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In this paper we consider the following problem, which is in a sense inverse to the problem whether a given map admits a probability section, namely:
Given a Markov kernel $(\nu_x)_{x \in X}$ from $(X, \mathcal{B}(X))$ to $(Y, \mathcal{B}(Y))$; does there exist a measurable map $u: Y \to X$ such that $\nu_x(u^{-1}(x))=1$ for all $x \in X$.
In this situation $x \mapsto \nu_x$ is called a probability section for u.

<u>Notation</u>: For a Polish space P let $\mathcal{B}(P)$ be the Borel field of P, M(P) the vector lattice of real-valued signed measures on P, $M_+(P)$ the cone of positive elements of M(P), and $M_+^1(P)$ the set of probability measures in $M_+(P)$. These spaces will always carry the narrow topology (Schwartz [5],p.370, Def.1).

Our setting will be as follows.
X and Y are Polish spaces, $\nu.: X \to M_+^1(Y)$, $x \mapsto \nu_x$ is a Markov kernel. i.e. a $\mathcal{B}(X)$-$\mathcal{B}(M_+^1(Y))$-measurable map. For such a kernel $x \mapsto \nu_x(B)$ is Borel measurable for all $B \in \mathcal{B}(Y)$.

Definition:

$(\nu_x)_{x\in X}$ is called

(i) orthogonal if for all $a,b \in X$ with $a\neq b$ we have $\nu_a \perp \nu_b$, i.e. there is a set $S_{a,b} \in \mathcal{B}(Y)$ with
$\nu_a(S_{a,b})=1=\nu_b(Y\setminus S_{a,b})$,

(ii) uniformly orthogonal if for all $a \in X$ there is a set $S_a \in \mathcal{B}(Y)$ with $\nu_a(S_a)=1=\nu_b(Y\setminus S_a)$ for all $b \in X\setminus\{a\}$.

(iii) completely orthogonal if for all $A \in \mathcal{B}(X)$ there is a set $S_A \in \mathcal{B}(Y)$ with

$$\nu_a(S_A)=1=\nu_b(Y\setminus S_A)$$

for all $a \in A$, $b \notin A$.

(iv) implemented if $\nu.$ is a probability section for some $\mathcal{B}(Y)$-$\mathcal{B}(X)$-measurable map $u\colon Y \to X$.

Remarks:

a) Obviously each of these properties implies the preceding ones.

b) Property (ii) was introduced by D.Maharam [3] and can easily be seen to be strictly stronger than (i). Consider for instance the family $(\nu_x)_{x\in[0,1]}$ on the unit interval where

$$\nu_x = \begin{cases} \text{Dirac measure } \delta_x, & x \neq 0 \\ \text{Lebesgue measure } \lambda, & x = 0. \end{cases}$$

Using the continuum hypothesis Maharam [3] gave an example of a uncountable family $(\nu_i)_{i\in I}$ of pairwise orthogonal measures on the unit square such that none of its uncountable subfamilies is uniformly orthogonal.

c) It seems to be unknown whether (ii) implies (iii).

d) For an uncountable space X Burgess and Mauldin [1] showed that for an orthogonal kernel $(\nu_x)_{x\in X}$ there exists a Cantor subset C of X and an analytic subset D of Y such that $(\nu_x)_{x\in C}$ is a probability section for some Borel measurable map $u: D \to C$.

In our situation properties (iii) and (iv) are equivalent as the following proposition shows.

Proposition 1:

Every completely orthogonal kernel is implemented.

Proof: The system $\mathfrak{N} = \{B \in \mathfrak{B}(Y) \mid \forall\, x \in X: \nu_x(A) = 0\}$ is obviously a σ-ideal in $\mathfrak{B}(Y)$. Let $\mathfrak{B}(Y)/_{\mathfrak{N}}$ be the corresponding quotient algebra and let $\Phi: \mathfrak{B}(X) \to \mathfrak{B}(Y)/_{\mathfrak{N}}$ be the map which sends $A \in \mathfrak{B}(X)$ to the equivalence class $[S_A]$ of S_A in $\mathfrak{B}(Y)/_{\mathfrak{N}}$. It is easy to see that Φ is a well-defined σ-homomorphism. By a theorem of Sikorski ([6], p.139, 32.5) there exists a $\mathfrak{B}(Y)$-$\mathfrak{B}(X)$-measurable map $u: Y \to X$ such that $\Phi(A) = [u^{-1}(A)]$ for all $A \in \mathfrak{B}(X)$. A straightforward calculation shows that u has the required properties.

Our next aim is to give a condition on the kernel which is easier to handle than (iii) but still implies (iii).

Definition:

(i) For $\mu \in M_+(X)$ let $\int \nu_x \, d\mu(x)$ be the measure on $\mathcal{B}(Y)$ defined by $B \to \int \nu_x(B) \, d\mu(x)$.

(ii) A kernel $(\nu_x)_{x \in X}$ is said to preserve orthogonality if $\int \nu_x \, d\mu \perp \int \nu_x \, d\mu'$ holds for all $\mu, \mu' \in M_+(X)$ with $\mu \perp \mu'$.

Remarks:

a) The kernel $(\nu_x)_{x \in X}$ preserves orthogonality if and only if $\mu \mapsto \int \nu_x \, d\mu(x)$ defines a lattice isomorphism from $M(X)$ into $M(Y)$.

b) If $(\nu_x)_{x \in X}$ is implemented, it preserves orthogonality.

c) A kernel which preserves orthogonality is obviously orthogonal.

d) A kernel $(\nu_x)_{x \in X}$ that preserves orthogonality also has the following properties:

1) $\nu.$ is an injective map, hence a Borel isomorphism from X onto $\{\nu_x | x \in X\}$ which is a Borel subset of $M_+^1(Y)$ (Schwartz, [5], p.107, Lemma 14 and following remark).

2) Given two disjoint sets $A, B \in \mathcal{B}(X)$ let $\mathcal{A} = \{\nu_x | x \in A\}$ and $\mathcal{B} = \{\nu_x | x \in B\}$. Then, for any two probability measures m on $\mathcal{A}$ and n on $\mathcal{B}$, we have $r(n) \perp r(m)$, where $r(m)(E) = \int_{\mathcal{A}} \tau(E) \, dm(\tau)$

and r(n) is defined in the same way.

This means that any two measures from the respective measure convex hulls of $\mathfrak{A}$ and $\mathfrak{S}$ are orthogonal.

From the last remark we can conclude that a kernel that preserves orthogonality is completely orthogonal provided the following is true:

If $\mathcal{M}_1, \mathcal{M}_2 \subset M_+^1(Y)$ are measure convex Borel sets which are mutually orthogonal (i.e. $\mu_1 \in \mathcal{M}_1$ and $\mu_2 \in \mathcal{M}_2$ implies $\mu_1 \perp \mu_2$) then there exists a set $S \in \mathcal{B}(Y)$ with $\mu_1(S)=1=\mu_2(Y \setminus S)$ for all $\mu_1 \in \mathcal{M}_1$, $\mu_2 \in \mathcal{M}_2$. This, however, is not true in general as a counterexample by D. Preiss [4] shows. But if the kernel meets additional requirements it is enough to have separation only for mutually orthogonal compact convex sets. The next proposition states that those sets can be separated.

<u>Proposition 2</u>:

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be compact convex subsets of $M_+^1(Y)$ such that $\mu_1 \perp \mu_2$ for any two measures $\mu_1 \in \mathcal{K}_1$ and $\mu_2 \in \mathcal{K}_2$. Then there is a Borel set $S \subset Y$ such that $\mu_1(S)=1=\mu_2(Y \setminus S)$ holds for all $\mu_1 \in \mathcal{K}_1$ and $\mu_2 \in \mathcal{K}_2$.

Before we shall proceed a few comments are in order:

The question whether a result of this type is true was asked by H. von Weizsäcker on the 8th Winter School. Subsequently v. Weizsäcker [8] and ourselves succeeded in proving it independently. Later we learned that the result was already obtained by A.Goullet de Rugy [2] in 1971.

Goullet de Rugy's and v. Weizsäcker's proofs both use the usual separation theorem for compact convex subsets of a locally convex

Hausdorff space. Our proof also uses the Hahn-Banach theorem but in a rather different argument which we are going to outline later.

Before we do that, we want to point out that Prop. 2 implies that certain kernels which preserve orthogonality are completely orthogonal. This was first observed by v. Weizsäcker ([8], Satz 1); here we will give a different proof of this fact.

Theorem:

Suppose that the kernel $(\nu_x)_{x\in X}$ preserves orthogonality and that $\mathcal{E} = \{\nu_x | x \in X\}$ is σ-compact. Then $(\nu_x)_{x\in X}$ is completely orthogonal

Proof: For $\mathfrak{A}, \mathfrak{B} \subset \mathcal{E}$ we write $\mathfrak{A} \perp\!\!\!\perp \mathfrak{B}$ if there exists $S \in \mathfrak{B}(Y)$ such that $\alpha(S)=1=\beta(Y\setminus S)$ for all $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$. We first show that

(*) $\mathcal{K} \perp\!\!\!\perp \mathcal{E} \setminus \mathcal{K}$ for every compact set $\mathcal{K} \subset \mathcal{E}$. Because $\mathcal{E} \setminus \mathcal{K}$ is an open subset of a σ-compact metrizable space(Schwartz [5], p.385, Thm. 7) it is the union of a sequence, $(\mathcal{K}_n)_{n\in\mathbb{N}}$ say, of compact sets. It is obvious that $\mathcal{K} \perp\!\!\!\perp \mathcal{K}_n$ for all n implies $\mathcal{K} \perp\!\!\!\perp \bigcup_n \mathcal{K}_n = \mathcal{E} \setminus \mathcal{K}$. Hence, to prove (*), it is enough to show $\mathcal{K} \perp\!\!\!\perp \mathcal{K}'$ for any two disjoint compact subsets of $\mathcal{E}$.

So let $\mathcal{K}, \mathcal{K}' \subset \mathcal{E}$ be disjoint and compact.

Observe that the closed convex hulls $\overline{co}\,\mathcal{K}$ and $\overline{co}\,\mathcal{K}'$ of $\mathcal{K}$ and $\mathcal{K}'$ are compact and convex and, by the Krein-Milman theorem, are contained in $\{r(m) | m \in M^1_+(\mathcal{K})\}$ and $\{r(n) | n \in M^1_+(\mathcal{K}')\}$ respectively.

As already mentioned $x \mapsto \nu_x$ defines a Borel isomorphism of X onto $\mathcal{E}$. Therefore there exist disjoint Borel sets $A, A' \subset X$ such that $\mathcal{K} = \{\nu_x | x \in A\}$ and $\mathcal{K}' = \{\nu_x | x \in A'\}$. By Remark d2) preceding Prop.2 $\overline{co}\,\mathcal{K}$ and $\overline{co}\,\mathcal{K}'$ satisfy the assumptions of Prop.2 from which (*) is an immediate consequence. Now consider the collection $\{\mathfrak{A} \subset \mathcal{E} : \mathfrak{A} \perp\!\!\!\perp \mathcal{E}\setminus\mathfrak{A}\}$

which is easily seen to be a σ-field containing the compact sets by (*). Since $\mathcal{E}$ is σ-compact it therefore contains all Borel subsets of $\mathcal{E}$. Observing that, for every $A \in \mathcal{B}(X)$, $\{\nu_x | x \in A\}$ is a Borel subset of $\mathcal{E}$ with complement $\{\nu_x | x \in X \setminus A\}$ completes the proof.

One easily deduces the following.

Corollary.

Let $(\nu_x)_{x \in X}$ be a Markov kernel such that $\{\nu_x : x \in X\}$ is σ-compact. Then the following statements are equivalent:

(i) $(\nu_x)_{x \in X}$ preserves orthogonality

(ii) $(\nu_x)_{x \in X}$ is completely orthogonal

(iii) $(\nu_x)_{x \in X}$ is implemented.

Remark:

The assumptions of the corollary are satisfied if, in addition, X is σ-compact and $x \mapsto \nu_x$ is continuous.

We will now describe the steps in the proof of Prop. 2. To do this we need some more notation.

For a subset $\mathcal{M}$ of $M_+(Y)$ let $p_{\mathcal{M}} : \mathcal{B}(Y) \to \mathbb{R}_+$ be defined by

$p_{\mathcal{M}}(B) = \sup \{\mu(B) | \mu \in \mathcal{M}\}$ and

$\tilde{\mathcal{M}} = \{\mu \in M_+(Y) | \forall B \in \mathcal{B}(Y) : \mu(B) \leq p_{\mathcal{M}}(B)\}$

$p_{\mathcal{M}}$ is always subadditive and monotone, and for compact $\mathcal{M}$, $p_{\mathcal{M}}$ is regular in the following sense:

$$p_{\mathcal{M}}(K) = \inf\{p_{\mathcal{M}}(U) | K \subset U,\ U \text{ open}\}$$

for all compact $K \subset Y$.

Lemma 1:
Let $\mathcal{M}$ and $\mathcal{N}$ be compact convex subsets of $M_+(Y)$ such that $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$ are mutually orthogonal. Then there exists an $A \in \mathcal{B}(Y)$ with

$$\mu(A) = 0 = \nu(Y \setminus A)$$

for all $\mu \in \mathcal{M}$ and $\nu \in \mathcal{N}$.

Proof: Let $B^\infty(Y)$ be the space all bounded real-valued Borel measurable functions on Y. Define $p,q\colon B^\infty(Y) \to \mathbb{R}$ by

$$p(f) = \sup \{\textstyle\int f_+ \, d\mu \mid \mu \in \tilde{\mathcal{M}}\}$$

and

$$q(f) = \sup \{\textstyle\int f_+ \, d\nu \mid \nu \in \tilde{\mathcal{N}}\}.$$

Then p and q are sublinear and monotone.
If we define $w\colon B^\infty(Y) \to \mathbb{R}$ by

$$w(f) = \inf\{p(g) + q(f_+ - g) \mid 0 \le g \le f_+\},$$

w is sublinear and monotone with

$$w(f) \le \min(p(f), q(f))$$

for all $f \in B^\infty(Y)$.

(1) We will show that, for all $K \subset Y$ compact, $w(1_K)=0$. Assume there is a compact set $K_o \subset Y$ with $w(1_{K_o}) \neq 0$. Then, by the Hahn-Banach theorem, there exists a linear functional φ on $B^\infty(Y)$ dominated by

w such that

$$\varphi(1_{K_o}) = w(1_{K_o}).$$

From a result of Topsøe ([7], Theorem 2) it follows that

$$\mu(B) = \sup\{\inf\{\varphi(1_U) \mid K \subset U\ ,\ U \text{ open}\} \mid K \subset B,\ K \text{ comp.}\}$$

defines a positive Radon measure on Y.

Then

$$\mu(k_o) \geq \varphi(1_{K_o}) > 0$$

and for open $U \subset Y$,

$$\mu(U) \leq \varphi(1_U) \leq w(1_U) \leq \min(p(1_U),\ q(1_U))$$
$$= \min(p_{\mathcal{M}}(U),\ p_{\mathcal{N}}(U)).$$

Since $p_{\mathcal{M}}$ and $p_{\mathcal{N}}$ are regular and μ is a Radon measure this implies

$$\mu(K) \leq \min(p_{\mathcal{M}}(K),\ p_{\mathcal{N}}(K))$$

for all compact $K \subset Y$,

hence $\mu \in \tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}$.

Since $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$ are mutually orthogonal this leads to $\mu \perp \mu$, contradicting $\mu \neq 0$.

Thus (1) is proved.

(2) We now claim that $w(1_Y)=0$.

Let $\varepsilon > 0$ be given. Since $\mathcal{N}$ is compact we may choose $K \subset Y$ compact with

$$q(1_{Y\setminus K}) = p_{\mathcal{N}}(Y\setminus K) < \varepsilon$$

(cf. Schwartz [5], p.381, Theorem 4).

Now

$$w(1_Y) \leq w(1_K) + w(1_{Y\setminus K}) = w(1_{Y\setminus K})$$
$$\leq q(1_{Y\setminus K}) < \varepsilon \, .$$

(3) Finally we will construct the separating set A. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers with $\sum_{n=1}^{\infty} n\,\varepsilon_n < \infty$.
By (2) there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $B^\infty(Y)$ such that $\quad 0 \leq f_n \leq 1 \quad$ and

$$0 \leq p(f_n) + q(1-f_n) \leq \varepsilon_n .$$

Define

$$A := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} \{y \in Y \mid f_n(y) \geq \tfrac{1}{n}\}.$$

This is obviously a Borel set and it is not difficult to show that it has the desired properties.

<u>Lemma 2</u>:

Let $\mathcal{M} \subset M_+(Y)$ be compact convex and $\nu \in \widetilde{\mathcal{M}}\setminus\{0\}$.
Then there is a measure $\mu \in \mathcal{M}$ such that μ and ν are not orthogonal.

<u>Proof</u>: Let $\mathcal{M}^* := \{\mu \otimes \delta_\mu \mid \mu \in \mathcal{M}\} \subset M_+(Y\times\mathcal{M})$.

(1) We claim that

$$p_{\mathcal{M}^*}(C \times \mathcal{K}) = \inf\{p_{\mathcal{M}^*}(U \times \mathcal{G}) \mid C \subseteq U \;, \mathcal{K} \subseteq \mathcal{G} \;, U, \mathcal{G} \quad \text{open}\}.$$

for compact $C \subset Y$, $\mathcal{K} \subset \mathcal{M}$.

Let $C \subset Y$ and $\mathcal{K} \subset \mathcal{M}$ be compact and $\varepsilon > 0$. Then $\mu \mapsto \mu(C)$ is an upper semi-continuous function on $\mathcal{M}$. Thus $\mathcal{Z} := \{\varrho \in \mathcal{M} \mid \varrho(C) < p_{\mathcal{M}^*}(C \times \mathcal{K}) + \varepsilon\}$ is open and contains $\mathcal{K}$. Since $\mathcal{M}$ is compact we can find an open set $\mathcal{G}$ in $\mathcal{M}$ with $\mathcal{K} \subset \mathcal{G} \subset \bar{\mathcal{G}} \subset \mathcal{Z}$. Now $p_{\bar{\mathcal{G}}}$ is regular. Hence there exists an open set $U \supset C$ such that $p_{\bar{\mathcal{G}}}(U) \leq p_{\bar{\mathcal{G}}}(C) + \varepsilon < p_{\mathcal{M}}^*(C \times \mathcal{K}) + 2\varepsilon$. Obviously

$$
\begin{aligned}
p_{\mathcal{M}^*}(U \times \mathcal{G}) &= \sup\{\varrho(U) \mid \varrho \in \mathcal{G}\} \\
&\leq \sup\{\varrho(U) \mid \varrho \in \bar{\mathcal{G}}\} = p_{\bar{\mathcal{G}}}(U).
\end{aligned}
$$

Thus our claim is proved.

(2) Define $p: B^\infty(Y \times \mathcal{M}) \to \mathbb{R}$ by

$$p(f) = \sup\{\int f_+ d(\mu \otimes \delta_\mu \mid \mu \in \mathcal{M}^*\}$$

Then p is sublinear and monotone and extends $p_{\mathcal{M}^*}$. Let $\pi: Y \times \mathcal{M} \to Y$ be the canonical projection and let $F = \{h \circ \pi \mid h \in B^\infty(Y)\}$. Moreover, let $\varphi: F \to R$ be the linear functional given by

$$\varphi(h \circ \pi) = \int h \, d\nu$$

Then φ is dominated by p. Thus, by the Hahn-Banach theorem there exists an extension ψ of φ to $B^\infty(Y \times \mathcal{M})$ which is still dominated by p. Applying Topsøe's procedure to ψ we get a measure n on $\mathfrak{B}(Y \times \mathcal{M})$ with

a) $n(K) \geq \psi(1_K)$ for K compact

and

b) $n(V) \leq \psi(1_V)$ for V open in $Y \times \mathcal{M}$.

From this we deduce

c) $n(A \times \mathcal{M}) = \nu(A)$ for all $A \in \mathcal{B}(Y)$.

Combining a) and (1) yields, in addition,

d) $n(A \times B) \leq p(1_{A\times B}) = p_{\mathcal{M}}*(A \times B)$

for all $A \in \mathcal{B}(Y)$, $B \in \mathcal{B}(\mathcal{M})$.

Let m be the normalized image measure of n under the canonical projection to $\mathcal{M}$. Let $\mu = r(m)$. Then $\mu \in \mathcal{M}$ (since $\mathcal{M}$ is compact convex) and we claim that μ and ν are not orthogonal. Assume the contrary. Then there is a set $A \in \mathcal{B}(Y)$ with $\mu(A) = 0 = \nu(Y \setminus A)$. The definition of μ together with c) yields that n is supported by $A \times \{\varsigma \in \mathcal{M} : \varsigma(A) = 0\} \in \mathcal{B}(Y \times \mathcal{M})$. But d) implies

$$n(A \times \{\varsigma \in \mathcal{M} : \varsigma(A) = 0\}) = 0.$$

Hence $n=0$ and therefore $\nu=0$, a contradiction.

Proof of Prop. 2:

Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are mutually orthogonal and compact convex we can apply Lemma 2 twice to deduce that $\tilde{\mathcal{K}}_1$ and $\tilde{\mathcal{K}}_2$ are mutually orthogonal, too. Prop. 2 then follows from Lemma 1.

Remarks:

a) According to oral communication by M. Talagrand, G.Mokobodzki proved the following result assuming continuum hypothesis (or Martin's axiom):

Any two mutually orthogonal measure convex Borel subsets of $M_+^1(Y)$ can be separated by a universally measurable subset of Y.

This result implies:

(MA or CH) A kernel that preserves orthogonality is uniformly orthogonal and is implemented by a $\mathcal{B}_u(Y)$ - $\mathcal{B}(X)$-measurable map (where $\mathfrak{G}(Y)$ is the σ-field of universally measurable sets in Y).

b) It is worth mentioning that the measure convex hull of a Borel set of discrete measures can be separated (by an analytic set) from any measure convex Borel set provided the two are mutually orthogonal.

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