z))/2$ so that the section s_1 is continuous. We will show it in general.

First, note that if C is a compact subset of Z, then the function f restricted to $T \times C$ is uniformly continuous. Hence. if $z_n \longrightarrow z$ in Z, then $f(\cdot,z_n) \longrightarrow f(\cdot,z)$ uniformly for $a \in T$, $n \rightarrow \infty$. Therefore, since $s(z_n) \longrightarrow s(z)$, we have

$$s_1(z_n) = s(z_n) + \int_T f(a,z_n)\, da \longrightarrow s(z) + \int_T f(a,z)\, da = s_1(z),$$

and s_1 is continuous. Obviously, s_1 is the desired section.

(c): It is known (see for example [3] or [7], Theorem 1.1) that every twisted sum of Y and Z is locally bounded whenever Y and Z are locally bounded. By the part (b), every twisted sum of Y and Z possesses a section s such that $s(az) = as(z)$ for every $a \in K$, $|a| = 1$, $z \in Z$. Let V be a closed bounded r-convex 0-neighbourhood in Z such that s(V) is bounded and let p_V be its gauge functional. Obviously it is a continuous r-norm. We can define a new homogeneous section s_1 as follows:

$$s_1(z) = \begin{cases} 0 & \text{for } z = 0; \\ p_V(z)s(z/p_V(z)) & \text{for } z \neq 0. \end{cases}$$

By the continuity of p_V, s_1 is continuous at every point $z \neq 0$. But $s(z/p_V(z)) \subset s(V)$ and the latter set is bounded. Hence, if $z \longrightarrow 0$. then $p_V(z) \longrightarrow 0$ and $s_1(z) \longrightarrow 0$, too. Therefore s_1 is continuous at every point and it is the desired section.

PROPOSITION 1.3 ([4], Corollary 2.1). Let Y be a semimetrizable tvs and let Z be a Hausdorff tvs. If

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0$$

is a twisted sum, then

$$0 \longrightarrow \tilde{Y} \xrightarrow{\tilde{j}} \tilde{X} \xrightarrow{\tilde{q}} \tilde{Z} \longrightarrow 0$$

is a twisted sum too.

Now, the following theorem establishes the existence of sections for some twisted sums. The parts (a) and (c) can be strenghthened, comp. [4], Theorem 2.1.

THEOREM 1.1. Let Y, Z be tvs. Then every twisted sum of Y and Z possesses a section s which is:

(a) continuous at zero, whenever Y is an arbitrary product of F-spaces;

(b) continuous, whenever Y is an arbitrary product of Fréchet spaces;

(c) continuous at zero and homogeneous, whenever Y is an arbitrary product of locally bounded F-spaces;

(d) continuous and homogeneous, whenever Y is an arbitrary product of Banach spaces and Z is locally pseudoconvex.

P r o o f. Proposition 1.1 shows that it is enough to prove the theorem omitting the phrases "an arbitrary product of". Hence the parts (a) and (c) are contained in [4], Theorem 2.1. Part (b) is an easy consequence of Propositions 1.2 (a), 1.3 and Lemma 1.2.

(d): Let us consider a twisted sum

$$0 \longrightarrow (Y,\|\cdot\|) \xrightarrow{j} (X,\tau) \xrightarrow{q} (Z,\gamma) \longrightarrow 0,$$

where $(Y,\|\cdot\|)$ is a Banach space and (Z,γ) is a locally pseudoconvex space. By [4], Theorem 1.1, (X,τ) is locally pseudoconvex. Let U be an r-convex (balanced) 0-neighbourhood in (X,τ) such that $j^{-1}(U)$ is contained in the unit ball of $(Y,\|\cdot\|)$.

By Lemma 1.2, it is enough to prove the existence of a required section for the twisted sum

$$0 \longrightarrow (Y,\|\cdot\|) \xrightarrow{j} (X,\tau_1) \xrightarrow{q} (Z,\gamma_1) \longrightarrow 0,$$

where τ_1 (γ_1, respectively) is the linear topology generated by the 0-neighbourhood base $\{n^{-1}U: n\in N\}$ ($\{n^{-1}q(U): n\in N\}$, respectively).

Let us observe that $q(\ker \tau_1) = \ker \gamma_1$ (where $\ker \tau_1$ is the closure of $\{0\}$ in (X,τ_1)) because $(Y,\|\cdot\|)$ is complete. Hence we have topological direct sums

$(X,\tau_1) = (X_1,\tau_1\cap X_1) \oplus (\ker \tau_1, \tau_1 \cap \ker \tau_1)$ for some $X_1 \supset j(Y)$,
and $(Z,\gamma_1) = (Z_1,\gamma_1\cap Z_1) \oplus (\ker \gamma_1, \gamma_1\cap \ker \gamma_1)$ for $Z_1 = q(X_1)$.
In consequence, the following diagram forms a twisted sum of Hausdorff locally r-convex spaces:

$$0 \longrightarrow (Y,\|\cdot\|) \xrightarrow{j} (X_1, \tau_1 \cap X_1) \xrightarrow{q_1} (Z_1, \gamma_1 \cap Z_1) \longrightarrow 0,$$

where $q_1 = q|_{X_1}$. By Proposition 1.3, also the next diagram forms a twisted sum:

$$0 \longrightarrow (Y,\|\cdot\|) \xrightarrow{j} (\widetilde{X}_1, \widetilde{\tau_1 \cap X_1}) \xrightarrow{\widetilde{q}_1} (\widetilde{Z}_1, \widetilde{\gamma_1 \cap Z_1}) \longrightarrow 0.$$

This twisted sum fulfils the assumptions of Proposition 1.2 (c), hence it admits a continuous and homogeneous section s_1. It is easily seen that $s_1|_{Z_1}: Z_1 \longrightarrow X_1$ is a section for the previous twisted sum. Let $p: Z \longrightarrow \ker \gamma_1$ be a linear (and obviously continuous) projection with $\ker p = Z_1$, and let $s_2: \ker \gamma_1 \longrightarrow \ker \tau_1$ be the linear map such that

$$\{s_2(z)\} = \ker \tau_1 \cap q^{-1}(z)$$

for every $z \in \ker \gamma_1$. Of course, s_2 is continuous. Now, we can define a homogeneous continuous section s as follows:

$$s = s_2 \cdot p + s_1 \cdot (id_Z - p).$$

This completes the proof.

R e m a r k. It is not known to the author if the assumption that Z is locally pseudoconvex is necessary in the part (d). The above method of proof is not applicable in general because the existence of p_V with the properties as required in the above proof implies that there is a 0-neighbourhood in Z_1 which does not contain any line. It is not the case, for instance, when $Z = L_0(0,1)/K$, and so the author does not know if the twisted sum

$$0 \longrightarrow K \longrightarrow L_0(0,1) \longrightarrow L_0(0,1)/K \longrightarrow 0$$

admits a homogeneous continuous section (it has a homogeneous section continuous at zero and a continuous section, comp. the parts (b) and (c)).

The following result may be called an "extension theorem":

THEOREM 1.2. Let us consider a tvs Y_1, the following twisted sum of tvs Y and Z:

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0,$$

and a topological linear embedding $i: Y \hookrightarrow Y_1$. Then there exist a tvs X_1, a topological embedding $T: X \hookrightarrow X_1$ and maps j_1, q_1 such that the following diagram commutes and the second row forms a twisted sum:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow T & & \downarrow id & & \\ 0 & \longrightarrow & Y_1 & \xrightarrow{j_1} & X_1 & \xrightarrow{q_1} & Z & \longrightarrow & 0. \end{array}$$

R e m a r k. This theorem is an easy consequence of [4], Corollary 3.2 (d). The twisted sum X_1 may be called an extension of the twisted sum X to the twisted sum of Y_1 and Z.

P r o o f. Algebraically, we define $X_1 = Y_1/Y \times Y \times Z$, let us choose a linear isomorphism j_1 from Y_1 onto $Y_1/Y \times Y = Y_1/Y \times Y \times \{0\}$ with $j_1(Y) = \{0\} \times Y$, and let us denote by q_1 the natural projection from X_1 onto Z. Then there is a unique linear isomorphism T from X onto $Y \times Z = \{0\} \times Y \times Z$ such that the above diagram commutes. Next, we equip X_1 with the strongest linear topology such that T and j_1 are continuous. The family of sets of the form

$$(\{0\} \times T(U)) + (j_1(V) \times \{0\}) ,$$

where U, V are 0-neighbourhoods in X, Y_1, respectively, forms a 0-neighbourhood base for this topology. It can be easily seen that X_1 is as desired.

The following theorem is the main result concerning the quasilinear technique; it is an immediate consequence of Lemma 1.1, Theorem 1.1 and Theorem 1.2 (comp. Remark 1.1).

THEOREM 1.3. Let $Y_1 \supset Y$, Z be tvs. Then every twisted sum of Y and Z is equivalent to a twisted sum of the form $Y \oplus_F Z$, where $F \in Q^i(Z,Y,Y_1)$ and

(a) $i = 0$ when Y_1 is an arbitrary product of F-spaces;
(b) $i = 2$ when Y is locally convex and Y_1 is an arbitrary product of Fréchet spaces;
(c) $i = 1$ when Y is locally pseudoconvex and Y_1 is an arbitrary product of locally bounded F-spaces;
(d) $i = 3$ when Y is locally convex, Y_1 is an arbitrary product of Banach spaces and Z is locally pseudoconvex.

R e m a r k. Comp. [4], Theorem 3.1 for a weaker version of the parts (a) and (c).

The last fact is proved in [4], Corollary 3.1:

THEOREM 1.4. The twisted sum:

$$0 \longrightarrow Y \xrightarrow{j} Y \oplus_F Z \xrightarrow{q} Z \longrightarrow 0,$$

where $Y \subset Y_1$, Z are tvs and $F \in Q(Z,Y,Y_1)$ splits iff there is a linear map $L: Z \longrightarrow Y$ such that $F - L: Z \longrightarrow Y_1$ is continuous at zero.

2. Characterization of TSC-spaces. Let us recall that a lcs Z is a TSC-space if for every lcs Y every twisted sum of Y and Z is locally convex. By $l_\infty(A)$ we denote the Banach space of all bounded functions $f: A \longrightarrow K$ with the sup-norm. First, we prove an easy Lemma:

LEMMA 2.1. Let B be a Hamel basis of Z and let I be an arbitrary family of quasilinear homogeneous maps $F: Z \longrightarrow K$ such that $F(e) = 0$ for every $e \in B$, $F \in I$. Assume that the associated family of mappings $A_F: Z \times Z \longrightarrow K$, $F \in I$, is equicontinuous at zero. Then for every $x \in Z$, $\sup_{F\in I} |F(x)| < \infty$.

P r o o f. Let r be a continuous seminorm on Z such that if $x, y \in Z$ and $r(x) \leqslant 1$, $r(y) \leqslant 1$, then $|A_F(x,y)| \leqslant 1$ for every $F \in I$. Thus $|A_F(x,y)| \leqslant r(x) + r(y)$ for all $x, y \in Z$ and $F \in I$. Now, if

$$x = \sum_{k=1}^{n} a_k e_k \in Z \quad \text{for } e_k \in B,\ a_k \in K,\ k=1,\dots,n,$$

then (as easily checked)

$$F(x) = \sum_{k=2}^{n} A_F(a_k e_k, \sum_{j=1}^{k-1} a_j e_j)$$

and hence

$$|F(x)| \leqslant m = \sum_{k=2}^{n} r(a_k e_k) + \sum_{k=2}^{n} r(\sum_{j=1}^{k-1} a_j e_j) < \infty,$$

where the constant m is independent of $F \in I$.

The following theorem characterizes TSC-spaces.

THEOREM 2.1. For every lcs Z the following conditions are equivalent:

(a) Z is a TSC-space.

(b) $Z \in S(l_\infty(I))$ for every set I.

(c) $Z \in S(l_\infty(I))$ for a set I the cardinality of which is equal to the density character of Z.

(d) $Z \in S(l_\infty(I))$ for a set I the cardinality of which is equal to the cardinality of a 0-neighbourhood base of Z.

(e) If $I \subset Q^1(Z,K)$ is a family of maps such that the associated family $(A_F)_{F\in I}$ is equicontinuous at zero, then there is a family of linear (not necessarily continuous) mappings $L_F: Z \longrightarrow K$, $F \in I$, such that the family $(F - L_F)_{F\in I}$ is equicontinuous at zero.

(f) If $I \subset Q^3(Z,K)$ is a family of maps such that the associated family $(A_F)_{F\in I}$ is equicontinuous at every point in $Z \times Z$, then there is a family of linear (not necessarily continuous) mappings $L_F: Z \longrightarrow K$, $F \in I$ such that the family $(F - L_F)_{F \in I}$ is equicontinuous at zero (or equivalently: equicontinuous at every point in Z).

P r o o f. (a) $\Longrightarrow$(b): This is an immediate consequence of the well-known fact that isomorphic copies of $l_\infty(I)$'s are complemented in every locally convex space.

(b) $\Longrightarrow$(a): Every locally convex space Y may be embedded into a suitable product $\prod_{b \in B} l_\infty(I_b) = Y_1$. Let us consider the following twisted sum

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0.$$

By Theorem 1.2, it may be extended to a twisted sum in the upper row of the diagram below

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_1 & \xrightarrow{j_1} & X_1 & \xrightarrow{q_1} & Z & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & Y & \xrightarrow{j} & X & \xrightarrow{q} & Z & \longrightarrow & 0 \end{array}$$

so that the diagram commutes. By Corollary 1.1 and (b), $Z \in S(Y_1)$ and then $X_1 \simeq Y_1 \oplus Z$ is locally convex. Hence so is X as a subspace of X_1.

(a)$\Longleftrightarrow$(c): The proof is quite similar as above.

(b)$\Longrightarrow$(d): It is obvious.

(d)$\Longrightarrow$(e): Let C be a 0-neighbourhood base in Z. Let us assume that (e) does not hold: there is a 0-neighbourhood $U \in C$ in Z such that $|A_F(U \times U)| \leqslant 1$ for every $F \in I$ but

for every $U \in C$ there exists $F_U \in I$ such that for every linear map $L: Z \longrightarrow K$ one can find $x \in U$ with $|(F_U - L)(x)| > 1$.

Obviously, for the family J, $J = \{F_U: U \in C\}$, the condition (e) does not hold, too.

Choose any Hamel basis B for Z. Then for each $F \in J$ there is a linear map $h_F: Z \longrightarrow K$ such that $(F - h_F)(e) = 0$ for all $e \in B$; clearly $A_{(F-h_F)} = A_F$. Using Lemma 2.1 we get

$$\sup_{F \in J} |(F - h_F)(z)| < \infty \text{ for every } z \in Z.$$

We can define a map $F_0: Z \longrightarrow l_\infty(J)$ by

$$F_0(z) = ((F - h_F)(z))_{F \in J} ;$$

clearly it is quasilinear and homogeneous. By (d), the twisted sum

$$0 \longrightarrow l_\infty(J) \longrightarrow l_\infty(J) \oplus_{F_0} Z \longrightarrow Z \longrightarrow 0$$

splits; hence, by Theorem 1.4, there exists a linear map

$L: Z \longrightarrow l_\infty(J)$ such that $F_\rho - L: Z \longrightarrow l_\infty(J)$ is continuous at zero. Now, if $(x_F^*)_{F \in J}$ are the coordinate functionals on $l_\infty(J)$, then for the linear maps $L_F = h_F + x_F^* \circ L: Z \longrightarrow K$ we have $x_F^* \circ (F - L) = F - L_F$ and the family $(F - L_F)_{F \in J}$ is equicontinuous at zero; a contradiction.

(e) $\Longrightarrow$ (f): This is obvious.

(f) $\Longrightarrow$ (b): Consider a twisted sum (B is an arbitrary set)

(2.1) $$0 \longrightarrow l_\infty(B) \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0.$$

By Theorem 1.3 (d), it is equivalent to

(2.2) $$0 \longrightarrow l_\infty(B) \longrightarrow l_\infty(B) \oplus_F Z \longrightarrow Z \longrightarrow 0$$

for a suitably chosen homogeneous quasilinear map $F: Z \longrightarrow l_\infty(B)$ such that $A_F: Z \times Z \longrightarrow l_\infty(B)$ is continuous at every point. Let $(x_b^*)_{b \in B}$ be the coordinate functionals on $l_\infty(B)$. Then $I = \{F_b = x_b^* \circ F: Z \longrightarrow K : b \in B\}$ form a family of quasilinear maps such that $A_{F_b} = x_b^* \circ A_F$ are equicontinuous at every point. By (f), there is a family of linear maps $L_b: Z \longrightarrow K$, such that $(F_b - L_b)_{b \in B}$ is equicontinuous at zero. It follows easily that $L: x \longmapsto (L_b(x))_{b \in B}$ is a linear map from Z to $l_\infty(B)$ and that $F - L$ is continuous at zero. By Theorem 1.4, (2.2) splits and then (2.1) splits. This completes the proof.

3. <u>Permanence properties</u>. In this section we will consider the permanence properties of the class of all TSC-spaces. The respective theorems will be obtained as immediate consequences of Theorem 2.1 ((a)$\Longleftrightarrow$(b)) and of "permanence theorems" for the classes S(Y) for some tvs Y. The first fact is very simple.

THEOREM 3.1. For an arbitrary tvs Y, the class S(Y) is closed under arbitrary direct sums (in the category of all tvs).

P r o o f. Let us consider a twisted sum

$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} \bigoplus_{a \in A} Z_a \longrightarrow 0,$$

where $Z_a \in S(Y)$ for every $a \in A$. It is obvious that for every $a \in A$ the diagram

$$0 \longrightarrow Y \xrightarrow{j} X_a \xrightarrow{q_a} Z_a \longrightarrow 0,$$

where $X_a = q^{-1}(Z_a)$ and $q_a = q|_{X_a}$, forms also a twisted sum. By the assumption, there is a linear continuous section s_a for the latter twisted sum. Then, of course, $\bigoplus_{a \in A} s_a$ is a linear continuous section for the original twisted sum.

It is well known (see for example [1], 3(6)) that for countable families of lcs their locally convex direct sum and their direct sum (in the category of all tvs) coincide.

COROLLARY 3.1. Every countable locally convex direct sum of TSC-spaces is a TSC-space.

For twisted sums the above facts hold as well. The original author's proof of the following result was very complicated and therefore we will give much simpler proof due to L. Drewnowski.

THEOREM 3.2. For every tvs Y and every Z_1, $Z_2 \in S(Y)$ every twisted sum of Z_1 and Z_2 belongs to $S(Y)$.

P r o o f. (L. Drewnowski) Let us consider the following twisted sums

$$0 \longrightarrow Z_1 \xrightarrow{j_Z} Z \xrightarrow{q_Z} Z_2 \longrightarrow 0,$$
$$0 \longrightarrow Y \xrightarrow{j} X \xrightarrow{q} Z \longrightarrow 0.$$

Let $X_1 = q^{-1}(j_Z(Z_1))$, thus $j(Y) \subset X_1 \subset X$ and we get the following twisted sum:

$$0 \longrightarrow Y \xrightarrow{j} X_1 \xrightarrow{q_1} Z_1 \longrightarrow 0,$$

where $q_1 = j_Z^{-1} \circ q|_{X_1}$. By the assumption, the above twisted sum splits, and hence there is a subspace $Z_1' \subset X_1$ isomorphic to Z_1 such that X_1 is a topological direct sum of Y and Z_1'. Let $X_2 = X/Z_1'$ and let $q_0\colon X \longrightarrow X_2$ be the natural quotient map. It is easily seen that $j_2 = q_0 \circ j\colon Y \longrightarrow X_2$ is an isomorphic embedding. We can obtain the following twisted sum

$$0 \longrightarrow Y \xrightarrow{j_2} X_2 \xrightarrow{q_2} Z_2 \longrightarrow 0,$$

because $X_2/j_2(Y) = (X/Z_1')/(X_1/Z_1')$ is naturally isomorphic to $X/X_1 \simeq$ $\simeq (X/j(Y))/(X_1/j(Y)) \simeq Z/Z_1 \simeq Z_2$. Obviously, by the assumption, the latter twisted sum splits; let $p\colon X_2 \longrightarrow Y$ be a continuous linear projection for it, i.e. $p \circ j_2 = id_Y$. Then the map $P = p \circ q_0\colon X \longrightarrow Y$ is a continuous linear projection for the given twisted sum of Y and Z, because $P \circ j = p \circ q_0 \circ j = p \circ j_2 = id_Y$. This completes the proof.

The following table collects all the permanence properties of the classes of all TSC-spaces, $\mathcal{K}$-spaces and S(Y) known to the author. For instance, the word "YES" in the column "$\mathcal{K}$-spaces" and the row 3.(a) "finite products" means that every finite product of $\mathcal{K}$-spaces is a $\mathcal{K}$-space. References for proofs and counterexamples are given in Remarks below.

	S(Y)	$\mathcal{K}$-spaces	TSC-spaces
1. Subspaces			
(a) closed	NO	NO	NO
(b) complemented	YES	YES	YES
(c) dense	?, YES for Y F-space or q-minimal	YES	YES
2. Quotients	NO see remark below		YES
3. Products			
(a) finite	YES	YES	YES
(b) arbitrary	NO, YES for Y locally bounded complete	YES	YES
4. Reduced projective limits	NO, YES for Y locally bounded complete	YES	YES
5. Direct sums			
(a) countable locally convex	YES	YES	YES
(b) general	YES	YES	---
6. Countable locally convex inductive limits	?	YES for lcs	YES
7. Twisted sums	YES	YES	YES

R e m a r k s. Let us observe that the classes of all $\mathcal{K}$-spaces and all TSC-spaces are particular cases of the classes S(Y) (comp. Theorem 2.1 (a)$\Longleftrightarrow$(b)), therefore it is enough to prove positive results only for the first column of the table above.

1.(a): It was proved by N. J. Kalton and J. W. Roberts [11] that l_∞ is a $\mathcal{K}$-space and, by Theorem 2.1 and [7], Theorem 4.10, l_∞ is a TSC-space. On the other hand l_1 is isomorphic to a subspace of l_∞ and, by [7], [14] and [15], l_1 is not a $\mathcal{K}$-space and obviously not a TSC-space.

1.(b): This was proved very simply in [4], Theorem 4.1 (a).

1.(c): The answer YES for Y semimetrizable complete may be easily deduced from Proposition 1.3. L. Drewnowski ([5], Corollary 2.6) proved that every quotient of a complete tvs by its q-minimal subspace is complete. (A Hausdorff tvs X is called q-minimal if none of its Hausdorff quotients admits a strictly weaker Hausdorff vector topology). This result allows us to prove a strict analogue of Proposition 1.3 for Y q-minimal and Z an arbitrary tvs. Using this fact we may justify the answer YES for Y q-minimal. The only known q-minimal tvs are K^I for every set I.

2: N. J. Kalton and N. T. Peck ([9], Theorem 5.2 and 5.3) proved that a quotient Z/Z_1 of a $\mathcal{K}$-space Z is a $\mathcal{K}$-space iff the subspace Z_1 has the HBEP in Z, i.e., every continuous linear functional on Z_1 can be extended to the whole space Z. Of course, this assumption is satisfied when Z is locally convex. It is proved in [4], Lemma 4.1 that if Z_1 is a subspace of a tvs Z, $Z \in S(Y)$ and every continuous linear map L: $Z_1 \longrightarrow Y$ can be extended to the whole space Z, then $Z/Z_1 \in S(Y)$. Of course, by the Hahn-Banach theorem, for every lcs $Z_1 \subset Z$, every continuous linear map L: $Z_1 \longrightarrow l_\infty(A)$ can be extended to a map defined on Z. Hence, by Theorem 2.1, the class of all TSC-spaces is closed under quotients. For the direct proof of the latter fact see [4], Theorem 5.3 (c).

3.(a): This was shown very simply in [4], Theorem 4.1 (b).

3.(b): D. Vogt ([17], proof of Lemma 1.6 or [18], Theorem 2.4) constructed a twisted sum of s and s^N which is isomorphic to s (s is the Fréchet nuclear space of rapidly decreasing sequences). Of course, s^N is not isomorphic to s because the latter has a continuous norm. Hence $s^N \in S(s)$. By [17], Theorems 1.3 and 1.5, every locally convex twisted sum of s and s splits. But s is nuclear and all nuclear spaces are TSC-spaces ([4], Theorem 5.5 (c) comp. Corollary 4.1 below), so finally $s \in S(s)$. Therefore the answer is NO even for S(Y), where Y is a Fréchet space and for countable products. Nevertheless, the answer is YES for S(Y), where Y is locally bounded complete, as proved in [4], Theorem 4.3 (b). By Theorem 2.1, the class of all TSC-spaces is closed under arbitrary products.

4: Every product is a reduced projective limit of its finite subproducts. Hence part 3.(b) implies the answer NO for S(Y). The answer YES is proved in [4], Theorem 4.3 (a).

5: This is contained in Theorem 3.1 and Corollary 3.1.

6: For TSC-spaces this is a consequence of the parts 5 and 2.

7: This was proved in Theorem 3.2. For TSC-spaces it may be proved directly (and rather simply), comp. [4], Theorem 5.3 (a).

4. The main open problem. In view of Theorem 2.1 it may be interesting if $S(\mathcal{K}) = S(l_\infty(A))$ for every set A. This seems unlikely but all the examples of $\mathcal{K}$-spaces known to the author belong to $S(l_\infty(A))$. (For instance, by [9], Theorem 3.6 and [7], Theorem 3.6, $L_p(0,1) \in S(l_\infty(A))$, $0 \leq p < 1$, for every set A). For locally convex spaces Z this problem (by Theorem 2.1) is equivalent to the following one (comp. [4]).

PROBLEM. Do the classes of all TSC-spaces and all locally convex $\mathcal{K}$-spaces coincide?

S. Dierolf in [3], Theorem 2.4.1 (implicitly) and N. J. Kalton in [7], Theorem 4.10 have proved that every twisted sum of two metrizable lcs Y and Z, with $Z \in S(\mathcal{K})$, is locally convex. From this it may be easily deduced (see [4], Corollary 5.1) that locally convex metrizable $\mathcal{K}$-spaces are TSC-spaces. Using the latter fact and the permanence properties given in the previous section we get immediately (Part (b) is proved by a different argument in [4], Theorem 5.1):

THEOREM 4.1. (a) An arbitrary product or countable direct sum of metrizable lcs is a TSC-space iff it is a $\mathcal{K}$-space.

(b) Every reduced projective limit of metrizable locally convex $\mathcal{K}$-spaces is a TSC-space.

Every nuclear lcs is a reduced projective limit of Hilbert spaces ([16], 7.3 Corollary 3) but, by [7], Theorem 4.9, every Hilbert space is a $\mathcal{K}$-space. Hence we obtain:

COROLLARY 4.1. ([4], Theorem 5.5 (c)) Every nuclear lcs is a TSC-space.

The following theorem may be proved directly as well.

THEOREM 4.2. Every twisted sum of a lcs Y with a weak topology and a locally convex $\mathcal{K}$-space Z is locally convex.

P r o o f. The tvs Y can be embedded into the product $K^I = Y_1$

for some I. Using Theorem 1.2 we may extend our given twisted sum to a twisted sum X_1 of Y_1 and Z, which splits, by Corollary 1.1, and hence is locally convex. Obviously, the given twisted sum is locally convex as a subspace of X_1.

In view of Theorem 2.1 ((a)$\Longleftrightarrow$(e)), the positive answer to our problem implies a kind of "Banach-Steinhaus theorem" for quasilinear mappings. It is interesting that N. J. Kalton has shown directly ([7], Proposition 3.3 (iii)) the condition (e) from Theorem 2.1 for Banach $\mathcal{K}$-spaces. By Theorems 4.1 (a) and 2.1, the same is true for a larger class of tvs, in particular, for metrizable locally convex $\mathcal{K}$-spaces but the author's attempts to prove it directly were unsuccessful.

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