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In: Martin Čadek (ed.): Proceedings of the 25th Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 2006. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 79. pp. [129]--138.

Persistent URL: http://dml.cz/dmlcz/701772

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 79 (2006), pp. 129-138

A GENERALIZATION OF FUETER'S MONOGENIC FUNCTIONS TO FINE DOMAINS

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ABSTRACT. The so-called quaternionic analysis is a theory analogous to complex analysis but the complex numbers are replaced by the non-commutative 4-dimensional field $\mathbf{H}$ of quaternions. A role of holomorphic functions in quaternionic analysis is played by monogenic functions. In this note we extend the notion of monogeneity to functions defined on fine domains in $\mathbf{R}^4$, i.e., domains in the fine topology of classical potential theory. In fact, we generalize to some extent Fuglede's theory of finely holomorphic functions for dimension 4.

1. INTRODUCTION

At the beginning of the twentieth century É. Borel tried to extend holomorphic functions to more general domains (no longer open) in such a way that the unique continuation property was preserved, see [3]. But his domains were rather special and his theory never became too popular. On the other hand, it did give inspiration to the creation of the important theory of quasi-analytic classes (on the real line) by Denjoy, Carleman and Mandelbrojt. In the 1970-80's B. Fuglede and others developed more elegant and more general theory of finely holomorphic functions.

Once the theory of finely holomorphic functions became firmly established it became natural to ask how to extend this deep theory for higher dimensions. Such an extension to functions of several complex variables has been made by B. Fuglede alone, see [11] and cf. [10]. In this note we present another way how to do it at least for dimension 4. Namely, in the 1930's R. Fueter developed the so-called quaternionic analysis. Roughly speaking, quaternionic analysis is a theory analogous to complex analysis but the complex numbers $\mathbf{C}$ are replaced by the quaternions $\mathbf{H}$. It studies $\mathbf{H}$-valued functions of one quaternionic variable and a role of holomorphic functions in quaternionic analysis is played by functions we shall call monogenic. In this note we suggest a definition of finely monogenic functions and investigate their properties. Let us remark that in other dimensions monogenic functions are also studied in the so-called Clifford analysis and the corresponding finely monogenic functions will be investigated in a next note.

The paper is in final form and no version of it will be submitted elsewhere.

2. Finely holomorphic functions

The theory of finely holomorphic functions has been developed by B. Fuglede, A. Debiard and B. Gaveau, T. J. Lyons and A. G. O'Farrell and others. It generalizes the theory of holomorphic functions to domains in a topology finer than the Euclidean topology of the complex plane $\mathbf{C}$, namely, in the so-called fine topology.

Recall that the fine topology $\mathcal{F}$ in $\mathbf{R}^n$, $n \geq 2$, is the weakest topology making all subharmonic functions in $\mathbf{R}^n$ continuous, see e.g. [2, Chapter 7]. It is strictly finer than the Euclidean topology in $\mathbf{R}^n$. For example, if K is a dense countable subset of an open set $G \subset \mathbf{R}^n$, then $U := G \setminus K$ is a finely open set but it has no interior points in the usual sense. Let $U \subset \mathbf{R}^n$ be finely open and $f : U \to \mathbf{R}^m$. Then we call f finely continuous on U if

$$f : (U, \text{ fine top.}) \to (\mathbf{R}^m, \text{ Euclidean top.})$$

is continuous. Denote by $\mathcal{F}_z$ the family of all finely open sets containing a point $z \in \mathbf{R}^n$. The fine limit of f at a point $z_0 \in U$ can be understood as the usual limit along some fine neighourhood of z_0, i.e., there is $V \in \mathcal{F}_{z_0}$ such that

$$\operatorname*{fine-lim}_{z \to z_0} f(z) = \lim_{z \to z_0, z \in V} f(z)\,,$$

see [2, p. 207]. Moreover, we call a linear map $L : \mathbf{R}^n \to \mathbf{R}^m$ the fine differential of f at a point $z_0 \in U$ if

$$\operatorname*{fine-lim}_{z \to z_0} \frac{f(z) - f(z_0) - L(z - z_0)}{|z - z_0|} = 0\,.$$

We write fine-$df(z_0) := L$ and, for $l = 1, \ldots, n$,

$$\text{fine-}\frac{\partial f}{\partial x_l}(z_0) := \text{fine-}df(z_0)(e_l)$$

where a vector $e_l \in \mathbf{R}^n$ has 1 at the l-th place and 0's otherwise.

It turns out to be fruitful to deal with finely harmonic functions, see [6]. For our purposes, let us recall the following characterization. A real-valued function f is finely harmonic on a finely open set $U \subset \mathbf{R}^n$ if and only if for every $z \in U$ there is $V \in \mathcal{F}_z$ such that $f|_V$, the restriction of f to V, is a uniform limit of functions harmonic on open sets containing V. Let us remark that f is harmonic on a usual open set $G \subset \mathbf{R}^n$ if and only if f is finely harmonic and locally bounded (from above or below) on G. In case of $\mathbf{R}^2$ we need not assume local boundedness of f. Moreover, finely harmonic functions are finely continuous but have the first fine differential only almost everywhere (a.e.), in general, see e.g. [9]. By [15], a finely harmonic function f in a fine domain U can vanish in some fine neighbourhood of a point of U without being identically 0 on the whole U.

Let $U \subset \mathbf{C}$ be finely open and $f : U \to \mathbf{C}$. There is a few equivalent definitions of a finely holomorphic function f, see e.g. [7], [8] and [13]:

(AM) $\forall z \in U\ \exists V \in \mathcal{F}_z : f|_V$ is a uniform limit of functions holomorphic on open sets containing V.

(Der) f has a finely continuous fine derivative f' on U. Here

$$f'(z_0) = \operatorname*{fine-lim}_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \ z_0 \in U\,.$$

(fzf) f and $zf(z)$ are finely harmonic (componentwise) on U.

(CRC) $\forall z \in U\ \exists V \in \mathcal{F}_z\ \exists F \in \mathcal{C}^1(\mathbf{C}) : f = F$ on V and $\bar\partial F = 0$ on V.
Here $z = x_0 + ix_1$ and

$$\bar\partial F := \frac{1}{2}\left(\frac{\partial F}{\partial x_0} + i\frac{\partial F}{\partial x_1}\right).$$

(CRW) f is finely continuous on U and $\forall z \in U\ \exists V \in \mathcal{F}_z\ \exists F \in W^{1,2}(\mathbf{C}) : f = F$ on V and $\bar\partial F = 0$ on V.

(CRH) f is finely harmonic and fine-$\bar\partial f = 0$ a.e. on U.

Remark. Let us state some properties of finely holomorphic functions.

(a) By (fzf), a function f is holomorphic on a usual open set $G \subset \mathbf{C}$ if and only if f is finely holomorphic on G.

(b) If f is finely holomorphic, so is f'. Thus f is infinitely fine differentiable.

(c) Finely holomorphic functions have the unique continuation property, i.e., if f is finely holomorphic on a fine domain $U \subset \mathbf{C}$ and all its fine derivatives $f^{(n)}(q_0)$, $n \in \mathbf{N}$, vanish at a point $q_0 \in U$, then f is constant on U.

3. Monogenic functions

For an account of quaternionic analysis, we refer to [19], [5] or [4]. Denote by $\mathbf{H}$ the field of real quaternions. The field $\mathbf{H}$ can be viewed as the Euclidean space $\mathbf{R}^4$ endowed with a non-commutative multiplication. A quaternion q can be written in the form $q = x_0 + x_1 i + x_2 j + x_3 k$ where x_0, x_1, x_2, x_3 are real numbers and i, j, k are the imaginary units such that

$$i^2 = j^2 = k^2 = -1\,, \quad ij = -ji = k\,, \quad jk = -kj = i\,, \quad ki = -ik = j\,.$$

Moreover, denote $\bar q = x_0 - x_1 i - x_2 j - x_3 k$, $|q| = \sqrt{q\bar q}$, $\operatorname{Re} q = x_0$ and $\operatorname{Im} q = x_1 i + x_2 j + x_3 k$. Obviously, $q^{-1} = \bar q/|q|^2$ for $q \neq 0$.

For a given open set $G \subset \mathbf{H}$, $\mathcal{C}(G)$ and $\mathcal{C}^1(G)$ stand for the set of continuous and continuously differentiable functions $f : G \to \mathbf{H}$, respectively. Let $f \in \mathcal{C}^1(G)$. Define

$$\bar\partial = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\,,$$

$$\bar\partial f = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}\,,$$

$$f\,\bar\partial = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k$$

and

$$\partial = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\,,$$

and, in an analogous way, ∂f and $f\,\partial$. Let us mention that the operators $\bar\partial$ and ∂ have coefficients of $\mathbf{H} \setminus \mathbf{R}$, namely, i, j, k. Thus in contrast to the complex case if the operators $\bar{\partial}$ and ∂ are applied to $\mathbf{H}$-valued functions from the left and from the right, different results are obtained in general. Let us remark that $\Delta = \bar{\partial}\partial = \partial\bar{\partial}$ where

$$\Delta := \sum_{l=0}^{3} \frac{\partial^2}{\partial x_l^2}$$

is the Laplace operator.

Definition. Let $G \subset \mathbf{H}$ be open, $f : G \to \mathbf{H}$ and $f \in C^1(G)$. Then f is called monogenic if f satisfies the so-called Fueter equation $\bar{\partial} f = 0$ on G.

Remark. Let us mention some properties of monogenic functions.

(a) Monogenic functions do **not** form an algebra! Moreover, the zeros of a monogenic function are not necessarily isolated, and its range is not necessarily open. (Consider $f(q) = x_1 - x_0 i$. Obviously, $\bar{\partial} f = 0$ and $\bar{\partial}(jf) = 2k$.)

(b) Monogenic functions have the unique continuation property, i.e., if f is a monogenic function on a domain $U \subset \mathbf{H}$ and all partial derivatives of f of any order at a point $q_0 \in U$ are zero, then f is constant on U. It follows easily from [19, Theorem 10].

(c) A function f is monogenic if and only if both f and $qf(q)$ are harmonic. Indeed, each monogenic function f is infinitely differentiable, see [19], and it is easy to see that $\Delta(qf(q)) = 2\bar{\partial} f(q) + q\Delta f(q)$.

Now we introduce further spaces of $\mathbf{H}$-valued functions. If $M \subset \mathbf{H}$ is (Lebesgue) measurable, then the Lebesgue space $L^2(M)$ is the set of measurable functions whose the second power is integrable on M. Let us mention that an inner product on $L^2(M)$ can be expressed as

$$\langle f, g \rangle_M := \sum_{l=0}^{3} \int_M f_l g_l = \mathrm{Re}\left(\int_M \bar{f} g \right)$$

where $f = (f_0, f_1, f_2, f_3)$ and $g = (g_0, g_1, g_2, g_3)$. Let $G \subset \mathbf{H}$ be open. Denote by $\mathcal{D}(G)$ the set of infinitely differentiable functions with compact support contained in G. The Sobolev space $W^{1,2}(G)$ is defined as the set of functions $f \in L^2(G)$ whose first distributional derivatives

$$\left\langle \frac{\partial f}{\partial x_l}, \varphi \right\rangle_G = -\left\langle f, \frac{\partial \varphi}{\partial x_l} \right\rangle_G, \quad \varphi \in \mathcal{D}(G), \quad l = 0,1,2,3$$

belong to $L^2(G)$ as well. An inner product on $W^{1,2}(G)$ is defined as $(f,g)_{1,2;G} := \langle f, g \rangle_G + (f,g)_G$ where

$$(f,g)_G := \sum_{l=0}^{3} \left\langle \frac{\partial f}{\partial x_l}, \frac{\partial g}{\partial x_l} \right\rangle_G .$$

In what follows we shall often omit subscript G. Finally, denote by $W_0^{1,2}(G)$ the closure of $\mathcal{D}(G)$ in $W^{1,2}(\mathbf{H})$. Of course, $W_0^{1,2}(G)$ is a closed subspace of $W^{1,2}(G)$. Moreover, it is easily seen that, for each $f \in W^{1,2}(G)$ and $g \in W_0^{1,2}(G)$, $\langle \partial f, g \rangle = -\langle f, \bar{\partial} g \rangle$ and $(f,g)_G = \langle \partial f, \partial g \rangle_G = \langle \bar{\partial} f, \bar{\partial} g \rangle_G$.

Now we generalize the so-called Weyl lemma for monogenic functions.

Theorem 1. *Let $G \subset \mathbf{H}$ be open and $f : G \to \mathbf{H}$ be locally integrable. Then $\bar{\partial} f = 0$ on G in the sense of distributions, i.e., $\langle f, \partial\varphi\rangle = 0$ for each $\varphi \in \mathcal{D}(G)$, if and only if f has a monogenic representative on G, i.e., there is a monogenic function h on G such that $f = h$ a.e. on G.*

Proof. The implication $\Leftarrow$ is trivial. We need to prove the converse. We apply the following characterization of harmonicity: $\Delta f = 0$ on G in the sense of distributions, i.e., $\langle f, \Delta\varphi\rangle = 0$ for each $\varphi \in \mathcal{D}(G)$, if and only if there is a harmonic function h on G such that $f = h$ a.e. on G, see [17, 2.61 Corollary, p. 95]. We assume that $\langle f, \partial\varphi\rangle = 0$ for each $\varphi \in \mathcal{D}(G)$. Then, of course,

$$\langle f, \Delta\varphi\rangle = \langle f, \partial(\bar{\partial}\varphi)\rangle = 0$$

for each $\varphi \in \mathcal{D}(G)$. Hence there is a harmonic function h on G such that $f = h$ a.e. on G. Moreover, for each $\varphi \in \mathcal{D}(G)$, it is easy to compute that $\Delta(\bar{q}\varphi(q)) = \bar{q}\Delta\varphi(q) + 2\partial\varphi(q)$ and thus

$$\langle qf(q), \Delta\varphi\rangle = \langle f, \bar{q}\Delta\varphi(q)\rangle = \langle f, \Delta(\bar{q}\varphi(q))\rangle - 2\langle f, \partial\varphi\rangle = 0\,. \tag{1}$$

Hence there is a harmonic function g on G such that $qf(q) = g(q)$ a.e. on G. Since $qh(q) = g(q)$ everywhere on G the function h is monogenic on G. □

Corollary 1. *Let $G \subset \mathbf{H}$ be open and $f \in L^2(G)$. Then the following statements are equivalent to each other:*

a) *$\bar{\partial} f = 0$ on G (in the sense of distributions);*
b) *f has a monogenic representative on G;*
c) *$\langle f, \partial\varphi\rangle = 0$ for each $\varphi \in W_0^{1,2}(G)$.*

Proof. Obvious. □

Lemma 1. *If $G \subset \mathbf{H}$ is bounded and open, $f \in W^{1,2}(G)$ and $\varphi \in W_0^{1,2}(G)$, then*

$$(qf(q), \varphi) = (f, \bar{q}\varphi(q)) + 2\langle f, \partial\varphi\rangle\,.$$

Proof. Let $f \in W^{1,2}(G)$ and $\varphi \in \mathcal{D}(G)$. In (1), we have proved that

$$\langle qf(q), \Delta\varphi\rangle = \langle f, \Delta(\bar{q}\varphi(q))\rangle - 2\langle f, \partial\varphi\rangle.$$

It is easy to see that $(f, \varphi) = -\langle f, \Delta\varphi\rangle$. Finally, by the density of $\mathcal{D}(G)$ in $W_0^{1,2}(G)$, we get

$$(qf(q), \varphi) = (f, \bar{q}\varphi(q)) + 2\langle f, \partial\varphi\rangle$$

even for $\varphi \in W_0^{1,2}(G)$, as required. □

4. Finely monogenic functions

We introduce finely monogenic functions as follows.

Definition. Let $U \subset \mathbf{H}$ be finely open and $f : U \to \mathbf{H}$. We call f finely monogenic if f and $qf(q)$ are both finely harmonic on U.

Remark. By Definition and known facts about finely harmonic functions, f is monogenic on a usual open set $G \subset \mathbf{H}$ if and only if f is finely monogenic and locally bounded on G. Moreover, fine monogeneity is a finely local property, i.e., f is finely monogenic on a finely open set $U \subset \mathbf{H}$ if and only if for each $q \in U$ there is $V \in \mathcal{F}_q$

such that f is finely monogenic on V. In fact, by definition, fine harmonicity is a finely local property.

The main results of this note are Theorems 2 and 4 below. Before stating these theorems, we introduce the Sobolev spaces $W^{1,2}(U)$ and $W_0^{1,2}(U)$ for a given finely open set $U \subset \mathbf{H}$, see [14] and [17]. First, define

$$W_0^{1,2}(U) := \bigcap \{W_0^{1,2}(G);\ G \subset \mathbf{H} \text{ open}, U \subset G\}.$$

Let us notice that the space $W_0^{1,2}(U)$ is closed in $W^{1,2}(\mathbf{H})$. Moreover, it is known that if $f \in W_0^{1,2}(U)$, then $f = 0$ and $\frac{\partial f}{\partial x_l} = 0$ a.e. on $\mathbf{H} \setminus U$ for $l = 0, 1, 2, 3$. Hence

$$(f,g)_{1,2;U} := \langle f,g\rangle_U + (f,g)_U$$

is an inner product on $W_0^{1,2}(U)$ where

$$(f,g)_U := \sum_{l=1}^{3} \Big\langle \frac{\partial f}{\partial x_l}, \frac{\partial g}{\partial x_l}\Big\rangle_U .$$

In what follows we shall often omit subscript U. Denote by $W_{\mathrm{ext}}^{1,2}(U)$ the set of functions $f \in L^2(U)$ for which there is an open set $G \subset \mathbf{H}$ and $F \in W^{1,2}(G)$ such that $U \subset G$ and $F = f$ on U. Moreover, define $W^{1,2}(U)$ as the completion of $W_{\mathrm{ext}}^{1,2}(U)$ with respect to $(\cdot,\cdot)_{1,2;U}$. For $f \in W^{1,2}(U)$ and $l = 0, 1, 2, 3$, set

$$\frac{\partial f}{\partial x_l} := \lim_{m\to\infty} \frac{\partial f_m}{\partial x_l} \text{ in } L^2(U)$$

provided $\{f_m\} \subset W_{\mathrm{ext}}^{1,2}(U)$ and $f_m \to f$ in $W^{1,2}(U)$. It is easy to see that each $\frac{\partial f}{\partial x_l}$ is defined correctly and, futhermore,

$$\Big\langle \frac{\partial f}{\partial x_l}, \varphi \Big\rangle = -\Big\langle f, \frac{\partial \varphi}{\partial x_l} \Big\rangle$$

for each $\varphi \in W_0^{1,2}(U)$. Clearly, if U is open, the above defined spaces $W_0^{1,2}(U)$ and $W^{1,2}(U)$ coincide with the usual Sobolev spaces.

A role of negligible sets in potential theory is played by polar sets. We call $M \subset \mathbf{R}^n$ polar if for each $z \in \mathbf{R}^n$ there is $V \in \mathcal{F}_z$ such that $M \cap (V \setminus \{z\}) = \emptyset$. In particular, polar sets are finely closed and Lebesgue null. Let us recall that a countable union of polar sets is polar as well, see [2, Corollary 5.1.4]. We shall say that a function f is finely monogenic quasi everywhere (q.e.) on a finely open set $U \subset \mathbf{H}$ if there is a polar set $M \subset U$ such that f is finely monogenic on $U \setminus M$.

Remark. A function f is finely monogenic q.e. on U if and only if f is finely monogenic q.e. finely locally on U. Indeed, it is true even for finely harmonic functions. Namely, let f be finely harmonic q.e. finely locally on U, i.e., for each $q \in U$ there is $V_q \in \mathcal{F}_q$ and a polar set $M_q \subset V_q$ such that f is finely harmonic on $V_q \setminus M_q$. Now let us recall that the fine topology is quasi-Lindelöf, which means that the union of any family of finely open sets differs by at most a polar set from the union of some countable subfamily, see [2, Theorem 7.3.11]. Hence there is a countable set $N \subset U$ such that, setting $V := \bigcup \{V_q;\ q \in N\}$, $M_0 := U \setminus V$ is polar. Moreover,

$$M := M_0 \cup \bigcup \{M_q;\ q \in N\}$$

is polar and f is finely harmonic on $U \setminus M$.

Remark. By [17, 2.152 Theorem, p. 149], each $f \in W^{1,2}(U)$ has a representative $\hat{f}$ (i.e., $\hat{f} = f$ a.e. on U) which is finely continuous q.e. on U. The $\hat{f}$ is uniquely determined up to a polar set.

Theorem 2. *Let $U \subset \mathbf{H}$ be finely open and $f \in W^{1,2}(U)$. Then the following statements are equivalent to each other:*

a) $\bar{\partial} f = 0$ *on* U;

b) $\hat{f}$ *is finely monogenic q.e. on* U,

c) $\langle f, \partial\varphi \rangle = 0$ *for each* $\varphi \in W_0^{1,2}(U)$.

Before proving the theorem let us recall the following key result essentially due to B. Fuglede, see [9] and [14].

Theorem 3. *If $f \in W^{1,2}(U)$, then $\hat{f}$ is finely harmonic q.e. on U if and only if $(f, \varphi) = 0$ for each $\varphi \in W_0^{1,2}(U)$.*

Proof. By Fuglede's theorem [9, 11. Théorème], the statement of Theorem 3 is true if there is an open bounded set $G \subset \mathbf{H}$ and $F \in W_0^{1,2}(G)$ such that $U \subset G$ and $F = f$ on U. Now let $f \in W^{1,2}(U)$. Then it is known that for each $q \in U$ there is a bounded set $V_q \in \mathcal{F}_q$ and $F \in W_0^{1,2}(U)$ such that $F = f$ on V_q, see [14, 1.12 Theorem and 2.5 Lemma]. Setting $\mathcal{V} := \{V_q;\ q \in U\}$, by Fuglede's theorem, $\hat{f}$ is finely harmonic q.e. finely locally on U if and only if, for any $V \in \mathcal{V}$, $(f, \varphi) = 0$ for each $\varphi \in W_0^{1,2}(V)$. It is sufficient to prove that the latter condition implies that $(f, \varphi) = 0$ even for each $\varphi \in W_0^{1,2}(U)$. But it follows directly from the fact that for each $\varphi \in W_0^{1,2}(U)$ there is a sequence $\{\varphi_l\} \subset W_0^{1,2}(U)$ such that $\varphi_l \to \varphi$ in $W_0^{1,2}(U)$ and each φ_l is a finite sum of functions from $\bigcup \{W_0^{1,2}(V);\ V \in \mathcal{V}\}$, see [14, 2.4 Lemma]. □

Remark. The set $W_0^{1,2}(U)$ is dense in $L^2(U)$. Indeed, let $f \in L^2(U)$ and $\langle f, \varphi \rangle = 0$ for each $\varphi \in W_0^{1,2}(U)$. Then, by [17, 1.5 Theorem], there is $\varphi \in W_0^{1,2}(U)$ with $\varphi > 0$ on U. For each $\psi \in \mathcal{D}(\mathbf{H})$,

$$0 = \langle f, \varphi\psi \rangle = \langle \bar{\varphi} f, \psi \rangle .$$

Thus $\bar{\varphi} f = 0$ a.e. on U and hence $f = 0$ a.e. on U.

Proof of Theorem 2. Cf. [8, 2. Lemme] for the complex case.
Let $f \in W^{1,2}(U)$. Since $\langle \bar{\partial} f, \varphi \rangle = -\langle f, \partial\varphi \rangle$ for each $\varphi \in W_0^{1,2}(U)$, by the previous Remark, a) $\Leftrightarrow$ c). We need to prove a) $\Leftrightarrow$ b). Without loss of generality, we can asume that U is bounded. Using Lemma 1 it is easy to show that

$$(qf(q), \varphi) = (f, \bar{q}\varphi(q)) + 2\langle f, \partial\varphi \rangle , \quad \varphi \in W_0^{1,2}(U) .$$

So $\bar{\partial} f = 0$ on U if and only if $(f, \varphi) = \langle \bar{\partial} f, \bar{\partial}\varphi \rangle = 0$ and $(qf(q), \varphi) = 0$ for each $\varphi \in W_0^{1,2}(U)$. Now we can apply Theorem 3 to conclude the proof. □

Remark. By Corollary 1 and Theorem 2, given an open set $G \subset \mathbf{H}$ and $f \in W^{1,2}(G)$, the function f is monogenic on G if and only if f is finely monogenic on G. Indeed, if $f \in W^{1,2}(G)$ is finely monogenic on G, then there is a monogenic function h on G such that $f = h$ a.e. on G. Thus $f = h$ everywhere on G because f and h are both finely continuous on G and no non-empty finely open set is Lebesgue null. The converse is obvious.

Denote by $W^{1,2}_{\text{f-loc}}(U)$ the set of functions f on U such that for each $q \in U$ there is $V \in \mathcal{F}_q$ with $f|_V \in W^{1,2}_{\text{ext}}(V)$. For $f \in W^{1,2}_{\text{f-loc}}(U)$ and $l = 0,1,2,3$, set $\frac{\partial f}{\partial x_l} := \frac{\partial (f|_V)}{\partial x_l}$ on V if $V \subset U$ is finely open and $f|_V \in W^{1,2}_{\text{ext}}(V)$. It is easily seen that each $\frac{\partial f}{\partial x_l}$ is an a.e. defined function on U. Moreover, it is known that

$$W^{1,2}(U) = \left\{ f \in W^{1,2}_{\text{f-loc}}(U);\ f \in L^2(U),\ \frac{\partial f}{\partial x_l} \in L^2(U),\ l = 0,1,2,3 \right\},$$

see [14].

Remark. By [17, 2.152 Theorem, p. 149], each $f \in W^{1,2}_{\text{f-loc}}(U)$ has approximate differential ap-df a.e. on U and, for almost every $q \in U$,

$$\text{ap-}df(q)(h) = \sum_{l=0}^{3} \frac{\partial f}{\partial x_l}(q)\, h_l,\ \ h = (h_0, h_1, h_2, h_3) \in \mathbf{R}^4 .$$

Recall that $q_0 \in \mathbf{R}^n$ is a density point for a (Lebesgue) measurable set $V \subset \mathbf{R}^n$ if

$$\lim_{r \to 0+} \frac{|B(q_0, r) \cap V|}{|B(q_0, r)|} = 1$$

where $|A|$ stands for the Lebesgue measure of a measurable set $A \subset \mathbf{R}^n$ and $B(q_0, r)$ is the ball with center q_0 and radius r. A linear map L on $\mathbf{R}^n$ is called the approximate differential of f at q_0 if there is a measurable set $V \subset \mathbf{R}^n$ such that q_0 is a density point for V and

$$\lim_{q \to q_0, q \in V} \frac{f(q) - f(q_0) - L(q - q_0)}{|q - q_0|} = 0 .$$

Theorem 4. *Let $U \subset \mathbf{H}$ be finely open and $f : U \to \mathbf{H}$. The following statements are equivalent to each other:*
(FM) *f is finely monogenic on U.*
(FH) *f is finely harmonic and* fine-$\bar{\partial} f = 0$ *a.e. on U. Here*

$$\text{fine-}\bar{\partial} f = \text{fine-}\frac{\partial f}{\partial x_0} + i\, \text{fine-}\frac{\partial f}{\partial x_1} + j\, \text{fine-}\frac{\partial f}{\partial x_2} + k\, \text{fine-}\frac{\partial f}{\partial x_3}$$

at each point where f is finely differentiable.
(FW) *f is finely continuous on U, $f \in W^{1,2}_{\text{f-loc}}(U)$ and $\bar{\partial} f = 0$ on U.*

Proof. Cf. [8, 3. Définition] for the complex case.
(FH) or (FM) $\Rightarrow$ (FW): Let $U \subset \mathbf{H}$ be finely open and $f : U \to \mathbf{H}$ be finely harmonic. Then f is, of course, finely continuous on U. By [9, 2. Théorème], $f \in W^{1,2}_{\text{f-loc}}(U)$. First assume (FH). If f is finely differentiable at $q \in U$, then f is approximately differentiable as well and fine-$df(q)$ = ap-$df(q)$. Indeed, it follows from definitions and the fact that each $q \in \mathbf{H}$ is a density point for any $V \in \mathcal{F}_q$, see [20, 3.3.5. Remark]. By the previous remark, $\bar{\partial} f$ = fine-$\bar{\partial} f = 0$. On the other hand, let (FM) be supposed. Then, by Theorem 2, $\bar{\partial} f = 0$ finely locally on U, i.e., for each $q \in U$ there is $V \in \mathcal{F}_q$ such that $\bar{\partial} f = 0$ on V. Hence $\bar{\partial} f = 0$ on U.

(FW) $\Rightarrow$ (FH) and (FM): Let $f \in W^{1,2}_{\text{f-loc}}(U)$ and $\bar{\partial} f = 0$ on U. By Theorem 2, f is finely monogenic q.e. finely locally on U. Hence the f is finely monogenic q.e. on U. Since f is finely continuous on U the f is finely monogenic even everywhere on U. Indeed, it is true even for finely harmonic functions, see [6, 9.15 Theorem]. Finally, as we know finely harmonic functions are finely differentiable a.e. and fine-$\bar{\partial} f = \bar{\partial} f = 0$ a.e. on U, which concludes the proof. □

5. Concluding remarks

We have mentioned that each finely holomorphic function is infinitely fine differentiable everywhere. As to a finely monogenic function f, we know so far only that f is finely differentiable a.e. because f is, by definition, finely harmonic. Moreover, we can ask whether a finely monogenic function on a fine domain U is uniquely determined by its values in some fine neighbourhood of a point of U.

In the second section, we recall various but equivalent ways how to introduce finely holomorphic functions. As to the quaternionic case, we have not considered all possibilities yet. If $U \subset \mathbf{H}$ is finely open and $f : U \to \mathbf{H}$, then there are other possible definitions:

(FF) $f \in$ fine-$\mathcal{C}^1(U)$ and fine-$\bar{\partial} f = 0$ on U. Here fine-$\mathcal{C}^1(U)$ is the set of functions having fine continuous fine differential on U.

(FC) $\forall q \in U\ \exists V \in \mathcal{F}_q\ \exists F \in \mathcal{C}^1(\mathbf{H}) : f = F$ on V and $\bar{\partial} F = 0$ on V.

(AM) $\forall q \in U\ \exists V \in \mathcal{F}_q : f|_V$ is a uniform limit of functions monogenic on open sets containing V.

Now it is quite natural to ask about relations between these conditions and which of them characterize finely monogenic functions. Let us end with the following

Theorem 5. *Let $U \subset \mathbf{R}^2$ be finely open and $f : U \to \mathbf{R}$. Then $f \in$ fine-$\mathcal{C}^1(U)$ if and only if $\forall q \in U\ \exists V \in \mathcal{F}_q\ \exists F \in \mathcal{C}^1(\mathbf{R}^2) : f = F$ on V.*

Proof. $\Leftarrow$ is trivial. B. Fuglede in [7] shows $\Rightarrow$ for finely holomorphic functions but it is easy to see that his proof works for fine-$\mathcal{C}^1$ functions as well. □

A basic question arises whether the previous theorem is true for $\mathbf{R}^n$, $n > 2$. Let us remark that, obviously, if the theorem were true for $n = 4$, then the conditions (FF) and (FC) above would be equivalent.

Acknowledgements. I am grateful to S. Gardiner, J. Malý, I. Netuka, A. G. O'Farrell and V. Souček for useful conversations. I acknowledge the financial support provided through the European community's human potential programme under contract HPRN-CT-2000-00116 (Analysis and Operators) and from the grants GAUK 447/2004 and GA 201/03/0935. This work is also a part of the research plan MSM 0021620839, which is financed by the Ministry of Education of the Czech Republic.

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