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# A Characterization of Midconvex Set-Valued Functions

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In [7] H. Rådström has proved that every additive set-valued function defined on $(0, \infty)$ with compact values in a locally convex topological vector space $Y$ is of the form $A(x) = a(x) + xK$, $x \in (0, \infty)$, where $a: (0, \infty) \to Y$ is an additive function and $K$ is a compact convex subset of $Y$. The purpose of this paper is to prove an analogous representation theorem for midconvex set-valued functions.

Let $X$ and $Y$ be vector spaces and $D$ be a convex subset of $X$. A set-valued function (abbreviated to s. v. function in the sequel) $F: D \to 2^Y$ is said to be convex if

$$t\,F(x) + (1 - t)\,F(y) \subset F(tx + (1 - t)\,y)$$

for all $x, y \in D$ and all $t \in [0, 1]$. We say that $F$ is midconvex (or Jensen convex) if

$$\frac{1}{2}F(x) + \frac{1}{2}F(y) \subset F\left(\frac{x + y}{2}\right) \tag{1}$$

for all $x, y \in D$ (cf. [1] and the bibliography therein). It is apparent that an s.v. function $F$ is convex (midconvex) if and only if the graph of $F$, $Gr\,F := \{(x, y) \in \in X \times Y: x \in D, y \in F(x)\}$, is a convex (midpoint convex) subset of $X \times Y$. A function $a: X \to Y$ is said to be additive if it satisfies the Cauchy functional equation

$$a(x + y) = a(x) + a(y)\,, \quad x, y \in X\,.$$

Given a topological vector space $Y$ (which always is assumed to be Hausdorff) we denote by $C(Y)$ the family of all compact non-empty subsets of $Y$ and by $CC(Y)$ the family of all compact convex and non-empty subsets of $Y$. The symbol $\mathbb{R}$ stands for the set of all reals. We say that an s.v. function $F: \mathbb{R} \to 2^Y$ is continuous if it is continuous with respect to the Hausdorff topology on $2^Y$.

The main result of this paper is the following

**Theorem.** Let $I \subset \mathbb{R}$ be an open interval and $Y$ be a locally convex space. An s.v. function $F: I \to C(Y)$ is midconvex if and only if there exist an additive functions

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$a\colon \mathbb{R} \to Y$ and a convex continuous s.v. function $G\colon I \to CC(Y)$ such that $F(x) = a(x) + G(x)$ for all $x \in I$.

We shall start from three lemmas which play a crucial role in the proof of this theorem. Recall that a function $f\colon D \to Y$ is said to be a selection of an s.v. function $F\colon D \to 2^Y$ if $f(x) \in F(x)$ for all $x \in D$. We say that a function $f\colon D \to Y$ is a Jensen function if it satisfies the Jensen functional equation

$$f\left(\frac{x+y}{2}\right) = \frac{1}{2}\left[f(x) + f(y)\right], \quad x, y \in D. \tag{2}$$

If the equality sign in (2) is replaced by "$\leqq$" ("$\geqq$") we say that $f$ is midconvex (midconcave).

**Lemma 1.** Every midconvex s.v. function $F\colon I \to C(\mathbb{R})$, where $I \subset \mathbb{R}$ is an open interval, admits a Jensen selection.

**Proof.** Assume that $F\colon I \to C(\mathbb{R})$ is midconvex and consider the functions $f_1, f_2\colon I \to \mathbb{R}$ defined by

$$f_1(x) := \inf F(x), \quad f_2(x) := \sup F(x), \quad x \in I.$$

It is easy to check that $f_1$ is midconvex and $f_2$ is midconcave. Moreover $f_1 \leqq f_2$ on $I$ If $I = \mathbb{R}$, then $f_1$ must be of the form $f_1(x) = a(x) + c$, $x \in \mathbb{R}$, where $a\colon \mathbb{R} \to \mathbb{R}$ is an additive function and $c$ is a real constant (cf. [5, Th. 2]). Therefore $f_1$ is a Jensen selection of $F$. Now let us assume that $I = (\alpha, \beta)$ where $\alpha > -\infty$ and $\beta \leqq +\infty$ (the proof in the case where $\alpha \geqq -\infty$ and $\beta < +\infty$ is analogous). Then there exist an additive function $a\colon \mathbb{R} \to \mathbb{R}$, a convex function $g_1\colon I \to \mathbb{R}$ and a concave function $g_2\colon I \to \mathbb{R}$ such that

$$f_1(x) = a(x) + g_1(x) \quad \text{and} \quad f_2(x) = a(x) + g_2(x) \tag{3}$$

for all $x \in I$ (cf. [3] or [5]). Let us extend the functions $g_1$, $g_2$ on $[\alpha, \beta)$ putting

$$g_1(\alpha) := \lim_{x \to \alpha^+} g_1(x) \quad \text{and} \quad g_2(\alpha) := \lim_{x \to \alpha^+} g_2(x)$$

(these limits exist and are finite because $g_1$ is convex, $g_2$ is concave and $g_1 \leqq g_2$ on $I$). Using the fact that the differences quotiens of convex (concave) functions are increasing (decreasing) we get for all $x \in I$

$$\lim_{x \to \beta^-} \frac{g_1(x) - g_1(\alpha)}{x - \alpha} \geqq \frac{g_1(x) - g_1(\alpha)}{x - \alpha} \tag{4}$$

and

$$\lim_{x \to \beta^-} \frac{g_1(x) - g_1(\alpha)}{x - \alpha} \leqq \lim_{x \to \beta^-} \frac{g_2(x) - g_2(\alpha)}{x - \alpha} + \lim_{x \to \beta^-} \frac{g_2(\alpha) - g_1(\alpha)}{x - \alpha} \leqq \tag{5}$$

$$\leqq \frac{g_1(x) - g_2(\alpha)}{x - \alpha} + \frac{g_2(\alpha) - g_1(\alpha)}{x - \alpha} = \frac{g_2(x) - g_1(\alpha)}{x - \alpha}.$$

Let us put

$$m := \lim_{x \to \beta^-} \frac{g_1(x) - g_1(\alpha)}{x - \alpha}$$

and consider the function $f: I \to \mathbb{R}$ defined by

$$f(x) := a(x) + m(x - \alpha) + g_1(\alpha), \quad x \in I .$$

Clearly, $f$ is a Jensen function. Moreover, by (3), (4) and (5),

$$(6) \qquad f_1(x) \leqq f(x) \leqq f_2(x) \quad \text{for all} \quad x \in I .$$

Since for every $x \in I$ the set $F(x)$ is closed and, in view of (1),

$$F(x) = F\left(\frac{x + x}{2}\right) \subset \tfrac{1}{2}[F(x) + F(x)] ,$$

we infer that $F(x)$ is convex. Therefore $F(x) = [f_1(x), f_2(x)]$, $x \in I$. This, together with (6), shows that $f$ is a selection of $F$ ::

**Remark 1.** A midconvex s.v. function $F$: $D \to C(\mathbb{R})$, where $D$ is a convex subset of $\mathbb{R}^n$, $n \geqq 2$, need not possesses any Jensen selection. For instance, let $D := \{(x, y) \in \in \mathbb{R}^2: |x| + |y| \leqq 1\}$ and let $S \subset \mathbb{R}^3$ be the simplex with vertices $(-1, 0, 0)$, $(1, 0, 0)$, $(0, -1, 1)$ and $(0, 1, 1)$. Then the s.v. function $F$: $D \to C(\mathbb{R})$ whose graph is equal to $S$ is convex and has no Jensen selection.

The idea of the proof of the next lemma is due to A. Smajdor and W. Smajdor (cf. [8]).

**Lemma 2.** Let $D$ be a convex subset of a vector space and let $Y$ be a locally convex space. If every midconvex s.v. function $F$: $D \to C(\mathbb{R})$ has a Jensen selection, then every midconvex s.v. function $F$: $D \to C(Y)$ has a Jensen selection.

**Proof.** Let $F$: $D \to C(Y)$ be a midconvex s.v. function. Consider the family $\mathscr{F} := \{G\colon D \to C(Y) : G$ is midconvex and $G(x) \subset F(x)$, $x \in D\}$ endowed with a partial order $\prec$ defined by $G_1 \prec G_2 :\Leftrightarrow G_1(x) \subset G_2(x)$, $x \in D$. Every chain $\mathscr{L}$ of elements of $\mathscr{F}$ is lower bounded by the s.v. function $H$: $D \to C(Y)$ given by $H(x) := \bigcap_{G \in \mathscr{L}} G(x)$, $x \in D$. So, by the lemma of Kuratowski-Zorn, there exists a minimal element $F_0$ of $\mathscr{F}$. We shall show that $F_0$ is single-valued. For the indirect proof suppose that for some $x_0 \in D$ there exist two points $y_1, y_2 \in F_0(x_0)$, $y_1 \neq y_2$. Take a linear continuous functional $y^*$: $Y \to \mathbb{R}$ such that $y^*(y_1) \neq y^*(y_2)$ and put $F^*(x) := y^*(F_0(x))$, $x \in D$. Then $F^*$: $D \to C(\mathbb{R})$ and it is midconvex. Therefore, by the assumption, there exists a Jensen selection $f$: $D \to \mathbb{R}$ of $F^*$. Consider the s.v. function $F_1$: $D \to C(Y)$ defined by $F_1(x) := F_0(x) \cap y^{*-1}(f(x))$, $x \in D$. This s.v. function is midconvex, $F_1(x) \subset F_0(x)$ for all $x \in D$ and $F_1(x_0) \neq F_0(x_0)$, which contradicts the minimality of $F_0$. Thus $F_0$, being midconvex and single-valued, is a Jensen selection of $F$ ::

The next lemma gives some condition under which midconvex s.v. functions are continuous.

**Lemma 3.** ([6, Cor. 3.1 for $K = \{0\}$). Let $X$, $Y$ be topological vector spaces and $D$ be an open convex subset of $X$. Assume that $F\colon D \to C(Y)$ is midconvex s.v. function and $f\colon D \to Y$ is its selection. If $f$ is continuous at a point of $D$, then $F$ is continuous on $D$.

**Proof of Theorem.** Let $F\colon I \to C(Y)$ be a midconvex s.v. function. Notice first that $F$ is convex-valued. Indeed, for every $x \in I$ the set $F(x)$ is closed and $F(x) = F(\frac{1}{2}(x + x)) \subset \frac{1}{2}[F(x) + F(x)]$. This implies that $F(x)$ is convex. In view of Lemma 1 and Lemma 2 there exists a Jensen selection $f\colon I \to Y$ of $F$. Being a Jensen function $f$ is of the form $f(x) = a(x) + c$, $x \in I$, where $a\colon \mathbb{R} \to Y$ is an additive function and $c \in Y$ (cf. [2, Lemma 2]; to be sure, the lemma is formulated for real-valued functions but the proof given there holds for vector-valued functions, too). Consider the s.v. function $G\colon I \to G(Y)$ defined by $G(x) := F(x) - a(x)$, $x \in I$. This s.v. function is midconvex and the constant function $c$ yields its continuous selection. Therefore, by Lemma 3, $G$ is continuous on $I$ and hence it is convex ([4, Th. 2]). Thus $F$ is of the required form. The converse implication is obvious ::

**Remark 2.** Recently A. Smajdor and W. Smajdor proved [8] that every midconvex s.v. function $F\colon K \cup \{0\} \to C(Y)$, where $Y$ is a locally convex space and $K$ is an open convex cone in a locally convex space, has a Jensen selection. Using the same method as in the proof of our Theorem we can show that such s.v. functions can be also representated in the form $F = a + G$ with an additive function $a$ and a convex and continuous on $K$ s.v. function $G$.

**Remark 3.** If an s.v. function $A\colon (0, \infty) \to C(Y)$, where $Y$ is a locally convex space, is additive, then it is convex-valued (cf. [7]). Consequently, it is midconvex because

$$2A\left(\frac{x+y}{2}\right) = A\left(\frac{x+y}{2}\right) + A\left(\frac{x+y}{2}\right) = A(x+y) = A(x) + A(y),$$
$$x, y \in (0, \infty).$$

On the other hand, additive and continuous s.v. functions $A\colon (0, \infty) \to CC(Y)$ are of the form $A(x) = x\,A(1)$, $x \in (0, \infty)$. Using these facts we can obtain the theorem of Râdström mentioned at the beginning as a consequence of our Theorem.

## References


[1] BORWEIN J. M.: Convex relations in analysis and optimization. In: Generalized concavity in optimization and economics (ed. S. Schaible, W. Ziemba) Academic Press, New York 1981, 335—377.

[2] GER R., KOMINEK Z.: Boundedness and continuity of additive and convex functionals, Aequantions Math. (to appear).

[3] NG C. T.: On midconvex functions with midconcave bounds, Proc. Amer. Math. Soc. 102 (1988), 538—540.

[4] NIKODEM K.: On midpoint convex set-valued functions, Aequationes Math. 33 (1987), 46—56.

[5] NIKODEM K.: Midpoint convex functions majorized by midpoint concave functions, Aequationes Math. 32 (1987), 45—51.

[6] NIKODEM K.: K-convex and K-concave set-valued functions, Zeszyty Nauk. Politech. Łódz. Mat. (to appear).

[7] RÅDSTRÖM H.: One-parameter semigroups of subsets of a real linear space, Ark. Mat. 4 (1960), 87—97.

[8] SMAJDOR A., SMAJDOR W.: Affine selections of midpoint convex set-valued functions (to appear).