Acta Universitatis Carolinae. Mathematica et Physica

Andrzej Spakowski

On superpositionally measurable multifunctions

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 30 (1989), No. 2, 149--151

Persistent URL: http://dml.cz/dmlcz/701808

On Superpositionally Measurable Multifunctions

ANDRZEJ SPAKOWSKI

Opole*)

Received 15 March 1989

We prove a theorem on measurability of the superposition $F(t, G(t))$, where F is a Carathéodory multifunction and G is a measurable one.

1. Introduction

The problem of measurability of the superposition $F(t, G(t))$ arise in many situations, lately in the study of differential inclusions and random differential inclusions (see e.g. [3, 6]).

The classical result on superpositional measurability is due to Carathéodory [4] and states the following: Let T be an arbitrary measurable space, X a separable metric space and Y a metric space. If $f: T \times X \to Y$ is a Carathéodory function, i.e. measurable in the first variable and continuous in the second one, then for every measurable function $x: T \to X$, the superposition $f(t, x(t))$ is a measurable function. There are some results of this type for multifunctions, see [9, 3, 13, 2, 6, 8]. For other references see the survey paper [1] by Appell.

Now, we recall some definitions from the multifunctions theory. Throughout this note let (T, Σ) be a measurable space and X, Y be two metric spaces. Denote by 2^X and 2^Y the families of all nonempty subsets of X and Y, respectively. A multifunction $G: T \to 2^X$ is said to be mesurable if for every open $A \subset X$ the set $G^{-1}(A) = \{t \in T: G(t) \cap A \neq \emptyset\} \in \Sigma$. Note that measurability of G is equivalent to the measurability of $\bar{G}$, where $\bar{G}(t) = \overline{G(t)}$ for $t \in T$.

A multifunction $H: X \to 2^Y$ is said to be continuous if it is both lower and upper semicontinuous. Lower (upper) semicontinuity of H means that for every open $B \subset Y$ the set $\{x \in X: H(x) \cap B \neq \emptyset\}$ ($\{x \in X: H(x) \subset B\}$) is open in X.

We say that a multifunction $F: T \times X \to 2^Y$ is a Carathéodory multifunction if for every $x \in X$ the multifunction $F(\cdot, x)$ is measurable, and for every $t \in T$ the multifunction $F(t, \cdot)$ is continuous.

In the next section we will prove the following theorem which generalize Caljuk's result [3].

*) Instytut Matematyki WSP, Oleska 48, 45-951 Opole, Poland

Theorem. Let X be complete and separable and $F: T \times X \to 2^Y$ be a Carathéodory multifunction with relatively compact values. Then for every measurable multifunction $G: T \to 2^X$ the superposition $F(t, G(t))$ is a measurable multifunction, where $F(t, G(t))$ is the sum of sets $F(t, x)$ over $x \in G(t)$.

2. Proof of the theorem

Our first step is a reduction of the problem to the measurability of the superposition $F(t, x(t))$, where x is an arbitrary measurable function from T to X. Indeed, let (g_n) be a Castaing representation of the multifunction G (see [12, Theorem 4.2] of [7, Corollary 2.2]), and observe that the lower semicontinuity of $F(t, \cdot)$ implies that

$$\{t \in T: F(t, G(t)) \cap A \neq \emptyset\} = \{t \in T: F(t, \overline{G(t)}) \cap A \neq \emptyset\} = = \bigcup_n \{t \in T: F(t, g_n(t)) \cap A \neq \emptyset\}$$

for every open $A \subset Y$.

Let $x: T \to X$ be an arbitrary measurable function. Since X is separable there exists a sequence (x_n) of measurable simple functions which converges pointwise to x([5, p. 61]). The superpositions $F(t, x_n(t))$ are measurable multifunctions because

$$\{t \in T: F(t, x_n(t)) \cap A \neq \emptyset\} = \bigcup_a \{t \in T: x_n(t) = a \text{ and } F(t, a) \cap A \neq \emptyset\}$$

for every open $A \subset Y$, and the sum over a is finite.

In view of [11, Theorem 4.7] it is sufficient to prove that the sequence $(F(t, x_n(t)))$ converges (with respect to the Hausdorff metric) to the compact set $F(t, x(t))$. However, the convergence follows from the continuity of the multifunction $F(t, \cdot)$.

3. Concluding remarks

Another version of the theorem can be formulated. Namely, if X is separable (not necessarily complete) and the values of G are complete subsets of X then the superposition $F(t, G(t))$ is a measurable multifunction too.

Bocsan [2] (see also [10]) consider the following condition (c): There exists a Castaing representation (g_n) of $G: T \to 2^X$ such that the superpositions $f(t, g_n(t))$ are measurable functions, where $f: T \times X \to Y$. He remarked that this condition holds provided X is separable, f is a Carathéodory function and G is a measurable and complete valued multifunction (see [10, Proposition 2]). In other words: the superposition $f(t, G(t))$ is measurable provided X is separable, f a Carathéodory function and G a measurable and complete valued multifunction.

The following two examples show that the assumptions on lower and upper semicontinuity in the theorem of section 1 cannot be omitted.

Example 1. Let $E \subset \mathbb{R}$ be non-Lebesgue measurable. Define $F: \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ as follows: $F(t, x) = \{0, 1\}$ if $x \neq t$, $F(t, x) = \{0\}$ if $x = t$ and $t \in E$, and $F(t, x) = \{1\}$ if $x = t$ and $t \notin E$. The multifunctions $F(t, \cdot)$ are lower semicontinuous but not upper semicontinuous. The multifunctions $F(\cdot, x)$ are measurable. However, the multifunction $F(t, t)$ is not measurable.

Example 2. Let E be as above and F be defined as follows: $F(t, x) = \{0\}$ if $x \neq t$, $F(t, x) = \{0, 1\}$ if $x = t$ and $t \in E$, and $F(t, x) = [0, 1]$ if $x = t$ and $t \notin E$. The multifunctions $F(t, \cdot)$ are upper semicontinuous but not lower semicontinuous. The multifunctions $F(\cdot, x)$ are measurable but the multifunction $F(t, t)$ is not measurable.

References

[1] Appell J., The superposition operator in function spaces. A survey, Expo. Math., 6 (1988), 209—270.

[2] Bocşan G., Some properties of continuous random functions on random domains with applications to continuous extensions and random fixed point theorems, Semin. Teor. Functiilor Mat. Appl. Ser. A, no 45, Univ. Timisoara 1979.

[3] Caljuk V. Z., On superpositionally measurable multifunctions (in Russian), Mat. Zametki, 43 (1988), 98—102.

[4] Carathéodory C., Vorlesungen über reelle Funktionen, De Gruyter, Leipzig—Berlin 1918.

[5] Castaing C., Valadier M., Convex analysis and measurable multifunctions, Lecture Notes in Math. 580, Springer, Berlin—Heidelberg—New York 1977.

[6] De Blasi F. S., Myjak J., Random differential inclusions with nonconvex right hand side, Funkc. Ekvac. ,30 (1987), 1—8.

[7] Graf S., Selected results on measurable selections, Rend. Circ. Mat. Palermo, 31 (1982), suppl., 87—122.

[8] Himmelberg C. J., Measurable relations, Fund. Math., 87 (1975), 53—72.

[9] Kikuchi N., Control problems of contingent equation, Publ. Res. Inst. Math. Sci. Ser. A, 3 (1967), 85—89.

[10] Kucia A., Nowak A., Some results and counterexamples on random fixed points, Trans. 10th Prague Conf. Inform. Theory, Statist. Decis. Functions and Random Processes, Prague 1986, Academia, Prague 1988, 75—82.

[11] Lechicki A., On continuous and measurable multifunctions, Comment. Math., 21 (1979), 141—156.

[12] Wagner D. H., Survey on measurable selection theorems, SIAM J. Control and Optim., 15 (1977), 859—903.

[13] Zygmunt W., Remarks on superpositionally measurable multifunctions (in Russian), to appear.