Martin Markl
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HOMOTOPY DIAGRAMS OF ALGEBRAS

MARTIN MARKL

ABSTRACT. In [7] we proved that strongly homotopy (sh) algebras are homotopy invariant concepts in the category of chain complexes. Our arguments were based on the fact that strongly homotopy algebras are algebras over minimal cofibrant operads and on the principle that algebras over cofibrant operads are homotopy invariant. In our approach, algebraic models for colored operads describing diagrams of homomorphisms played an important rôle.

The aim of this paper is to give an explicit description of these models. This description is necessary for practical applications of some of the conceptual results of [7]. Another possible application is an appropriate formulation of the homological perturbation lemma for chain complexes with algebraic structures in the spirit of [8]. Our results also provide a conceptual approach to 'homotopies through homomorphisms' for strongly homotopy algebras. We also argue that strongly homotopy algebras form an honest (not only weak Kan) category.

As an 'application of an application,' we find the homotopy structure on the category of strongly homotopy associative algebras and their strongly homotopy homomorphisms described in [3, 5.4].

The paper can be understood as a continuation of our program to translate [1] to algebra. We recommend looking at Section 6 first to get a broader perspective on methods, results and implications of this technical paper.

PLAN OF THE PAPER (in place of Introduction)

Section 1: Notation and terminology. We recall standard facts concerning (colored) operads and their (minimal) models.

Section 2: Minimal model for a homomorphism. We describe the minimal model $\mathfrak{M}_{\mathtt{B} \to \mathtt{W}}$ of the colored operad $\mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ describing a homomorphism f of $\mathcal{A}$-algebras. The minimal operad $\mathfrak{M}_{\mathtt{B} \to \mathtt{W}}$ encapsulates strongly homotopy homomorphisms of strongly homotopy algebras. The main result is Theorem 7, the rest of this section is occupied by its proof.

Section 3: Some applications of Theorem 7 are given. We prove a theorem about extensions of strongly homotopy algebra structures (Theorem 15). From this we derive that each multiplication homotopic to a strictly associative one extends to

an $A(\infty)$-structure (Corollary 16). We also prove that the abelianization of an associative homotopy commutative multiplication can be extended to a balanced strongly homotopy associative algebra (Corollary 17). The remaining part of the paper is independent on this section.

Section 4: Homotopy through homomorphisms. We construct, in Theorem 18, another model $\mathfrak{M}_{\mathsf{B}\rightleftarrows\mathsf{u}}$ of $\mathcal{A}_{\mathsf{B}\to\mathsf{u}}$ with two independent generators for f. Representations of $\mathfrak{M}_{\mathsf{B}\rightleftarrows\mathsf{u}}$ describe homotopies of homomorphisms of homotopy algebras. While it is clear what this 'homotopy through homomorphisms' means in topology, we will see that the situation in algebra is much subtler.

Section 5: Strong homotopy equivalences of algebras. In Theorem 24 we describe a model $\mathfrak{M}_{\mathcal{I}so}$ of the colored operad $\mathcal{A}_{\mathcal{I}so}$ for two mutually inverse homomorphism of $\mathcal{A}$-algebras. We call colored algebras over $\mathfrak{M}_{\mathcal{I}so}$ strong homotopy equivalences of strongly homotopy algebras (Definition 27).

Section 6: Final remarks and challenges. We discuss possible applications and generalizations of the methods developed in this paper. Namely, we discuss the 'category' of strongly homotopy algebras, 'good' homotopy equivalences and the ideal homological perturbation lemma, and homotopies through homomorphisms. Then we propose a possible generalization of the main results of the paper.

1. NOTATION AND TERMINOLOGY

As a reference for standard terminology concerning operads, collections, ideals, presentations etc., we recommend [9, 2] and [6]. Algebraic models of colored operads were studied in [7, 8]. A map will be called a *quasi-isomorphism* or a *quism* if it induces an isomorphism of cohomology.

If not said otherwise, all algebraic objects will be defined over a field k of *characteristic zero*. This assumptions is necessary because maps between spaces of operatoins of arity n must be Σ_n-invariant, which is achieved by 'averaging' over Σ_n with $n!$ in the denominator. In non-Σ situations some objects, however, exist over an arbitrary field or even over the integers. This is the case of $A(\infty)$-algebras and their diagrams presented in Examples 4, 12 and 21.

Let us recall *colored* operads describing diagrams of algebras. Fix a (finite) *set of colors* $\mathfrak{C}$ and consider an operad $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 1}$ such that each $\mathcal{P}(n)$ decomposes as the direct sum

$$(1) \qquad\qquad \mathcal{P}(n) = \bigoplus \mathcal{P}\begin{pmatrix} c \\ c_1, \dots, c_n \end{pmatrix},$$

where the summation runs over all colors $c, c_1, \dots, c_n \in \mathfrak{C}$. We require the decomposition (1) to be, in the obvious sense, Σ_n-equivariant.

We also demand the following. Let

$$x \in \mathcal{P}\begin{pmatrix} c \\ c_1, \dots, c_n \end{pmatrix} \text{ and } x_i \in \mathcal{P}\begin{pmatrix} d_i \\ d_1^i, \dots, d_{k_i}^i \end{pmatrix}, \ 1 \leq i \leq n.$$

Then we require that the non-triviality of the composition $x(x_1, \dots, x_n)$ implies that

$$(2) \qquad\qquad d_i = c_i, \text{ for } 1 \leq i \leq n,$$

in which case

$$x(x_1, \ldots, x_n) \in \mathcal{P} \begin{pmatrix} & & c & & \\ d_1^1, \ldots, d_{k_1}^1, \ldots, d_1^n, \ldots, d_{k_n}^n \end{pmatrix}.$$

The intuitive meaning of (2) is that one may plug the element x_i into the i-th slot of the element x if and only if the color of the output of x_i is the same as the color of the i-th input of x, otherwise the composition is defined to be zero.

An example is provided by the *colored endomorphism operad* $\mathcal{E}nd_{\mathbf{U}}$ on a 'colored' chain complex $\mathbf{U} = \bigoplus_{c \in C} U_c$ where we put, in (1),

$$\mathcal{E}nd_{\mathbf{U}} \begin{pmatrix} c \\ c_1, \ldots, c_n \end{pmatrix} := \mathrm{Hom}(U_{c_1} \otimes \cdots \otimes U_{c_n}, U_c).$$

A *colored algebra* over a colored operad $\mathcal{P}$ is then a map of operads $A : \mathcal{P} \to \mathcal{E}nd_{\mathbf{U}}$. We also sometimes say that A is a *representation* of $\mathcal{P}$.

Example 1. For an ordinary dg (= differential graded) operad $\mathcal{A} = (\mathcal{A}, d_{\mathcal{A}})$, define the $\{\mathsf{B}, \mathsf{W}\}$-colored operad $\mathcal{A}_{\mathsf{B} \to \mathsf{W}}$ as

$$(3) \qquad \mathcal{A}_{\mathsf{B} \to \mathsf{W}} := \frac{\mathcal{A}_{\mathsf{B}} * \mathcal{A}_{\mathsf{W}} * \mathcal{F}(f)}{(fa_{\mathsf{B}} = a_{\mathsf{W}} f^{\otimes n}, \ \forall a \in \mathcal{A}(n), \ n \geq 1)}.$$

In this formula, $\mathcal{A}_{\mathsf{B}}$ (resp. $\mathcal{A}_{\mathsf{W}}$) denotes the copy of $\mathcal{A}$ 'concentrated' in the color B (resp. W). The symbol f is a new generator, $f : \mathsf{B} \to \mathsf{W}$, and $\mathcal{F}(f)$ denotes the free colored operad generated by f. In fact, $\mathcal{F}(f)$ consist only of f, since there is no way to compose f with itself. The asterix $*$ denotes the free product of colored operads and the equation $fa_{\mathsf{B}} = a_{\mathsf{W}} f^{\otimes n}$ generating the ideal in the denominator of (3) expresses the fact that f commutes with all operations of the operad $\mathcal{A}$.

There exists a differential on $\mathcal{A}_{\mathsf{B}} * \mathcal{A}_{\mathsf{W}} * \mathcal{F}(f)$ induced by $d_{\mathcal{A}}$ on $\mathcal{A}_{\mathsf{B}}, \mathcal{A}_{\mathsf{W}}$ and trivial on $\mathcal{F}(f)$ which descends to a differential $d = d_{\mathcal{A}_{\mathsf{B} \to \mathsf{W}}}$ on $\mathcal{A}_{\mathsf{B} \to \mathsf{W}}$. An algebra over $\mathcal{A}_{\mathsf{B} \to \mathsf{W}}$ consists of two $\mathcal{A}$-algebras U and V and a chain map $f : U \to V$ which is also an $\mathcal{A}$-homomorphism.

The following definition, introduced for ordinary operads in [6], is crucial.

Definition 2. Let $\mathcal{A} = (\mathcal{A}, d_{\mathcal{A}})$ be an (ordinary or colored) dg operad. A minimal model of $\mathcal{A}$ is a differential graded operad $\mathfrak{M}_{\mathcal{A}} = (\mathcal{F}(E), d_{\mathfrak{M}})$, where $\mathcal{F}(E)$ is the free operad on a collection E, together with a map $\alpha_{\mathcal{A}} : \mathfrak{M}_{\mathcal{A}} \to \mathcal{A}$ of dg operads such that

(i) $\alpha_{\mathcal{A}} : \mathfrak{M}_{\mathcal{A}} \to \mathcal{A}$ is a quasi-isomorphism, and

(ii) $d_{\mathfrak{M}}(E)$ consists of decomposable elements of the free operad $\mathcal{F}(E)$ (the minimality).

In [6, Theorem 2.1] we proved the following theorem.

Theorem 3. *Let* $\mathcal{A} = (\mathcal{A}, d_{\mathcal{A}})$ *be an (ordinary) operad with*

$$(4) \qquad H_*(\mathcal{A}(1), d_{\mathcal{A}}) \cong \mathbf{k}.$$

Then there exists a minimal model $\rho : (\mathcal{F}(X), d_{\mathfrak{M}}) \to (\mathcal{A}, d_{\mathcal{A}})$, *unique up to isomorphism.*

In fact, in [6, Theorem 2.1] we assumed that $\mathcal{A}(1) \cong \mathbf{k}$, but it is immediate to see that each $\mathcal{A}$ satisfying (4) is quasi-isomorphic to some $\bar{\mathcal{A}}$ with $\bar{\mathcal{A}}(1) \cong \mathbf{k}$. The lifting property of minimal models guarantee that the minimal model of $\bar{\mathcal{A}}$ is also a minimal model of $\mathcal{A}$.

It can be immediately seen that the generators of the minimal model from Theorem 3 satisfy $X(1) = 0$, that is, there are no generators of arity 1.

As we explained in [7], the minimal model of the operad $\mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ (unique up to an isomorphism, by an easy modification of [6, Theorem 2.1]) describes *strongly homotopy $\mathcal{A}$-algebras* and their *strongly homotopy maps*. This is illustrated in the following example, taken again from [7].

Example 4. Let $Ass := \mathcal{F}(\mu)/(\mu(\mu \otimes \mathbb{1}) - \mu(\mathbb{1} \otimes \mu))$ be the operad for associative (non-unital) algebras. Then the minimal model of the operad

$$Ass_{\mathtt{B} \to \mathtt{W}} = \mathcal{F}(\mu, \nu, f)/(\mu(\mu \otimes \mathbb{1}) = \mu(\mathbb{1} \otimes \mu), \ \nu(\nu \otimes \mathbb{1}) = \nu(\mathbb{1} \otimes \nu), \ f\mu = \nu f^{\otimes 2})$$

is given by

$$(5) \qquad Ass_{\mathtt{B} \to \mathtt{W}} \xleftarrow{\alpha} \left(\mathcal{F}\Big(\mu_2, \mu_3, \mu_4, \dots, f_1, f_2, f_3, \dots, \nu_2, \nu_3, \nu_4, \dots \Big), \partial \right),$$

where

$$
\begin{aligned}
(6) \qquad & \mu_n : \mathtt{B}^{\otimes n} \to \mathtt{B} \text{ is a generator of degree } n - 2, \ n \geq 2, \\
& f_n : \mathtt{B}^{\otimes n} \to \mathtt{W} \text{ is a generator of degree } n - 1, \ n \geq 1, \text{ and} \\
& \nu_n : \mathtt{W}^{\otimes n} \to \mathtt{W} \text{ is a generator of degree } n - 2, \ n \geq 2.
\end{aligned}
$$

The map α is defined by $\alpha(\mu_2) = \mu$, $\alpha(\nu_2) = \nu$, $\alpha(f_1) = f$, while α is trivial on the remaining generators. The differential is given by

$$(7) \qquad \partial(\mu_n) := \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} \mu_i(\mathbb{1}^{\otimes s} \otimes \mu_j \otimes \mathbb{1}^{\otimes i-s-1}),$$

$$(8) \qquad \partial(f_n) := -\sum_{k=2}^{n} \sum_{r_1 + \cdots + r_k = n} (-1)^{\sum_{1 \leq i < j \leq k} r_i(r_j+1)} \nu_k(f_{r_1} \otimes \cdots \otimes f_{r_k})$$

$$\qquad \qquad - \sum_{\substack{i+j=n+1 \\ i \geq 1, \, j \geq 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} f_i(\mathbb{1}^{\otimes s} \otimes \mu_j \otimes \mathbb{1}^{\otimes i-s-1}),$$

$$(9) \qquad \partial(\nu_n) := \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} \nu_i(\mathbb{1}^{\otimes s} \otimes \nu_j \otimes \mathbb{1}^{\otimes i-s-1}),$$

Let us denote the operad on the right hand side of (5) by $A(\infty)_{\mathtt{B} \to \mathtt{W}}$. Algebras over $A(\infty)_{\mathtt{B} \to \mathtt{W}}$ are clearly $A(\infty)$-algebras and their $A(\infty)$-homomorphisms in the sense of [5].

The fact that the above object is indeed a minimal model of the bi-colored operad $A(\infty)_{\mathtt{B} \to \mathtt{W}}$ is fairly nontrivial. While it is clear that all the objects above are well-defined, we need Proposition 11 below to show that α is a quism. See also Example 12.

2. Minimal model for a homomorphism

The aim of this section is to generalize Example 4 and describe the minimal model of the $\{B,W\}$-colored operad $\mathcal{A}_{B\to W}$ for an arbitrary operad $\mathcal{A}$ satisfying (4) in terms of the minimal model of $\mathcal{A}$. We need to introduce some notation first.

Let $j_c : \mathcal{A} \to \mathcal{A}_{B\to W}$ be, for $c \in \{B,W\}$, the map induced by the identification of $\mathcal{A}$ with $\mathcal{A}_c \subset \mathcal{A}_B * \mathcal{A}_W * \mathcal{F}(f)$. It is easy to see that the map j_c is an inclusion. Construction (3) is clearly functorial, that is, every map $\beta : \mathcal{A} \to \mathcal{B}$ induces a natural map $\beta_{B\to W} : \mathcal{A}_{B\to W} \to \mathcal{B}_{B\to W}$ of colored dg operads with the property that $\beta_{B\to W}(j_c(a)) = j_c(\beta(a))$, for $a \in \mathcal{A}$ and $c \in \{B,W\}$. The following proposition describes the structure of $\mathcal{A}_{B\to W}$.

Proposition 5. *Let $c, c_1, \ldots, c_n \in \{B,W\}$ be a sequence of colors. The component*

$$(10) \qquad \mathcal{A}_{B\to W}\begin{pmatrix} c \\ c_1, \ldots, c_n \end{pmatrix}$$

of the operad $\mathcal{A}_{B\to W}$ is trivial for $c = B$ and $(c_1, \ldots, c_n) \neq (B, \ldots, B)$, while it is canonically isomorphic to $\mathcal{A}(n)$ for all other choices of colors. These canonical isomorphisms are functorial and commute with the differentials.

Proof. It is clear that (10) is trivial for $c = B$ and $(c_1, \ldots, c_n) \neq (B, \ldots, B)$, because there is no way to create a 'map' $(c_1, \ldots, c_n) \to B$ in (10) from the generators. The only nontrivial possibility with $c = B$ is

$$\mathcal{A}_B(n) \cong \mathcal{A}_{B\to W}\begin{pmatrix} B \\ B, \ldots, B \end{pmatrix},$$

which is clearly isomorphic to $\mathcal{A}(n)$.

It follows from relations in the denominator of (3) that, for $c = W$, any element a of (10) can be naturally and uniquely presented as

$$a = \overline{a}(\xi(c_1) \otimes \cdots \otimes \xi(c_n)),$$

where $\overline{a} \in \mathcal{A}_W(n) \cong \mathcal{A}(n)$ and

$$\xi(c_i) := \begin{cases} f, & \text{if } c_i = B, \text{ and} \\ \mathbb{1}_B, & \text{if } c_i = W. \end{cases}$$

Since $d(f) = 0$, the functorial correspondence $a \leftrightarrow \overline{a}$ commutes with the differentials. $\square$

Proposition 5 has the following important corollary.

Corollary 6. *Suppose that $\beta : \mathcal{A} \to \mathcal{B}$ is a quasi-isomorphism. Then the induced map $\beta_{B\to W} : \mathcal{A}_{B\to W} \to \mathcal{B}_{B\to W}$ is a quasi-isomorphism, too.*

Let $\mathcal{A} = (\mathcal{A}, d_{\mathcal{A}})$ be a dg operad satisfying (4) and let $\rho : (\mathcal{F}(X), d_{\mathfrak{M}}) \to (\mathcal{A}, d_{\mathcal{A}})$ be its minimal model (see Definition 2). Let X_c be, for $c \in \{B,W\}$, another copy of the space of generators X and let $i_c : X \overset{\cong}{\to} X_c$ be the identification. Let $\overline{X} := \uparrow X$ be the suspension of X and let $\uparrow : X \overset{\cong}{\to} \overline{X}$ be the canonical map. For $x \in X$, let $x_c := i_c(x)$, $c \in \{B,W\}$, and $\overline{x} := \uparrow x$.

Let us introduce some notation which we will need in Theorem 7. The minimal model of the colored operad $\mathcal{A}_{B\to W}$ will be described in terms of the free colored operad

$\mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}})$ generated by the collections $X_{\mathtt{B}}$, $X_{\mathtt{W}}$, $\overline{X}$ and a generator $f : \mathtt{B} \to \mathtt{W}$. Generators X_c will be considered 'concentrated' in a color $c \in \{\mathtt{B},\mathtt{W}\}$ and $\overline{X}$ as a collection of 'maps' $\mathtt{B} \otimes \cdots \otimes \mathtt{B} \to \mathtt{W}$. We will again have canonical inclusions $j_c :$ $\mathcal{F}(X) \to \mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}})$, $c \in \{\mathtt{B},\mathtt{W}\}$. We will denote, for $n \geq 1$, by

$$(11) \qquad\qquad \mathfrak{I}^{<n} \subset \mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}})$$

the ideal generated by $\overline{X}(<n) := \bigoplus_{k<n} \overline{X}(k)$. The main theorem of this section reads:

Theorem 7. *Let be an (ordinary) operad satisfying (4) and let $\rho : \mathfrak{M} \to (\mathcal{A}, d_{\mathcal{A}})$, $\mathfrak{M} = (\mathcal{F}(X), d_{\mathfrak{M}})$, be its minimal model. Then the minimal model $\mathfrak{M}_{\mathtt{B} \to \mathtt{W}}$ for $\mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ is of the form*

$$(12) \qquad\qquad \alpha : (\mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}), D) \to (\mathcal{A}_{\mathtt{B} \to \mathtt{W}}, d)$$

such that, for each $n \geq 2$ and $x \in X(n)$,

$$(13) \qquad \begin{aligned} &D(x_{\mathtt{B}}) = j_{\mathtt{B}}(d_{\mathfrak{M}}(x)), \ D(x_{\mathtt{W}}) = j_{\mathtt{W}}(d_{\mathfrak{M}}(x)), \\ &D(f) = 0 \ and \ D(\overline{x}) = f x_{\mathtt{B}} - x_{\mathtt{W}} f^{\otimes n} + \omega, \end{aligned}$$

for some $\omega = \omega_x \in \mathfrak{I}^{<n}$ which linearly depends on x. The map α is given by

$$(14) \qquad \alpha(x_{\mathtt{B}}) = j_{\mathtt{B}}(\rho(x)), \ \alpha(x_{\mathtt{W}}) = j_{\mathtt{W}}(\rho(x)), \ \alpha(f) = f \ and \ \alpha(\overline{x}) = 0, \ for \ x \in X.$$

We call $f x_{\mathtt{B}} - x_{\mathtt{W}} f^{\otimes n}$ the *principal part* and ω the *tail* of $D(\overline{x})$.

Remark 8. It is well-known that if the operad $\mathcal{A}$ is quadratic Koszul [2], its minimal model $(\mathcal{F}(X), d_{\mathfrak{M}})$ is the cobar construction over the quadratic dual $\mathcal{A}^!$ of the operad $\mathcal{A}$ [6, Proposition 2.6]. Something similar is true also here – it can be shown that for $\mathcal{A}$ quadratic Koszul, there exists a 'closed' formula for the tail ω. An example is provided by the minimal model $A(\infty)_{\mathtt{B} \to \mathtt{W}}$ of $Ass_{\mathtt{B} \to \mathtt{W}}$ described in Example 4, compare also Example 12. We are lead to:

Problem 9. *Is there a notion of (quadratic) duality, Koszulness and of the cobar construction for colored operads such that colored operad $\mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ is Koszul if $\mathcal{A}$ is and the minimal model of $\mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ is the cobar construction on the dual $(\mathcal{A}_{\mathtt{B} \to \mathtt{W}})^!$?*

Note that the differential D described in Theorem 7 is not quadratic, so one cannot expect the solution of Problem 9 to be a straightforward one.

The **proof of Theorem 7** will occupy the rest of this section. We show first that conditions (13) and (14) of Theorem 7 already guarantee that both α and D are well defined (Lemma 10) and that α is a quasi-isomorphism (Proposition 11). We then show that the differential D indeed exists.

Lemma 10. *For any differential D as in (13), the map α defined by (14) commutes with the differentials, that is*

$$(15) \qquad\qquad d_{\mathcal{A}} \alpha(a) = \alpha(Da), \ \forall a \in \mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}).$$

Proof of the lemma. It is enough to verify (15) on generators. It obviously holds for $a \in X_{\mathtt{B}}$, $a \in X_{\mathtt{W}}$ or $a = f$. For $a = \overline{x} \in \overline{X}$, equation (15) reduces to $0 = \alpha(D(\overline{x}))$. We have

$$\alpha(D(\overline{x})) = \alpha(f x_{\mathtt{B}} - x_{\mathtt{W}} f^{\otimes n} + \omega) = f j_{\mathtt{B}}(\rho(x)) - j_{\mathtt{W}}(\rho(x)) f^{\otimes n} + \alpha(\omega),$$

because α is a homomorphism. It follows from the definition of $\mathcal{A}_{\mathtt{B}\to\mathtt{W}}$ that $fj_\mathtt{B}(\rho(x)) - j_\mathtt{W}(\rho(x))f^{\otimes n} = 0$. Moreover, $\alpha(\omega) = 0$, since $\omega \in \mathfrak{I}^{<n}$ and α is trivial on the generators $\overline{X}(<n)$ of $\mathfrak{I}^{<n}$. Thus $\alpha(D(\overline{x})) = 0$ which finishes the proof. $\qquad\square$

Proposition 11. *Suppose that the differential D is as in (13). Then the map α defined by (14) is a quasi-isomorphism.*

Example 12. The minimal model of the operad *Ass* for associative algebras is $(\mathcal{F}(X), d_{\mathfrak{M}})$ with X generated by $\mu_2, \mu_3, \ldots$ as in (6) and the differential $d_{\mathfrak{M}}$ given by (7). Equation (8) is clearly of the form

$$\partial f_n = f_1 \mu_n - \nu_n f_1^{\otimes n} + \omega$$

where ω belongs to the ideal generated by $f_2, f_3, \ldots, f_{n-1}$. Therefore the model for $Ass_{\mathtt{B}\to\mathtt{W}}$ described in Example 4 is of the type predicted by Theorem 7, with $X_\mathtt{B} = \mathrm{Span}(\mu_2, \mu_3, \ldots)$, $X_\mathtt{W} = \mathrm{Span}(\nu_2, \nu_3, \ldots)$ and $\overline{X} = \mathrm{Span}(f_2, f_3, \ldots)$. In particular, the map α constructed in this example is a quasi-isomorphism.

To prove Proposition 11, observe that the map α of (12) decomposes as

$$(16) \qquad (\mathcal{F}(X_\mathtt{B}; f, \overline{X}; X_\mathtt{W}), D) \xrightarrow{\gamma} \left(\frac{\mathcal{F}(X_\mathtt{B}; f; X_\mathtt{W})}{(fx_\mathtt{B} - x_\mathtt{W}f^{\otimes n})}, d \right) = \mathfrak{M}_{\mathtt{B}\to\mathtt{W}} \xrightarrow{\rho_{\mathtt{B}\to\mathtt{W}}} \mathcal{A}_{\mathtt{B}\to\mathtt{W}}.$$

Since the map $\rho_{\mathtt{B}\to\mathtt{W}}$ is a quism by Corollary 6, it is enough to prove that the map γ is also a quism:

Lemma 13. *For each minimal operad $(\mathcal{F}(X), d_{\mathfrak{M}})$ the map*

$$\gamma : (\mathcal{F}(X_\mathtt{B}; f, \overline{X}; X_\mathtt{W}), D) \longrightarrow (\mathcal{F}(X), d_{\mathfrak{M}})_{\mathtt{B}\to\mathtt{W}}$$

given by $\gamma(x_\mathtt{B}) = j_\mathtt{B}(x)$, $\gamma(x_\mathtt{W}) = j_\mathtt{W}(x)$, $\gamma(f) = f$ and $\gamma(\overline{x}) = 0$, is a quasi-isomorphism.

Proof of the lemma is based on a repeated spectral sequence argument. Let us define an ascending filtration of $\mathcal{F}(X_\mathtt{B}; f, \overline{X}; X_\mathtt{W})$,

$$(17) \qquad \mathfrak{F}_1 : \quad \cdots \subseteq F_{-2} \subseteq F_{-1} \subseteq F_0 = F_1 = F_2 = \cdots = \mathcal{F}(X_\mathtt{B}; f, \overline{X}; X_\mathtt{W}),$$

by postulating F_p to be the subspace spanned by expressions having at least $-p$ occurrences of generators from $X_\mathtt{B}, X_\mathtt{W}$ or $\overline{X}$. Since clearly $D(F_p) \subset F_p$, we can consider the related spectral sequence $\mathbf{E} = (E^r_{pq}, d^r)$. This spectral sequence converges because, for any fixed arity n, $F_p(n) = 0$ for p sufficiently small. The initial term of $\mathbf{E}$ is easy to describe,

$$(E^0, d^0) \cong (\mathcal{F}(X_\mathtt{B}; f, \overline{X}; X_\mathtt{W}), d^0),$$

with d^0 denoting the 'linear part' of d, namely

$$(18) \qquad d^0(x_\mathtt{B}) = d^0(x_\mathtt{W}) = d^0(f) = 0 \text{ and } d^0(\overline{x}) = fx_\mathtt{B} - x_\mathtt{W}f^{\otimes n}, \text{ for } x \in X(n).$$

We claim that

$$(19) \qquad H_*(E^0, d^0) \cong \frac{\mathcal{F}(X_\mathtt{B}; f; X_\mathtt{W})}{(fx_\mathtt{B} - x_\mathtt{W}f^{\otimes n})}.$$

Let us denote, just for the purpose of this proof, $\mathcal{F} := \mathcal{F}(X_\mathtt{B}; f, \overline{X}; X_\mathtt{W})$. The colored operad $\mathcal{F}$ has also another 'upper' grading, $\mathcal{F} = \bigoplus_{s \geq 0} \mathcal{F}^s$, induced by the number of

generators from $\overline{X}$. The differential d^0 lowers this upper degree by 1. It is immediate to see that

$$H^0_*(\mathcal{F}^*_*, d^0) \cong \frac{\mathcal{F}(X_{\text{B}}; f; X_{\text{W}})}{(f x_{\text{B}} - x_{\text{W}} f^{\otimes n})},$$

equation (19) will thus follow from $H^{>0}_*(\mathcal{F}^*_*, d^0) = 0$. To prove this, observe that there is yet another ascending filtration of $\mathcal{F}$ induced by the number of occurrences of the generator f, i.e. $\mathcal{F} = \bigcup G_p$, where G_p is spanned by expressions having at least $-p$ occurrences of the generator f. The first term of the related spectral sequence is $(\mathcal{F}^*_*, \overline{d}_0)$, where

$$\overline{d}_0(x_{\text{B}}) = \overline{d}_0(x_{\text{W}}) = \overline{d}_0(f) = 0 \text{ and } \overline{d}_0(\overline{x}) = f x_{\text{B}}, \text{ for } x \in X.$$

We shall prove that $(\mathcal{F}^*_*, \overline{d}_0)$ is acyclic in positive 'upper' gradings. This means that

(20)
$$\left(\mathcal{F}^{>0}_* \begin{pmatrix} c \\ c_1, \ldots, c_n \end{pmatrix}, \overline{d}_0 \right)$$

is acyclic for each choice of colors $c, c_1, \ldots, c_n \in \{\text{B}, \text{W}\}$. If $c = \text{B}$ then, as in the proof of Proposition 5, we conclude that (20) cannot contain a generator from $\overline{X}$, thus it is not only acyclic, but even trivial in positive upper dimensions.

If $c = \text{W}$, each element a of (20) can be expanded as

(21)
$$a = \sum_{m \geq 2} (f e^{m,i}_{\text{B}} r^{\text{B}}_{m,i} + \overline{e}^{m,i} \underline{r}_{m,i} + e^{m,i}_{\text{W}} r^{\text{W}}_{m,i})$$

where $\{e^{m,i}\}$ is a basis of $X(m)$ and $r^{\text{B}}_{m,i}, r^{\text{W}}_{m,i}, \underline{r}_{m,i} \in \mathcal{F}^{\otimes m}$, $m \geq 2$ (the summation convention assumed). We prove, by induction on the arity n, that

(22)   if a is of a positive upper grading and $\overline{d}_0(a) = 0$, then a is a $\overline{d}_0$-boundary.

For $n = 2$, equation (21) reduces to $a = \sum_i \alpha_i \overline{e}^{2,i}$ for some $\alpha_i \in \mathbb{k}$, thus $\overline{d}_0(a) = \sum_i \alpha_i f e^{2,i}_{\text{B}} = f(\sum_i \alpha_i e^{2,i}_{\text{B}})$. The condition $\overline{d}_0(a) = 0$ then implies that $\alpha_i = 0$ for each i, thus $a = 0$.

Suppose we have proved (22) for all arities $< n$ and suppose that a has arity n. If $\overline{d}_0(a) = 0$, then

$$0 = \sum_{m \geq 2} ((-1)^{e^{m,i}} \cdot f e^{m,i}_{\text{B}} \overline{d}_0(r^{\text{B}}_{m,i}) + f e^{m,i}_{\text{B}} \underline{r}_{m,i} - (-1)^{e^{m,i}} \cdot \overline{e}^{m,i} \overline{d}_0(\underline{r}_{m,i}) + (-1)^{e^{m,i}} \cdot e^{m,i}_{\text{W}} \overline{d}_0(r^{\text{W}}_{m,i})),$$

which implies that, for all $m \geq 2$,

$$\overline{d}_0(\underline{r}_{m,i}) = 0, \quad \overline{d}_0(r^{\text{B}}_{m,i}) = -(-1)^{e^{m,i}} \cdot \underline{r}_{m,i} \text{ and } \overline{d}_0(r^{\text{W}}_{m,i}) = 0.$$

Since each $r^{\text{W}}_{m,i}$ is a product of elements of arity $< n$, by induction assumption there exists $b^{\text{W}}_{m,i}$ such that $r^{\text{W}}_{m,i} = \overline{d}_0(b^{\text{W}}_{m,i})$. It is easy to verify that then

$$a = \overline{d}_0 \left\{ \sum_{m \geq 2} (\overline{e}^{m,i} r^{\text{B}}_{m,i} + (-1)^{e^{m,i}} e^{m,i}_{\text{W}} b^{\text{W}}_{m,i}) \right\},$$

which proves (22) for arity n. Thus $(\mathcal{F}^*_*, \overline{d}_0)$ is acyclic in positive upper dimensions, and so is, by a spectral sequence argument, also $(\mathcal{F}^*_*, d^0)$. This proves (19).

Define an ascending filtration $\mathfrak{F}_2$ of the operad

$$(\mathcal{F}(X), d_{\mathfrak{M}})_{\mathtt{B}\to\mathtt{W}} = \left(\frac{\mathcal{F}(X_{\mathtt{B}}; f; X_{\mathtt{W}})}{(fx_{\mathtt{B}} - x_{\mathtt{W}}f^{\otimes n})}, d \right) ,$$

by the number of generators from $X_{\mathtt{B}}$ and $X_{\mathtt{W}}$. The differential induced on the first term of the related spectral sequence is trivial. The map γ is clearly a morphism of $\mathfrak{F}_1$-$\mathfrak{F}_2$ filtered operads. It follows from (19) that γ induces a quism of the initial terms of the spectral sequences, thus γ is a quism as well. This finishes the proof of the lemma.   □

To finish the proof of Theorem 7, we will need also a 'restricted' version of Lemma 13:

Lemma 14. *Let $K \geq 2$ and let $d_{\mathfrak{M}}^{\leq K}$ be the restriction of $d_{\mathfrak{M}}$ to $X(<K)$. Let $D^{<K}$ be the similar obvious restriction of the differential D. Then the map*

$$(23) \qquad \gamma_{<K} : (\mathcal{F}(X_{\mathtt{B}}(<K); f, \overline{X}(<K); X_{\mathtt{W}}(<K)), D^{<K}) \longrightarrow (\mathcal{F}(X(<K)), d_{\mathfrak{M}}^{\leq K})_{\mathtt{B}\to\mathtt{W}}$$

given by $\gamma_{<K}(x_{\mathtt{B}}) = j_{\mathtt{B}}(x)$, $\gamma_{<K}(x_{\mathtt{W}}) = j_{\mathtt{W}}(x)$, $\gamma_{<K}(f) = 0$ and $\gamma_{<K}(\overline{x}) = 0$ is a quasi-isomorphism.

Proof of the lemma. Since all objects in the lemma are well-defined and the restricted differential $D^{<K}$ has the form (13), the map $\gamma_{<K}$ is a quism by Lemma 13 applied to the minimal operad $(\mathcal{F}(X(<K)), d_{\mathfrak{M}}^{\leq K})$.   □

Proof of Theorem 7. In the light of Lemma 10 and Proposition 11, it suffices to construct a differential D satisfying (13). We proceed by induction on the arity n. For $\overline{x} \in \overline{X}(2)$, let $D(\overline{x}) := fx_{\mathtt{B}} - x_{\mathtt{W}}f^{\otimes 2}$. Suppose we have already defined D on $\overline{X}(<n)$. If we put, for $\overline{x} \in \overline{X}(n)$ and for some $\omega \in \mathfrak{J}^{<n}$,

$$D(\overline{x}) := fx_{\mathtt{B}} - x_{\mathtt{W}}f^{\otimes n} + \omega ,$$

then $(D \circ D)(\overline{x}) = 0$ implies that ω must satisfy

$$(24) \qquad D(x_{\mathtt{W}})f^{\otimes n} - fD(x_{\mathtt{B}}) = D(\omega) .$$

Let us denote the left hand side of the above equation by φ,

$$\varphi := D(x_{\mathtt{W}})f^{\otimes n} - fD(x_{\mathtt{B}}) = j_{\mathtt{W}}(d_{\mathfrak{M}}(x))f^{\otimes n} - fj_{\mathtt{B}}(d_{\mathfrak{M}}(x)) .$$

We need to solve the equation

$$(25) \qquad \qquad \varphi = D(\omega)$$

with some $\omega \in \mathfrak{J}^{<n}$. We are going to apply Lemma 14. To simplify the notation, denote

$$\mathcal{G}_{<n} := (\mathcal{F}(X_{\mathtt{B}}(<n); f, \overline{X}(<n); X_{\mathtt{W}}(<n)), D^{<n}) .$$

Observe that $\mathfrak{J}^{<n} \subset \mathcal{G}_{<n}$. It is also evident that $\varphi \in \mathcal{G}_{<n}$ and that $D^{<n}(\varphi) = 0$. Since $\phi \in \mathrm{Ker}(\gamma_{<n})$, by Lemma 14 with $K = n$ there exits $\alpha_1 \in \mathcal{G}_{<n}$ such that $\varphi = D^{<n}\alpha_1$. Clearly α_1 decomposes as $\alpha_1 = u_1 + \omega_1$, where $\omega_1 \in \mathfrak{J}^{<n}$ and u_1 does not contain generators from $\overline{X}$.

Let us apply the map $\gamma_{<n}$ to the equation $\varphi = D^{<n}u_1 + D^{<n}\omega_1$. Since $\gamma_{<n}(\varphi) = \gamma_{<n}(D^{<n}\omega_1) = 0$, $D^{<n}u_1 \in \mathrm{Ker}(\gamma_{<n})$ as well. Because u_1 does not contain elements of $\overline{X}$, $D^{<n}u_1 \in \mathcal{G}_{<n-1}$ thus, in fact, $D^{<n}u_1 \in \mathrm{Ker}(\gamma_{<n-1})$.

Lemma 14 with $K = n - 1$ gives some $\omega_2 \in \mathfrak{I}^{<n-1}$ and some $u_2 \in \mathcal{G}_{<n-1}$ which does not contain generators from $\overline{X}$ such that $D^{<n}u_1 = D^{<n-1}u_2 + D^{<n-1}\omega_2$. Repeating this process $n - 2$ times, we end up with $D^{<4}u_{n-3} = D^{<3}u_{n-2} + D^{<3}\omega_{n-2}$, where $\omega_{n-2} \in \mathfrak{I}^{<3}$ and $u_{n-2} \in \mathcal{G}_{<3}$ does not contain generators from $\overline{X}$. Thus $D^{<3}u_{n-2} = 0$ and $\omega := \omega_1 + \omega_2 + \ldots + \omega_{n-2}$ solves (25).   □

3. Some applications

In this section we present a couple of applications of the model studied in Section 2. Though the results are formulated for very specific examples of homotopy algebras, as we will see below, only the principal parts of the differential matter. Therefore similar statements can be formulated for any type of strongly homotopy algebras.

Recall that, for each $K \geq 1$, an $A(K)$-*algebra* is an object $\mathbf{W} = (W, \partial, n_2, n_3, \ldots, n_K)$ such that the multilinear maps $n_i : W^{\otimes i} \to W$, $2 \leq i \leq K$, satisfy all axioms of $A(\infty)$-algebras in arities $\leq K$. Similarly, a *morphism* $\mathbf{F}$ of two $A(K)$-algebras,

$$\mathbf{F} : \mathbf{V} = (V, \partial, m_2, m_3, \ldots, m_K) \longrightarrow \mathbf{W} = (W, \partial, n_2, n_3, \ldots, n_K)$$

is a sequence of multilinear maps $F_i : V^{\otimes i} \to W$, $1 \leq i \leq K$, that satisfies all axioms of a homomorphism of $A(\infty)$-algebras in arities $\leq K$. This notion slightly differs from the one of [5, p. 147].

Each $A(\infty)$-algebra $\mathbf{V}$ determines, by forgetting all structure operations of arities $> K$, an $A(K)$-algebra $\mathbf{V}_K$.

Theorem 15. *Suppose we are given, for some fixed $K \geq 1$, the following data:*
  (i) *An $A(\infty)$-algebra $\mathbf{V} = (V, \partial, m_2, m_3, \ldots)$,*
  (ii) *an $A(K)$-algebra $\mathbf{W} = (W, \partial, n_2, n_3, \ldots, n_K)$ and*
  (iii) *a morphism $\{F_n : V^{\otimes i} \to W\}_{n \leq K} : \mathbf{V}_K \to \mathbf{W}$ of $A(K)$-algebras.*
Suppose that the chain map $F_1 : (V, \partial) \to (W, \partial)$ is a quasi-isomorphism. Then the above data can be extended to an $A(\infty)$-structure $\mathbf{W}$ on (W, ∂) and to an $A(\infty)$-homomorphism $\mathbf{F} : \mathbf{V} \to \mathbf{W}$. A similar statement holds also for balanced sh associative ($C(\infty)$) and sh Lie ($L(\infty)$) algebras (see [5, p. 148] and [4] for the definitions of these objects).

Proof. Let us assume the notation of Example 4. The data of the theorem can be encoded by a map

$$\phi : \mathcal{F}(\mu_2, \mu_3, \ldots; f_1, f_2, \ldots, f_K; \nu_2, \nu_3, \ldots, \nu_K) \to \mathcal{E}nd_{V,W}$$

with $\phi(\mu_i) := m_i$, $2 \leq i$, $\phi(\nu_j) := n_j$, $2 \leq j \leq K$ and $\phi(f_i) := F_i$, $1 \leq i \leq K$. We need some $n_{K+1} \in \mathrm{Hom}(W^{\otimes(K+1)}, W)$ and $F_{K+1} \in \mathrm{Hom}(V^{\otimes(K+1)}, W)$ such that

$$(26) \qquad \partial n_{K+1} = \phi(\partial \nu_{K+1}) \text{ and}$$

$$(27) \qquad \partial F_{K+1} = \phi(\nu_{K+1} f_1^{\otimes(K+1)} - f_1 \mu_{K+1} + \omega),$$

$\phi(\nu_{K+1}) := n_{K+1}$ and $\phi(f_{K+1}) := F_{K+1}$ will then extend our data one step up. We start by (26). Applying ϕ to

$$0 = \partial^2 f_{K+1} = \partial \nu_{K+1} f_1^{\otimes(K+1)} - f_1 \partial \mu_{K+1} + \partial \omega$$

we obtain

$$\phi(\partial \nu_{K+1}) F_1^{\otimes(K+1)} = F_1 \phi(\partial \mu_{K+1}) - \phi(\partial \omega) = \partial(F_1 \phi(\mu_{K+1} - \omega)),$$

so $\phi(\partial\mu_{K+1})F_1^{\otimes(K+1)}$ is homologous to zero in $\mathrm{Hom}(V^{\otimes(K+1)}, W)$. Since F_1 is a quism, $\phi(\partial\nu_{K+1})$ is homologous to zero in $\mathrm{Hom}(W^{\otimes(K+1)}, W)$ and the existence of n_{K+1} satisfying (26) easily follows.

We solve (27) invoking a very useful trick of an 'additive renormalization'. Observe that if $\theta_{K+1} \in \mathrm{Hom}(W^{\otimes(K+1)}, V)$ is closed, then changing n_{K+1} to $n_{K+1} + \theta_{K+1}$ does not violate (26). For such a renormalized n_{K+1}, the right hand side of (27) reads

$$\chi := n_{K+1}F_1^{\otimes(K+1)} - F_1 m_{K+1} + \phi(\omega) + \theta_{K+1}F_1^{\otimes(K+1)}.$$

The first three terms of χ are closed. Since F_1 a quism, there exists θ_{K+1} such that χ is homologous to zero; the existence of F_{K+1} solving (27) follows. The induction may go on. □

For $K = 1$ the data of Theorem 15 consist of an $A(\infty)$-algebra $\mathbf{V} = (V, \partial, m_2, m_3, ...)$ and a chain equivalence $F_1 : (V, \partial) \to (W, \partial)$. For $K = 2$ we have also a bilinear product $n_2 : W \otimes W \to W$ and a homotopy $F_2 : V \otimes V \to W$ between $F_1 m_2$ and $n_2(F_1, F_1)$.

A statement similar to Theorem 15 can be clearly formulated for any type of strongly homotopy algebras, since all we needed was that the minimal model was of the form of Theorem 7.

Corollary 16. *Suppose that (U, μ, ∂) is a differential graded associative algebra and let $\nu : U \otimes U \to U$ be a bilinear multiplication chain homotopic to μ. Then ν can be extended to a strongly homotopy associative structure on (U, ∂).*

Proof. The corollary immediately follows from Theorem 15 with $\mathbf{V} = (U, \partial, \mu, 0, 0, ...)$, $(W, \partial) := (U, \partial)$, $F_1 := \mathrm{id}$ and F_2 the homotopy between μ and ν. □

As observed by Jim Stasheff, the above corollary must be 'spiritually' true if one believes that $A(\infty)$-structures are homotopically invariant versions of associative algebras. And indeed, it is very easy to prove that the multiplication ν, homotopic to an associative one, is associative up to a homotopy; it is even possible to give an explicit formula for the homotopy. But it is not clear whether this homotopy extends to a coherent hierarchy of higher homotopies, and this the hard part of Corollary 16. In the following corollary, the characteristic 0 assumption is very crucial.

Corollary 17. *Suppose that (U, ∂, μ) is an associative, homotopy commutative algebra. Let $\bar{\mu}(u, v) := \frac{1}{2}(\mu(u, v) + \mu(v, u))$ be the symmetrization. Then $\bar{\mu}$ can be extended to a balanced $A(\infty)$-structure on (U, ∂).*

Proof. Let $H := H_*(U, \partial)$ and let $*$ be the multiplication induced by μ on H. It is immediate to see that it is commutative associative and that it coincides with that induced by $\bar{\mu}$. Now choose a monomorphism $(H, 0) \overset{\iota}{\hookrightarrow} (U, \partial)$ inducing the identity map on homology. It can always be done, since we are over a field of characteristic zero. Then $\iota(*) : H \otimes H \to U$ and $\bar{\mu}(\iota, \iota) : H \otimes H \to U$ induce the same map in cohomology, thus (again the characteristic zero argument) there exists a homotopy h between $\iota(*)$ and $\bar{\mu}(\iota, \iota)$.

The corollary now follows from Theorem 15 with $\mathbf{V} := (H, 0, *, 0, ...)$, $(W, \partial, \nu_2) := (U, \partial, \bar{\mu})$, $F_1 = \iota$ and $F_2 := h$. □

4. Homotopy through homomorphisms

In Section 2 we described the minimal model for the operad $\mathcal{A}_{\mathtt{B}\to\mathtt{W}}$ and observed that it describes strongly homotopy homomorphisms of strongly homotopy algebras. The aim of this section is to understand homotopies between these homomorphisms. By our philosophy, we need to resolve the operad

$$\mathcal{A}_{\mathtt{B}\rightrightarrows\mathtt{W}} := \frac{\mathcal{A}_{\mathtt{B}} * \mathcal{A}_{\mathtt{W}} * \mathcal{F}(p,q)}{(pa_{\mathtt{B}} = a_{\mathtt{W}}p^{\otimes n}, \; qa_{\mathtt{B}} = a_{\mathtt{W}}q^{\otimes n}, \; p = q)}.$$

where $p, q : \mathtt{B} \to \mathtt{W}$ are generators of degree 0 and the first two relations in the denominator are satisfied for all $a \in \mathcal{A}(n)$ and $n \geq 1$. This operad describes two identical homomorphisms of $\mathcal{A}$-algebras and its resolution should replace the strict equality $p = q$ by a homotopy.

Since the operad $\mathcal{A}_{\mathtt{B}\rightrightarrows\mathtt{W}}$ is clearly isomorphic to $\mathcal{A}_{\mathtt{B}\to\mathtt{W}}$, its minimal model will not give what we want and we shall consider a resolution which has two different generators for the same map $p = q$ instead. This resolution will not be minimal, but it will still be cofibrant in a suitable sense [7]. A toy model for this resolution is the $\{\mathtt{B},\mathtt{W}\}$-colored operad

(28) $\qquad \mathcal{D} = (\mathcal{F}(p,q,h),d), \; d(p) = d(q) := 0 \text{ and } d(h) := p - q$

($\mathcal{D}$ for dull), where $p, q : \mathtt{B} \to \mathtt{W}$ are generators of degree 0 and $h : \mathtt{B} \to \mathtt{W}$ has degree 1. Operad (28) clearly resolves the free operad $(\mathcal{F}(f), d = 0)$ on one degree 0 generator $f : \mathtt{B} \to \mathtt{W}$. An algebra over $\mathcal{D}$ consists of two chain maps and a chain homotopy between them. For each $m \geq 1$ the space $\mathcal{D}(\mathtt{B}^{\otimes m}, \mathtt{W}^{\otimes m})$ contains a special degree 1 morphism $[\![h]\!]_{\mathrm{ns}}$ ("ns" for non-Σ) defined by

$$[\![h]\!]_{\mathrm{ns}} := h \otimes q^{\otimes m-1} + p \otimes h \otimes q^{\otimes m-2} + \cdots + p^{\otimes m-2} \otimes h \otimes q + p^{\otimes m-1} \otimes h$$

and, since $\mathrm{char}(\mathbf{k}) = 0$, also its symmetrization

$$[\![h]\!] := \frac{1}{m!} \sum_{\sigma \in \Sigma_m} [\![h]\!]\sigma$$

(Σ_m is the symmetric group on m symbols). The above morphisms satisfy the following differential equation in $\mathcal{D}$

$$d([\![h]\!]_{\mathrm{ns}}) = d([\![h]\!]) = p^{\otimes m} - q^{\otimes n}$$

and the morphism $[\![h]\!]$ is Σ_n-equivariant. The maps $[\![h]\!]$ and $[\![h]\!]_{\mathrm{ns}}$ are very simple examples of a polarization which we introduce in Definition 23.

Let X be a collection. We will need the free operad $\mathcal{F}(X_{\mathtt{B}}; p,q,h; X^p, X^q, X^h; X_{\mathtt{W}})$ generated by

(i) two copies $X_{\mathtt{B}}$ and $X_{\mathtt{W}}$ of X interpreted as in Theorem 7,

(ii) the generators p,q,h as in (28),

(iii) two copies of X^p and X^g of the suspension $\uparrow X$ interpreted as collections of multilinear maps from $\mathtt{B}$ to $\mathtt{W}$ and

(ii) a copy X^h of the double suspension $\uparrow^2 X$, again interpreted as a collection of multilinear maps from $\mathtt{B}$ to $\mathtt{W}$.

The notation used below is paralell to that of Theorem 7 and we believe its meaning is clear.

Theorem 18. *Let $\mathcal{A}$ be an (ordinary) operad satisfying (4) and let $\rho : \mathfrak{M} \to (\mathcal{A}, d_{\mathcal{A}})$, $\mathfrak{M} = (\mathcal{F}(X), d_{\mathfrak{M}})$, be the minimal model of $\mathcal{A}$. Then there exists a cofibrant model $\mathfrak{M}_{\mathtt{B} \rightrightarrows \mathtt{W}}$ of the colored operad $\mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ of the form*

$$\psi : (\mathcal{F}(X_{\mathtt{B}}; p, q, h; X^p, X^q, X^h; X_{\mathtt{W}}), D) \to (\mathcal{A}_{\mathtt{B} \to \mathtt{W}}, d)$$

where $\psi(x_{\mathtt{B}}) = j_{\mathtt{B}}(\rho(x))$ and $\psi(x_{\mathtt{W}}) = j_{\mathtt{W}}(\rho(x))$ for $x \in X$, $\psi(p) = \psi(q) = f$ and ψ is trivial on remaining generators. The differential is given by $D(p) = D(q) = 0$, $D(h) = p - q$ and, for $x \in X(n)$, $n \geq 2$, $D(x_{\mathtt{B}}) = j_{\mathtt{B}}(d_{\mathfrak{M}}(x))$, $D(x_{\mathtt{W}}) = j_{\mathtt{W}}(d_{\mathfrak{M}}(x))$. Moreover,

$$D(x^p) = p x_{\mathtt{B}} - x_{\mathtt{W}} p^{\otimes n} + \omega_p , \quad D(x^q) = q x_{\mathtt{B}} - x_{\mathtt{W}} q^{\otimes n} + \omega_q , \quad and$$
$$D(x^h) = x^p - x^q - h x_{\mathtt{B}} + (-1)^x x_{\mathtt{W}} [\![h]\!] + \omega_h ,$$

where the tails ω_p, ω_q and ω_h depend linearly on x and

(i) *ω_p belongs to the ideal of the suboperad $\mathcal{F}(x_{\mathtt{B}}; p, X^p; x_{\mathtt{W}})$ generated by $X^p(<n)$,*
(ii) *ω_q belongs to the ideal of the suboperad $\mathcal{F}(x_{\mathtt{B}}; q, X^q; x_{\mathtt{W}})$ generated by $X^q(<n)$, and*
(iii) *ω_h belongs to the ideal generated by $X^p(<n), X^q(<n)$ and $X^h(<n)$.*

Proof. The tails ω_p and ω_q can be constructed as follows. Let $(\mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}), D)$ be as in Theorem 7 and consider two homomorphisms

(29)
$$\theta_p : \mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}) \searrow$$
$$\theta_q : \mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}) \nearrow \mathcal{F}(X_{\mathtt{B}}; p, q, h; X^p, X^q, X^h; X_{\mathtt{W}})$$

defined by

$$\theta_p(f) = p, \ \theta_p(x_{\mathtt{B}}) = \theta_q(x_{\mathtt{B}}) := x_{\mathtt{B}}, \ \theta_q(f) = q,$$
$$\theta_p(x_{\mathtt{W}}) = \theta_q(x_{\mathtt{W}}) := x_{\mathtt{W}}, \ \theta_p(\overline{x}) := x^p \text{ and } \theta_q(\overline{x}) := x^q$$

for $x \in X$. We put then $\omega_p := \theta_p(\omega)$ and $\omega_q := \theta_q(\omega)$, where ω is as in (13). The construction of the differential D will be completed by proving the existence of some ω_h belonging to the ideal defined in (iii) and satisfying

(30)
$$D(\omega_h) = D(\omega_q - \omega_p) .$$

Assume, as in the proof of Theorem 7, that the differential D exists and prove that the map $\psi : \mathfrak{M}_{\mathtt{B} \rightrightarrows \mathtt{W}} \to \mathcal{A}_{\mathtt{B} \to \mathtt{W}}$ is a quism. Since $\alpha = \psi \theta_p = \psi \theta_q$, where α is the quism of (7), it suffices to prove that θ_p (or θ_q) is a quism.

Let $\mathfrak{F}_3$ be the filtration of $\mathfrak{M}_{\mathtt{B} \rightrightarrows \mathtt{W}}$ given by the number of generators from X^p, X^q and X^h. The differential d^0 of the first term of the corresponding spectral sequence is given by

$$d^0(h) = p - q , \ d^0(x_{\mathtt{B}}) = d^0(x_{\mathtt{W}}) = 0 , \ d^0(p) = d^0(q) = 0 ,$$
$$d^0(x^p) = p x_{\mathtt{B}} - x_{\mathtt{W}} p^{\otimes n} , \ d^0(x^q) = q x_{\mathtt{B}} - x_{\mathtt{W}} q^{\otimes n} \text{ and}$$
$$d^0(x^h) = x^p - x^q - h x_{\mathtt{B}} + (-1)^x x_{\mathtt{W}} [\![h]\!] ,$$

for $x \in X$. The substitution

$$p \mapsto p, \ h \mapsto h, \ q \mapsto -q + p, \ x_{\mathtt{B}} \mapsto x_B, \ x_{\mathtt{W}} \mapsto x_{\mathtt{W}} ,$$
$$x^p \mapsto x^p, \ x^h \mapsto x^h \text{ and } x^q \mapsto x^p - x^q - h x_{\mathtt{B}} + (-1)^x x_{\mathtt{W}} [\![h]\!]$$

converts d^0 to the differential $\underline{d}^0$ with

$$\underline{d}^0(p) = \underline{d}^0(q) = 0\,, \ \underline{d}^0(h) = q\,, \ \underline{d}^0(x_{\mathtt{B}}) = \underline{d}^0(x_{\mathtt{W}}) = 0\,,$$
$$\underline{d}^0(x^p) = px_{\mathtt{B}} - x_{\mathtt{W}}p^{\otimes n}\,, \ \underline{d}^0(x^q) = 0 \text{ and } \underline{d}^0(x^h) = x^q\,.$$

The dg operad $(\mathcal{F}(X_{\mathtt{B}}; p, q, h; X^p, X^q, X^h; X_{\mathtt{W}}), \underline{d}^0)$ is clearly isomorphic to the free product

(31) $\qquad (\mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}), d^0) * (\mathcal{F}(q, h, ; X^q, X^h),\ dh = q,\ dx^h = x^q)\,,$
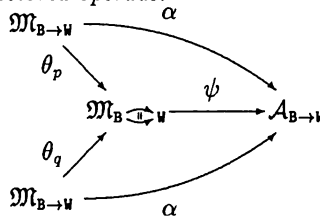
where the differential d^0 of the first factor is as in (18) and the second factor is acyclic.

Recall that in (17) we defined filtration $\mathfrak{F}_1$ of $(\mathcal{F}(X_{\mathtt{B}}; f, \overline{X}; X_{\mathtt{W}}), D)$. The map θ_p is clearly an $\mathfrak{F}_1$-$\mathfrak{F}_3$ homomorphism of filtered operads. On the initial stages of the related spectral sequences it induces precisely the inclusion of the first factor of (31), which is a quism. Thus θ_p is a quism, by a standard spectral sequence argument.

The existence the tail ω_h satisfying (30) follows, as in the proof of Theorem 7, from a general nonsense. We leave the details to the reader. $\qquad\square$

In the proof of Theorem 18 we in fact proved:

Corollary 19. *The maps θ_p and θ_q of (29) fit into the following commutative diagram of quasi-isomorphisms of colored operads:*

$$
\begin{array}{ccc}
\mathfrak{M}_{\mathtt{B}\to\mathtt{W}} & \xrightarrow{\ \ \alpha\ \ } & \\
\ \ \searrow{\scriptstyle\theta_p} & & \\
& \mathfrak{M}_{\mathtt{B}\overset{\rightarrow}{\rightarrow}\mathtt{W}} \xrightarrow{\ \psi\ } & \mathcal{A}_{\mathtt{B}\to\mathtt{W}} \\
\ \ \nearrow{\scriptstyle\theta_q} & & \\
\mathfrak{M}_{\mathtt{B}\to\mathtt{W}} & \xrightarrow{\ \ \alpha\ \ } &
\end{array}
$$

Remark 20. If $\mathcal{A}$ is a non-Σ-operad, Theorem 18 holds with $[\![-]\!]_{\mathrm{ns}}$ instead of $[\![-]\!]$. Since $[\![-]\!]_{\mathrm{ns}}$ has integral coefficients, the model $\mathfrak{M}_{\mathtt{B}\overset{\rightarrow}{\rightarrow}\mathtt{W}}$ might in principle, as in Example 21, exist over the integers.

Example 21. Let us consider the free operad $\mathcal{F}(\{\mu_n, \nu_n\}_{n\geq 2}, \{p_n, q_n\}_{n\geq 1})$ generated by

  (i) generators μ_n and ν_n, $n \geq 2$, as in Example 4,
  (ii) generators $p_n, q_n : \mathtt{B}^{\otimes n} \to \mathtt{W}$ of degree $n - 1$, $n \geq 1$, and
  (ii) generators $h_n : \mathtt{B}^{\otimes n} \to \mathtt{W}$ of degree n, $n \geq 1$.

The differential of μ_n and ν_n is given by formulas (7) and (9), the differential of p_n and q_n by formula (8) (with f_i replaced by p_i resp. q_i, $i \geq 1$) and

$$
\partial(h_n) := p_n - q_n + \sum_{\substack{k=2r_1+\cdots+r_k=n \\ 0\leq s\leq k-1}}^{n} (-1)^\epsilon \nu_k(p_{r_1} \otimes \cdots \otimes p_{r_s} \otimes h_{r_{s+1}} \otimes q_{r_{s+2}} \otimes \cdots \otimes q_{r_k})
$$
$$
+ \sum_{\substack{i+j=n+1 \\ i\geq 1, j\geq 2}} \sum_{s=0}^{n-j} (-1)^{i+s(j+1)} h_i(\mathbb{1}^{\otimes s} \otimes \mu_j \otimes \mathbb{1}^{\otimes i-s-1})\,,
$$

where

$$
\epsilon := \sum_{1\leq i<j\leq k} r_i(r_j + 1) + (r_1 + \cdots + r_s) + k + s\,.
$$

An algebra over the colored operad $A(\infty)_{\mathtt{B} \rightleftarrows \mathtt{W}}$ described above consists of two $A(\infty)$-algebras $\mathbf{V} = (V, \partial, m_2, m_3, \dots)$ and $\mathbf{W} = (V, \partial, n_2, n_3, \dots)$, two strongly homotopy homomorphisms $\mathbf{P}, \mathbf{Q} : \mathbf{V} \to \mathbf{W}$, $\mathbf{P} = (P_1, P_2, \dots)$, $\mathbf{Q} = (Q_1, Q_2, \dots)$, and a sequence $\mathbf{H} = (H_1, H_2, \dots)$ which should be interpreted as a *homotopy through strongly homotopy homomorphisms* between $\mathbf{P}$ and $\mathbf{Q}$. This notion coincides with the homotopy structure considered by M. Grandis in [3, 5.4]. Observe that $\partial H_1 = P_1 - Q_1$ and that, for $a, b \in V$,

$$(\partial H_2)(a, b) = (-1)^a n_2(P_1(a), H_1(b)) + n_2(H_1(a), Q_1(b)) - H_1 m_2(a, b).$$

Writing m_2 and n_2 as multiplications, we can translate the above equation to

$$(\partial H_2)(a, b) = (-1)^a P_1(a) H_1(b) + H_1(a) Q_1(b) - H_1(ab),$$

thus H_1 is 'derivation homotopy up to homotopy' between P_1 and Q_1.

Remark 22. It is well-known that $A(\infty)$-algebras are the same as differentials on cofree coconnected coalgebras (that is, coderivations with square zero). In this language, strong homotopy homomorphisms are homomorphisms of these coalgebras commuting with the differentials. It can be shown that the homotopies of Example 21 translate to ordinary (co)derivation homotopies of these maps of coalgebras.

5. STRONG HOMOTOPY EQUIVALENCES OF ALGEBRAS

Let $\mathcal{A}$ be an ordinary operad and $f : \mathtt{B} \to \mathtt{W}$ and $g : \mathtt{W} \to \mathtt{B}$ be two degree zero generators. Consider the operad

$$(32) \qquad \mathcal{A}_{\mathcal{I}so} := \frac{\mathcal{A}_{\mathtt{B}} * \mathcal{A}_{\mathtt{W}} * \mathcal{F}(f, g)}{(fa_{\mathtt{B}} = a_{\mathtt{W}} f^{\otimes n},\ ga_{\mathtt{W}} = a_{\mathtt{B}} g^{\otimes n},\ fg = \mathbb{1}_W,\ gf = \mathbb{1}_{\mathtt{B}})}$$

describing two mutually inverse homomorphisms of $\mathcal{A}$-algebras. By our general philosophy, cofibrant resolutions of this operad will describe homotopy equivalences of strongly homotopy $\mathcal{A}$-algebras. If $\mathcal{A}$ is the operad $\mathbf{1}$ for trivial algebras, construction (32) gives the operad

$$\mathcal{I}so := \mathbf{1}_{\mathcal{I}so} = \frac{\mathcal{F}(f, g)}{(fg = \mathbb{1}_{\mathtt{W}},\ gf = \mathbb{1}_{\mathtt{B}})}$$

describing two mutually inverse chain maps. Operad $\mathcal{I}so$ was studied in great details in [7] where we described its minimal cofibrant resolution as a graded colored differential operad

$$(33) \qquad \mathcal{R}_{\mathrm{iso}} := (\mathcal{F}(f_0, f_1, \dots; g_0, g_1, \dots), d),$$

with two types of generators,

$$(34) \qquad
\begin{array}{l}
\text{(i) generators } \{f_k\}_{k \geq 0},\ \deg(f_k) = k, \left\{
\begin{array}{l}
f_k : \mathtt{B} \to \mathtt{W} \text{ if } k \text{ is even,} \\
f_k : \mathtt{B} \to \mathtt{B} \text{ if } k \text{ is odd,}
\end{array}\right. \\[2em]
\text{(ii) generators } \{g_k\}_{k \geq 0},\ \deg(g_k) = k, \left\{
\begin{array}{l}
g_k : \mathtt{W} \to \mathtt{B} \text{ if } k \text{ is even,} \\
g_k : \mathtt{W} \to \mathtt{W} \text{ if } k \text{ is odd.}
\end{array}\right.
\end{array}$$

The differential d was given by

$$(35) \qquad \begin{aligned} df_0 &:= 0, & dg_0 &:= 0, \\ df_1 &:= g_0 f_0 - 1, & dg_1 &:= f_0 g_0 - 1 \end{aligned}$$

and, on remaining generators, by the formula

$$(36) \qquad \begin{aligned} df_{2m} &:= \sum_{0 \le i < m} \left(f_{2i} f_{2(m-i)-1} - g_{2(m-i)-1} f_{2i} \right), \ m \ge 0, \\ df_{2m+1} &:= \sum_{0 \le j \le m} g_{2j} f_{2(m-j)} - \sum_{0 \le j < m} f_{2j+1} f_{2(m-j)-1}, \ m \ge 1, \\ dg_{2m} &:= \sum_{0 \le i < m} \left(g_{2i} g_{2(m-i)-1} - f_{2(m-i)-1} g_{2i} \right), \ m \ge 0, \\ dg_{2m+1} &:= \sum_{0 \le j \le m} f_{2j} g_{2(m-j)} - \sum_{0 \le j < m} g_{2j+1} g_{2(m-j)-1}, \ m \ge 1. \end{aligned}$$

The above formulas can be written in a shorter form by introducing 'formal generators'

$$(37) \qquad \begin{aligned} f_\bullet &:= f_0 + f_2 + f_4 + \cdots : \mathbf{B} \to \mathbf{W}, & h_\bullet &:= f_1 + f_3 + f_5 + \cdots : \mathbf{B} \to \mathbf{B}, \\ g_\bullet &:= g_0 + g_2 + g_4 + \cdots : \mathbf{W} \to \mathbf{B}, & l_\bullet &:= g_1 + g_3 + g_5 + \cdots : \mathbf{W} \to \mathbf{W}. \end{aligned}$$

Then $\mathcal{R}_{\mathrm{iso}} = \mathcal{F}(f_\bullet, g_\bullet, h_\bullet, l_\bullet)$ with the differential

$$df_\bullet = f_\bullet h_\bullet - l_\bullet f_\bullet, \quad dh_\bullet = g_\bullet f_\bullet - h_\bullet h_\bullet - 1_{\mathbf{B}}, \quad dg_\bullet = g_\bullet l_\bullet - h_\bullet g_\bullet, \quad dl_\bullet = f_\bullet g_\bullet - l_\bullet l_\bullet - 1_{\mathbf{W}}.$$

Warning. We will use the above abbreviation very often, but we shall always keep in mind that each formula of this type in fact represents infinitely many formulas for homogeneous parts. Observe also that in this section the symbols $f_0, f_1, \ldots$ have different meaning than in Example 4 and related examples and applications.

For the formulation of Theorem (24) we need the following definition.

Definition 23. Let $m \ge 1$. A polarization is a choice of Σ_m-equivariant morphisms $[\![f_\bullet]\!] \in \mathcal{R}_{\mathrm{iso}}(\mathbf{B}^{\otimes m}, \mathbf{W}^{\otimes m})$, $[\![g_\bullet]\!] \in \mathcal{R}_{\mathrm{iso}}(\mathbf{W}^{\otimes m}, \mathbf{B}^{\otimes m})$, $[\![h_\bullet]\!] \in \mathcal{R}_{\mathrm{iso}}(\mathbf{B}^{\otimes m}, \mathbf{B}^{\otimes m})$ and $[\![l_\bullet]\!] \in \mathcal{R}_{\mathrm{iso}}(\mathbf{W}^{\otimes m}, \mathbf{W}^{\otimes m})$ such that

(i) $[\![f_\bullet]\!]_0 = f_0^{\otimes m}$ and $[\![g_\bullet]\!]_0 = g_0^{\otimes m}$ (the subscript 0 denotes the degree zero part), and

(ii) the following differential equations in $\mathcal{R}_{\mathrm{iso}}$ are satisfied:

$$\begin{aligned} d[\![f_\bullet]\!] &= [\![f_\bullet]\!][\![h_\bullet]\!] - [\![l_\bullet]\!][\![f_\bullet]\!], & d[\![h_\bullet]\!] &= [\![g_\bullet]\!][\![f_\bullet]\!] - [\![h_\bullet]\!][\![h_\bullet]\!] - 1_{\mathbf{B}}, \\ d[\![g_\bullet]\!] &= [\![g_\bullet]\!][\![l_\bullet]\!] - [\![h_\bullet]\!][\![g_\bullet]\!], & d[\![l_\bullet]\!] &= [\![f_\bullet]\!][\![g_\bullet]\!] - [\![l_\bullet]\!][\![l_\bullet]\!] - 1_{\mathbf{W}}. \end{aligned}$$

The existence of a polarization easily follows from the acyclicity of the resolution $\mathcal{R}_{\mathrm{iso}}$ proved in [8]. See also Remark 25.

Let X be a collection. In the main theorem of this section we consider the free operad

$$\left(\mathcal{F}(X_{\mathbf{B}}; X_{\mathbf{W}}; \{f_k\}_{k \ge 0}; \{g_k\}_{k \ge 0}, \{X^{f_k}\}_{k \ge 0}; \{X^{g_k}\}_{k \ge 0}) \right),$$

generated by

(i) two copies $X_{\mathbf{B}}$ and $X_{\mathbf{W}}$ of X interpreted as in Theorem 7,

(ii) the generators f_k and $g_k, k \geq 0$, as in (34) and

(iii) two copies X^{f_k} and X^{g_k} of the suspension $\uparrow^{k+1} X$, $k \geq 0$.

A generator $x^{f_k} \in X^{f_k}$ is, for $x \in X(n)$, interpreted as a morphism $\mathtt{B}^{\otimes n} \to \mathtt{W}$ if k is even and as a morphism $\mathtt{B}^{\otimes n} \to \mathtt{B}$ if k is odd. Similarly, $x^{g_k} \in X^{g_k}$ is interpreted as a morphism $\mathtt{W}^{\otimes n} \to \mathtt{B}$ if k is even and as a morphism $\mathtt{W}^{\otimes n} \to \mathtt{W}$ if k is odd.

Theorem 24. *Let $(\mathcal{A}, d_\mathcal{A})$ be an (ordinary) operad satisfying (4) and let $\rho : \mathfrak{M} \to (\mathcal{A}, d_\mathcal{A})$, $\mathfrak{M} = (\mathcal{F}(X), d_\mathfrak{M})$, be its minimal model. Then the minimal model $\mathfrak{M}_{Iso}$ of the colored operad $\mathcal{A}_{Iso}$ is of the form*

$$\beta : (\mathcal{F}(X_\mathtt{B}; X_\mathtt{W}; \{f_k\}_{k \geq 0}; \{g_k\}_{k \geq 0}, \{X^{f_k}\}_{k \geq 0}; \{X^{g_k}\}_{k \geq 0}), D) \longrightarrow (\mathcal{A}_{Iso}, d)$$

with β given by $\beta(x_\mathtt{B}) = j_\mathtt{B}(\rho(x))$, $\beta(x_\mathtt{W}) = j_\mathtt{W}(\rho(x))$ for $x \in X$, $\beta(f_0) = f$, $\beta(g_0) = g$, while β is trivial on the remaining generators.

The differential D is on f_k and g_k, $k \geq 0$, given by formulas (35) and (36). For $n \geq 2$ and $x \in X(n)$, $D(x_\mathtt{B}) = j_\mathtt{B}(d_\mathfrak{M}(x))$, $D(x_\mathtt{W}) = j_\mathtt{W}(d_\mathfrak{M}(x))$ and, in the shorthand of (37),

$$D(x^{f\bullet}) = f_\bullet x_\mathtt{B} - x_\mathtt{W}[\![f_\bullet]\!] + f_\bullet x^{h\bullet} - (-1)^x x^{f\bullet}[\![h_\bullet]\!] - l_\bullet x^{f\bullet} - x^{l\bullet}[\![f_\bullet]\!] + \omega_{f\bullet},$$

$$D(x^{g\bullet}) = g_\bullet x_\mathtt{W} - x_\mathtt{B}[\![g_\bullet]\!] + f_\bullet x^{l\bullet} - (-1)^x x^{g\bullet}[\![l_\bullet]\!] - h_\bullet x^{g\bullet} - x^{h\bullet}[\![g_\bullet]\!] + \omega_{g\bullet},$$

$$D(x^{h\bullet}) = (-1)^x x_\mathtt{B}[\![h_\bullet]\!] - h_\bullet x_\mathtt{B} - h_\bullet x^{h\bullet} + (-1)^x x^{h\bullet}[\![h_\bullet]\!] - g_\bullet x^{f\bullet} + x^{g\bullet}[\![f_\bullet]\!] + \omega_{h\bullet},$$

$$D(x^{l\bullet}) = (-1)^x x_\mathtt{W}[\![l_\bullet]\!] - l_\bullet x_\mathtt{W} - l_\bullet x^{l\bullet} + (-1)^x x^{l\bullet}[\![l_\bullet]\!] - f_\bullet x^{g\bullet} + x^{f\bullet}[\![g_\bullet]\!] + \omega_{l\bullet}.$$

where $\omega_{f\bullet}$, $\omega_{g\bullet}$, $\omega_{h\bullet}$ and $\omega_{l\bullet}$ depend linearly on x and are elements of the ideal generated by $X^{f_k}(<n)$, $X^{g_k}(<n)$, $k \geq 0$.

Proof. Assume that the differential D exists. The map β then decomposes as

$$\mathfrak{M}_{Iso} \xrightarrow{\beta_1} \left(\frac{\mathcal{F}(X_\mathtt{B}; X_\mathtt{W}; f_\bullet, g_\bullet, h_\bullet, l_\bullet)}{I}, d \right) \xrightarrow{\beta_2} \mathcal{A}_{Iso},$$

where the ideal I is generated by

$$f_\bullet x_\mathtt{B} - x_\mathtt{W}[\![f_\bullet]\!], \quad g_\bullet x_\mathtt{W} - x_\mathtt{B}[\![g_\bullet]\!], \quad (-1)^x x_\mathtt{B}[\![h_\bullet]\!] - h_\bullet x_\mathtt{B}, \quad \text{and} \quad (-1)^x x_\mathtt{W}[\![l_\bullet]\!] - l_\bullet x_\mathtt{W}, \quad x \in X.$$

The definitions of β_1 and β_2 is obvious. To prove that β_1 is a quism, consider the filtration of $\mathfrak{M}_{Iso}$ given by the number of generators from X^{f_k}, X^{g_k}, $k \geq 0$ and use a standard spectral sequence argument. The nature of β_2 is similar to that of the map $\beta_{\mathtt{B} \to \mathtt{W}}$ in Corollary 6 and the same arguments as in its proof show that β_2 is a quism. The tails can be constructed by general nonsense. We leave the details to the reader. $\square$

Remark 25. There is a very explicit, though rather involved, formula for the polarization of Definition 23. If we do not demand the equivariance, there exists (in fact many) a polarization with *integral* coefficients. An example is given, for $m = 2$, by

$$[\![f_\bullet]\!] := \sum_{i \geq 0} f_\bullet (g_\bullet f_\bullet)^{\otimes i} \otimes f_{2i}, \qquad [\![g_\bullet]\!] := \sum_{i \geq 0} g_\bullet (f_\bullet g_\bullet)^{\otimes i} \otimes g_{2i},$$

$$[\![h_\bullet]\!] := h_\bullet \otimes \mathbb{1} + \sum_{i \geq 1} (g_\bullet f_\bullet)^{\otimes i} \otimes h_{2i-1}, \quad [\![l_\bullet]\!] := l_\bullet \otimes \mathbb{1} + \sum_{i \geq 1} (f_\bullet g_\bullet)^{\otimes i} \otimes l_{2i-1}.$$

and by similar formulas for $m > 2$. These formulas may be used for integral minimal models of $\mathcal{A}_{Iso}$ for non-Σ operads.

We believe that, for $\mathcal{A}$ quadratic Koszul, there exists a closed formula for the 'tails' of the differential D. It might be a challenging, though very involved, exercise to write this formula when $\mathcal{A} = Ass$, the operad describing associative algebras.

6. FINAL REMARKS AND CHALLENGES

In this section, "sh" will abbreviate "strongly (or strong) homotopy."

The 'category' of sh algebras. Let $\mathcal{A}$ be an ordinary operad. Recall that a *sh $\mathcal{A}$-algebra* is an algebra over the minimal model $\mathfrak{M} = (\mathcal{F}(X), d_{\mathfrak{M}})$ of $\mathcal{A}$. A *sh homomorphism of sh $\mathcal{A}$-algebras* is then a colored algebra over the minimal model $\mathfrak{M}_{\mathtt{B}\to\mathtt{W}}$ of $\mathcal{A}_{\mathtt{B}\to\mathtt{W}}$. See Example 4 for the case $\mathcal{A} = Ass$.

Let $\mathbf{P} : \mathfrak{M}_{\mathtt{B}\to\mathtt{W}} \to \mathcal{E}nd_{U,V}$ and $\mathbf{Q} : \mathfrak{M}_{\mathtt{W}\to\mathtt{T}} \to \mathcal{E}nd_{V,W}$, where $\mathtt{T}$ is a Third color, be two sh homomorphisms. Their 'composition' $\mathbf{Q} \circ \mathbf{P} : \mathfrak{M}_{\mathtt{B}\to\mathtt{T}} \to \mathcal{E}nd_{U,W}$ is given by an appropriate map

$$(38) \qquad\qquad \Xi : \mathfrak{M}_{\mathtt{B}\to\mathtt{T}} \to \mathfrak{M}_{\mathtt{B}\to\mathtt{W}} * \mathfrak{M}_{\mathtt{W}\to\mathtt{T}}$$

($*$ denotes the free product), whose existence is guaranteed by general nonsense.

A topological counterpart of this situation was studied by Boardman and Vogt. It turned out that the composition above need not be associative, thus topological sh-algebras do not form an honest category, but only a hazier object called a weak Kan category [1, Theorem 4.9].

We are sure that the situation in algebra is much better, because minimal models are 'reasonably' functorial. So we believe that the following problem has an affirmative answer.

Problem 26. *The map Ξ in (38) can be constructed in such a way that the composition $\circ$ is associative, that is, sh algebras and their sh homomorphisms form an honest category.*

Since sh algebras over quadratic Koszul operads may be interpreted as certain dg cofree coconnected coalgebras and their maps as homomorphisms of these coalgebras, for these operads such a map Ξ inducing an associative composition clearly exists.

Ideal perturbation lemma. In [8] we proved that a proper formulation of the homological perturbation lemma needs a good notion of chain homotopy equivalences. These 'good homotopy equivalences' were defined as representations of the colored operad $\mathcal{R}_{\mathrm{iso}}$ of (33) and called *strong* homotopy equivalences. Our theory was, however, tailored for chain complexes with no additional algebraic structure.

Constructions of Section 5 provide a similar notion for chain complexes with an additional algebraic structure:

Definition 27. Let $A : \mathfrak{M} \to \mathcal{E}nd_U$ and $B : \mathfrak{M} \to \mathcal{E}nd_V$ be two sh $\mathcal{A}$-algebras. A strong homotopy equivalence between A and B is given by an operad action $S : \mathfrak{M}_{\mathcal{I}so} \to \mathcal{E}nd_{U,V}$ such that $S \circ j_{\mathtt{B}} = A$ and $S \circ j_{\mathtt{W}} = B$, where

$$j_{\mathtt{B}} : \mathfrak{M} \hookrightarrow \mathfrak{M}_{\mathcal{I}so} \hookleftarrow \mathfrak{M} : j_{\mathtt{W}}$$

are obvious inclusions.

Problem 28. *Formulate an 'ideal perturbation lemma' for chain complexes with an additional algebraic structure, using sh equivalences of Definition 27.*

Homotopies through homomorphisms. While it is quite obvious what a "homotopy through homomorphisms" for maps of topological monoids should be, the situation in algebra is much subtler.

Suppose we have two sh $\mathcal{A}$-algebras $A : \mathfrak{M} \to \mathcal{E}nd_U$, $B : \mathfrak{M} \to \mathcal{E}nd_V$ and two sh homomorphisms $\mathbf{P}, \mathbf{Q} : A \to B$ given by actions $\mathbf{P}, \mathbf{Q} : \mathfrak{M}_{B\to\mathtt{W}} \dashrightarrow \mathcal{E}nd_{U,V}$. These data clearly define a representation of the outer square of the diamond
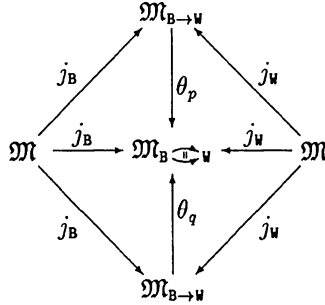
Definition 29. A homotopy through sh homomorphisms between $\mathbf{P}$ and $\mathbf{Q}$ is an operad action $\mathfrak{M}_{B\rightleftarrows\mathtt{W}} \to \mathcal{E}nd_{U,V}$ extending the data above. We write $\mathbf{P} \sim \mathbf{Q}$ if there exists a sh homotopy between $\mathbf{P}$ and $\mathbf{Q}$.

As we saw in Example 21, this notion generalizes the homotopy structure on the category of $A(\infty)$-algebras considered in [3, 5.4]. The relation $\sim$ is indeed an equivalence:

Proposition 30. *The relation $\sim$ is an equivalence on the set of sh homomorphisms of sh $\mathcal{A}$-algebras.*

Proof. The reflexivity ($\mathbf{P} \sim \mathbf{P}$) and symmetry ($\mathbf{P} \sim \mathbf{Q} \implies \mathbf{P} \sim \mathbf{Q}$) is obvious. If $\mathbf{P} \sim \mathbf{Q}$ and $\mathbf{Q} \sim \mathbf{T}$ then the homotopy between $\mathbf{P}$ and $\mathbf{T}$ can be constructed by standard homological methods (general nonsense). We leave the details to the reader. $\square$

Possible generalizations. The main results of this paper (Theorems 7, 18 and 24) seem to admit a generalization to resolutions of colored operads $\mathcal{A}_\mathfrak{D}$ describing $\mathfrak{D}$-diagrams of $\mathcal{A}$-algebras for arbitrary $\mathfrak{D}$.

We start with a cofibrant resolution $\mathcal{R}_\mathfrak{D} = (\mathcal{F}(F), d_\mathfrak{D})$ of the diagram $\mathfrak{D}$ considered as a V-colored operad, where V is the set of vertices of $\mathfrak{D}$. The second ingredient is the minimal model $\mathfrak{M} = (\mathcal{F}(X), d_\mathfrak{M})$ of $\mathcal{A}$. We believe that it is possible to prove:

Conjecture 31. *There exists a cofibrant model $\mathfrak{M}_\mathfrak{D}$ of the colored operad $\mathcal{A}_\mathfrak{D}$ of the form*

$$\mathfrak{M}_\mathfrak{D} = \left(\mathcal{F}(X \times V \sqcup F \sqcup X^F), D\right),$$

where $X \times \{v\} \subset X \times V$ is, for $v \in V$, a copy of X "concentrated" in the color v and elements of X^f are considered as multilinear maps with the same source and the same target as $f \in F$. The differential D on $X \times V$ and on F is given by the inclusions

$$j_\mathfrak{D} : \mathcal{R}_\mathfrak{D} \hookrightarrow \mathfrak{M}_\mathfrak{D} \text{ and } j_v : \mathfrak{M} \hookrightarrow \mathfrak{M}_\mathfrak{D}, \ v \in V,$$

while the differential of x^f, for $f \in F$ and $x \in X(n)$, $n \geq 2$, decomposes as

$$D(x^f) = Pr + \omega_x.$$

The principal part Pr further decomposes as $Pr = Pr_1 + Pr_2$, where Pr_1 is 'ideologically' the commutator of x and f and Pr_2 is given by a polarization of $d_{\mathfrak{D}}(f)$. The tail ω_x belongs to the ideal generated by $X^J(<n)$, $f \in F$. The model $\mathfrak{M}_{\mathfrak{D}}$ is minimal if and only if $\mathcal{R}_{\mathfrak{D}}$ is.

We leave as an exercise to check that all models described in this paper are of the above form (in all cases $V = \{\mathtt{B}, \mathtt{W}\}$).

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MATHEMATICAL INSTITUTE OF THE ACADEMY
ŽITNÁ 25, 115 67 PRAHA 1
CZECH REPUBLIC
E-mail: `markl@math.cas.cz`