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Polynomials and Holomorphic Functions on Interpolation Spaces

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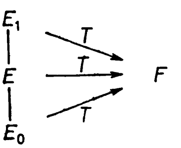
1. Introduction

Let E_0 and E_1 be any Banach spaces. For sake of simplicity we will suppose that E_0 is continuously embedded into E_1 via some linear bounded map $J: E_0 \to E_1$. Now, let E be any intermediate space between E_0 and E_1, i.e. $E_0 \subseteq E \subseteq E_1$. Suppose that we are given a diagram of the form



where T is some operator into F defined simultaneously on E_0, E and E_1, respectively. Now we can ask the question, to what extent the behaviour of T on E is determined by the properties of T on E_0 and E_1. For linear operator this question is anwered by the interpolation theory developped by Lions and Peetre during the early sixties. In this paper we are interested in nonlinear mappings. More precisely, we are interested in estimations of the norm of polynomials on intermediate spaces and in the property of the uniform boundedness of holomorphic functions. Other properties as compactness or σ-compactness have been studied in [4] and [6]. Recall that an intermediate Banach space E is called to be of interpolation type $K(\Theta, E_0, E_1)$ for some parameter $0 < \Theta < 1$, if there is some constant $C > 0$ such that

$$\|x\|_E \geqq C \inf \{t^{-\Theta}\|x_0\|_{E_0} + t^{1-\Theta}\|x_1\|_{E_1}: x = x_0 + x_1\}$$

holds true for all $t > 0$ and for all $x \in E$. In our context the following characterization is more important: An intermediate Banach space E is of type $K(\Theta, E_0, E_1)$ iff there is some constant $C > 0$ such that for all linear bounded operators $T \in \mathscr{L}(E_1, F)$

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the following estimation holds true

$$\|T: E \to F\| \leqq C\|T: E_0 \to F\|^{1-\theta} \|T: E_1 \to F\|^{\theta} .$$

Concerning a proof we refer to [1].

2. Polynomials and holomorphic functions of uniformly bounded type

Let us recall some necessary definitions. For all Banach spaces $E_1, \ldots, E_m$ and F we denote by $\mathscr{L}(E_1, \ldots, E_m; F)$ the Banach space of all m-linear continuous operators $T: E_1 \times \ldots \times E_m \to F$ endowed with the norm

$$\|T\| = \sup \{\|T(x_1, \ldots, x_m)\|: x_i \in E_i, \|x_i\| \leqq 1\} .$$

The class of all m-linear operators is denoted by $\mathscr{L}^m$. For any m-linear operator $T: E \times \ldots \times E \to F$ the function

$$\hat{T}(x) = T(x, \ldots, x) , \quad x \in E$$

is called the m-homogeneous polynomial associated to T. The space

$$\mathscr{P}(^m(E; F)) = \{P: P = \hat{T} \text{ for some } T \in \mathscr{L}(E, \ldots, E; F)\}$$

of all F-valued m-homogeneous continuous polynomials on E becomes a Banach space with respect to the norm

$$\|\hat{T}\| = \sup \{\|T(x, \ldots, x)\|: \|x\| \leqq 1\} .$$

For $F = \mathrm{C}$ we simply write $\mathscr{P}(^mE)$ instead of $\mathscr{P}(^mE, \mathrm{C})$. For details we refer to [3].

Let E be any complex Banach space and let G be any open subset of E. Recall that a function $f: G \to \mathrm{C}$ is called to be holomorphic on G, if it is Fréchet-differentiable at each point $x \in G$. Equivalently, f is holomorphic in G, if for each $x \in G$ there is a sequence of continuous m-homogeneous polynomials $P_{m,x}$ such that

$$f(x + h) = \sum_{m=0}^{\infty} P_{m,x}(h)$$

converges uniformly for h in some neighbourhood of x. According to Cauchy's formulae, the Taylor polynomials can be computed by

$$P_{m,x}(h) = \frac{1}{2\pi i} \oint_{|\lambda| = \delta} \frac{f(x + \lambda h)}{\lambda^{m+1}} \, d\lambda .$$

The function f is called an entire function on E if it is Fréchet-differentiable at each point of E. The set of all entire functions on E is denoted by $\mathscr{H}(E)$. The number

$$R(f, x) = \limsup_{m \to \infty} \|P_{m,x}\|^{-1/m}$$

is called the radius of uniform convergence of f at x. From now on we suppose $x = 0$. Cauchy's formulae implies

$$\|P_{m,0}\|_E \leqq \frac{1}{\varrho^m} \|f\|_{\varrho S_E},$$

where S_E denotes the unit ball in E. Therefore, the radius of convergence can also be computed by

$$R(f, 0) = \sup \{\varrho > 0 \colon f \text{ is bounded on } \varrho S_E\}.$$

The set $\mathscr{H}_{ub}(E) = \{f \in \mathscr{H}(E) \colon R(f, 0) = \infty\}$ is called the set of all uniformly bounded holomorphic functions. Now, it is very important that in contrast to the finite dimensional case not every entire function has an infinite radius of convergence. Let us consider an example. For all $x = (\xi_j) \in c_0$ we define

$$f(x) = \sum_{n=1}^{\infty} \xi_1 \cdot \ldots \cdot \xi_n .$$

It is easy to see that f is entire on c_0, but it is not bounded on the unit ball S_{c_0}, since $f(\sum_{i=1}^{n} e_i) = n$. Therefore, $R(f, 0) = 1$. But on l_1 we obtain $f \in \mathscr{H}_{ub}(l_1)$. In fact, if N is any natural number, then $\|x\|_1 \leqq N$ implies that at most N coordinates ξ_j of x are larger than 1. This implies $|\xi_1 \cdot \ldots \cdot \xi_l| \leqq N^N |\xi_l|$ for all $l \in \mathrm{N}$. Therefore,

$$|f(x)| \leqq N^N \sum |\xi_l| \leqq N^{N+1} .$$

This shows $R(f, 0) = \infty$. Obviously, the same argument works on each l_p for $1 \leqq p < \infty$. This indicates that interpolation could preserve uniformly boundedness. This problem will be studied now.

Proposition 1. Suppose that E_0 is embedded into E_1 via some linear compact operator T. Let E be any intermediate space of type $K(\Theta, E_0, E_1)$. Then each entire function $f \in \mathscr{H}(E_1)$ is of uniformly bounded type on E.

Proof. By Heinrich's result [5], the embedding of E into E_1 is also compact. Therefore, each ball ϱS_E is relatively compact in E_1. This implies the boundedness of $f(\varrho S_E)$, since f is continuous. Hence, $R_E(f, 0) = \infty$.

Next, let us study the more restrictive case, where the maps $f \in \mathscr{H}(E_0)$ admit only a local extension to E_1. To handle this case we will use the following definition.

Definition. Let E_0 be continuously emdedded into E_1. We will say that an intermediate space E is of polynomial type $K_P(\Theta, E_0, E_1)$, if there is some constant C not depending on m, such that

$$\|P\|_E \leqq C^m \|P\|_{E_0}^{1-\Theta} \|P\|_{E_1}^{\Theta}$$

holds true for all $m \in \mathrm{N}$ and all $P \in \mathscr{P}(^m E)$.

Proposition 2. Let E be a Banach space of polynomial type $K_P(\Theta, E_0, E_1)$. Suppose that $f \in \mathscr{H}_{ub}(E_0)$ admits a holomorphic extension g in some zero-neighbourhood W of E_1. Then f admits a uniformly bounded extension on E.

Proof. Let $g(x) = \sum_{m=0}^{\infty} P_m(x)$ be the Taylor series expansion of g on $W \subseteq E_1$. Since the Taylor polynomials are uniquely determined, this formulae can also be considered as the expansion of f on E_0. Now we can compute the radius of uniform convergence of f on E by the rstimation

$$R_E(f, 0) = \limsup \|P_m\|_E^{-1/m} \leqq \limsup C\|P_m\|_{E_0}^{-(1-\Theta)/m} \|P_m\|_{E_1}^{-\Theta/m} \leqq$$
$$\leqq C\, R_{E_0}(f, 0)^{1-\Theta} R_{E_1}(f, 0)^{\Theta} = \infty\,.$$

Next, let us consider two important cases of polynomial type interpolation. An interpolation space E is called to be of multilinear type $K_m(\Theta, E_0, E_1)$, if

$$\|M\|_E \leqq \|M\|_{E_0}^{1-\Theta} \|M\|_{E_1}^{\Theta}$$

holds true for each multilinear form $M \in \mathscr{L}^m(E_1)$. Because of

$$\|\hat{M}\| \leqq \|M\| \leqq \frac{m^m}{m!} \|\hat{M}\| \leqq e^m \|\hat{M}\|\,,$$

where $\hat{M}$ denotes the polynomial associated to M, each interpolation space of multilinear type is even of polynomial type. Although not any interpolation space E of type $K(\Theta, E_0, E_1)$ is of multilinear type $K_m(\Theta, E_0, E_1)$, so at least the complex interpolation method leads to spaces of multilinear type. This has been proved by Calderon in 1961 (cf. [10, 1.19.4] or [2, 4.4]). Since

$$l_p = [l_1, c_0]_{\Theta} \quad \text{for} \quad 1 \leqq p < \infty\,, \quad p = 1/\Theta\,,$$

we get from Proposition 2 the following Corollary.

Corollary. If $f \in \mathscr{H}(c_0)$ is of uniformly bounded type on l_1, then it is uniformly bounded on each l_p for $1 \leqq p < \infty$.

For other couples of spaces E_0 and E_1 we need some additional informations about the quality of the embedding map. Recall that a linear bounded operator A mapping a Hilbert space into another Hilbert space is called to be a Schatten class operator of type $0 < p < \infty$, if it admits a representation

$$A = \sum_{j=1}^{\infty} \lambda_j e_j \otimes y_j$$

with some orthonormal systems (e_j) and (y_j) and with some sequence $(\lambda_j) \in l_p$.

Proposition 3. Let E_0 and E_1 be any Hilbert spaces and suppose that the embedding map $A\colon E_0 \to E_1$ is a Schatten class operator of the type $\Theta/4$ for some number $0 < \Theta < 1$. If E is any intermediate space of type $K(\Theta/2, E_0, E_1)$ then E is of polynomial type $K_P(\Theta, E_0, E_1)$.

Proof. Let

$$A = \sum_{j=1}^{\infty} \lambda_j e_j \otimes y_j$$

be any representation of A, where (λ_j) is a decreasing sequence of reals belonging to $l_{\theta/4}$ and where (e_j) and (y_j) are orthonormal systems in E_0 and E_1, respectively. We may suppose that (e_j) and (y_j) are even complete orthonormal systems. Let P_1 be any m-homogeneous polynomial on E_1 and let $P_0 = P_1 A$ be the restriction of P_1 to E_0. Following an idea of Meise/Vogt [7], we will estimate the polynomial as follows. Let B be the linear hull of $\{y_j : j \in \mathrm{N}\}$. For every $y \in B$ the Fourier series

$$y = \sum_{j=1}^{\infty} (y, y_j)\, y_j$$

is actually a finite sum. Using Newton's formulae for m-homogeneous polynomials, we get

$$P_1(y) = P_1\Big(\sum_j (y, y_j)\, y_j\Big) = \sum_{|\mathfrak{m}|=m} a_{\mathfrak{m}} \prod_{j=1}^{\infty} (y, y_j)^{m_j} \tag{1}$$

for all $y \in B$, where $\mathfrak{m} = (m_1, m_2, \dots, m_n, 0, \dots) \in \mathrm{N}^{(\mathrm{N})}$ runs through all multi-indices of arbitrary length of the degree $|\mathfrak{m}| = \sum_j m_j = m$. To satisfy the use of the infinite product we settle $0^0 = 1$.

Now, for every fixed multi-index $\mathfrak{m}$ the coefficient $a_{\mathfrak{m}}$ can be computed by Cauchy's Integral formulae

$$a_{\mathfrak{m}} = \frac{1}{(2\pi i)^n} \oint_{|\varrho_j| = \mu_j} \cdots \oint \frac{P_1\big(\sum_{j=1}^{n} \varrho_j y_j\big)}{\varrho_1^{m_1+1} \cdot \ldots \cdot \varrho_n^{m_n+1}}\, d\varrho_1 \cdot \ldots \cdot d\varrho_n . \tag{2}$$

If we use the abbreviation

$$\mu^{\mathfrak{m}} = \prod_{j=1}^{\infty} \mu_j^{m_j},$$

then we get the estimation

$$|a_{\mathfrak{m}}| \leqq \frac{1}{\mu^{\mathfrak{m}}} \sup_{|\varrho_j| \leqq \mu_j} \Big|P_1\Big(\sum \varrho_j y_j\Big)\Big| .$$

If we put

$$\mu_j = \frac{1}{j} \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right)^{-1},$$

we obtain

$$\sum \|\varrho_j y_j\|^2 = \sum |\varrho_j|^2 \leqq 1 \quad \text{for} \quad |\varrho_j| \leqq \mu_j .$$

Thus we can majorize the supremum by the norm of P_1 on E_1. This leads to

$$|a_{\mathfrak{m}}| \leqq \frac{1}{\mu^{\mathfrak{m}}} \|P_1\|_{E_1} . \tag{3}$$

On the other hand, we can estimate the $a_{\mathfrak{m}}$ by the norm of $P_0 = P_1 A$ on E_0. Since $y_j = A\lambda_j^{-1}x_j$, we get from (1) by the integral transform $\varrho_j\lambda_j^{-1} = \sigma_j$ the equation

$$(4) \qquad a_{\mathfrak{m}} = \frac{1}{(2\pi i)^n}\frac{1}{\lambda^{\mathfrak{m}}}\oint_{|\sigma_j|=\mu_j}\cdots\oint \frac{P_0(\sum \sigma_i x_i)}{\sigma_1^{m_1+1}\cdot\ldots\cdot\sigma_n^{m_n+1}}\,d\sigma_1\cdot\ldots\cdot d\sigma_n\,,$$

which yields the inequality

$$(5) \qquad |a_{\mathfrak{m}}| \leqq \frac{1}{\lambda^{\mathfrak{m}}\mu^{\mathfrak{m}}}\,\|P_0\|_{E_0}\,.$$

Now, we estimate the restriction P of P_1 to E as follows. If J denotes the embedding map of E into E_1, we obtain from (1) the representation

$$(6) \qquad P(x) = \sum_{|\mathfrak{m}|=m} a_{\mathfrak{m}}\prod_{j=1}^{\infty}(Jx, y_j)^{m_j} = \sum_{|\mathfrak{m}|=m} a_{\mathfrak{m}}\prod_{j=1}^{\infty}\langle x, J^*y_j\rangle^{m_j}\,.$$

Since E is supposed to be of type $K(\Theta/2, E_0, E_1)$, the norm of the linear functional $J^*y_j \in E^*$ can be estimated by

$$\|J^*y_j\|_{E^*} \leqq C\|A^*y_j\|^{1-\Theta/2}\,\|y_j\|^{\Theta/2} = C\lambda_j^{1-\Theta/2}\,.$$

For $\|x\| \leqq 1$, the inequalities (3), (5) and (6) imply

$$|P(x)| \leqq \sum_{|\mathfrak{m}|=m}|a_{\mathfrak{m}}|\prod_{j=1}^{\infty}\|J^*y_j\|_{E^*}^{m_j} \leqq \sum_{|\mathfrak{m}|=m}|a_{\mathfrak{m}}|\,C^{\mathfrak{m}}\lambda^{\mathfrak{m}(1-\Theta/2)} \leqq$$

$$\leqq \|P_0\|_{E_0}^{1-\Theta}\,\|P_1\|_{E_1}^{\Theta}\sum_{|\mathfrak{m}|=m}\left(\frac{C\lambda^{\Theta/2}}{\mu}\right)^{\mathfrak{m}}.$$

The final sum is bounded by some constant $\varkappa > 0$ which is independent of the degree m as one can see by the following computation:

$$\sum_{|\mathfrak{m}|=m}\left(\frac{C\lambda^{\Theta/2}}{\mu}\right)^{\mathfrak{m}} \leqq \prod_{j=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{C\lambda_j^{\Theta/2}}{\mu_j}\right)^k = \prod_{j=1}^{\infty}\left(1-\frac{C\lambda_j^{\Theta/2}}{\mu_j}\right)^{-1} = \varkappa\,.$$

The number $\varkappa$ is indeed finite, since $(\lambda_j) \in l_{\Theta/4}$ implies $(\lambda_j^{\Theta/2}/\mu_j) \leqq 2(j\lambda_j^{\Theta/2}) \in l_1$.

Corollary. Under the assumptions of Proposition 3, each uniformly bounded holomorphic function on E_0 which admits a holomorphic extension to some zero-neighbourhood of E_1, is of uniformly bounded type on E.

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