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TRANSFORMATIONS OF CONFORMALLY INVARIANT σ-MODELS

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ABSTRACT. Elements of Poisson–Lie T-plurality transformations are reviewed. Explicit examples of three-dimensional conformal σ-models related by the Poisson–Lie T-plurality are presented. Spectacular relations of their geometric properties are illustrated.

1. INTRODUCTION

The main goal of this contribution is to present several examples of three-dimensional σ-models transformable by the so called Poisson–Lie T-plurality to another one with different geometric properties. The classical σ-models on Lie groups are given by the action functionals

$$S_F[\phi] = \int d^2x \partial_- \phi^i F_{ij}(\phi) \partial_+ \phi^j \tag{1}$$

that depend on a tensor field of the second order F on the group G where the functions $\phi : \mathbf{R}^2 \to \mathbf{R}^n$ are obtained by the composition $\phi^\mu = y^\mu \circ \gamma$ of a map $\gamma : \mathbf{R}^2 \to G$ and a coordinate map $y : U(g) \to \mathbf{R}^n$ in a neighbourhood of an element $g \in G$.

In [5], [4] the following extension of the T-duality [2] was found. Let $\tilde{f}_i^{jk}$ be structure coefficients of a Lie group $\tilde{G}$, $\dim \tilde{G} = \dim G$ so that F satisfies

$$\mathcal{L}_{v_i}(F)_{\mu\nu} = F_{\mu\kappa} v_j^\kappa \tilde{f}_i^{jk} v_k^\lambda F_{\lambda\nu}, \qquad i = 1, \ldots, \dim G \tag{2}$$

for a basis of left-invariant fields v_i on G. Then there is a relation between solutions of the equations of motion for S_F and $S_{\tilde{F}}$ where $\tilde{F}_{\mu\nu}$ is a second order tensor field on $\tilde{G}$. This relation is called Poisson–Lie T-duality and the corresponding σ-models are called PLT-dual. The Poisson–Lie T-plurality was introduced in [9] as an extension of the Poisson–Lie T-duality. The main idea of the Poisson–Lie T-duality or plurality consists in transformation of equations of motion of the σ-model to equations on a Drinfel'd double. The classification of the six-dimensional Drinfel'd doubles and corresponding Manin triples [7] enabled to find many examples of three-dimensional σ-models related by the Poisson–Lie T-plurality transformation [9], [3].

Quantization of the σ-models requires that they be made conformal invariant. This is achieved by addition of another term depending on a scalar (dilaton) field Φ to the

The paper is in final form and no version of it will be submitted elsewhere.

action (1). To guarantee the conformal invariance (at least at one-loop level) the fields F and Φ must satisfy the so called vanishing β equations.

$$0 = R_{ij} - \nabla_i \nabla_j \Phi - \frac{1}{4} H_{imn} H_j^{mn} \tag{3}$$

$$0 = \nabla^k \Phi H_{kij} + \nabla^k H_{kij} \tag{4}$$

$$0 = R - 2 \nabla_k \nabla^k \Phi - \nabla_k \Phi \nabla^k \Phi - \frac{1}{12} H_{kmn} H^{kmn} \tag{5}$$

where covariant derivatives ∇_k, Ricci tensor R_{ij} and scalar curvature R are calculated from the metric

$$G_{ij} = \frac{1}{2}(F_{ij} + F_{ji}) \tag{6}$$

that is also used for lowering and raising indices and

$$H_{ijk} = \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} \tag{7}$$

where the torsion potential is

$$B_{ij} = \frac{1}{2}(F_{ij} - F_{ji}) . \tag{8}$$

We shall present explicit examples of σ-models that satisfy the vanishing β equations. Such σ-models together with the transformed ones were calculated in [3] but the main goal of that paper was a classification of a certain class of models. For the lack of space, it was usually not possible to display the forms of the tensor F that governs the geometric properties of the target manifold of the model, only the constant matrix $F(0)$ from which F can be reconstructed together with dilaton field were presented.

2. General features of Poisson–Lie T-plurality

As mentioned, the equations of motion of the transformable σ-models can be rewritten as equations on the Drinfel'd double D, i.e. connected Lie group whose Lie algebra $\mathcal{D}$ is equipped with a bilinear, symmetric, nondegenerate, ad-invariant form $\langle\ ,\ \rangle$ such that $\mathcal{D}$ admits a decomposition

$$\mathcal{D} = \mathcal{G} + \tilde{\mathcal{G}}$$

into two subalgebras that are maximally isotropic with respect to $\langle\ ,\ \rangle$. Such decomposition is called Manin triple $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$. It can be shown that the dimensions of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ must be equal so that the dimension of $\mathcal{D}$ must be even. The rewritten equations of motion on D read [5], [4]

$$\langle (\partial_\pm l) l^{-1}, \mathcal{E}^\pm \rangle = 0 , \tag{9}$$

where

$$l = \gamma \cdot \tilde{h} : \mathbf{R}^2 \to D , \qquad \gamma(x_+, x_-) \in G , \ \tilde{h}(x_+, x_-) \in \tilde{G} \tag{10}$$

and

$$\mathcal{E}^+ = \mathrm{span}\,(X^i + E_{ij} \tilde{X}_j) , \quad \mathcal{E}^- = \mathrm{span}\,(X^i - E_{ji} \tilde{X}_j) ,$$

where E_{ij} is a constant matrix, $X^i, \tilde{X}_j$ are basis elements of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. The subspaces $\mathcal{E}^{\pm}$ are orthogonal w.r.t. $\langle\ ,\ \rangle$ and

$$\mathcal{D} = \mathcal{E}^+ + \mathcal{E}^- .$$

This means that the Poisson–Lie T-dualizable models on G are given by the Manin triples $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ and the constant matrix E_{ij}. The tensor field F is obtained as (see e.g. [8])

$$F(\phi) = e(\phi)^t (E^{-1} + \pi(\phi))^{-1} e(\phi)\,, \tag{11}$$

where the matrix $\pi(\phi)$ follows from the adjoint representation of G on $\mathcal{D}$ and $e(\phi)$ is the vielbein field on G.

Besides $(\mathcal{D}, \mathcal{G}, \tilde{\mathcal{G}})$ the Manin triple $(\mathcal{D}, \tilde{\mathcal{G}}, \mathcal{G})$ always exists for any Drinfel'd double and we can decompose l into

$$l = \tilde{\gamma} \cdot h\,, \qquad \tilde{\gamma}(x_+, x_-) \in \tilde{G}\,, \qquad h(x_+, x_-) \in G \tag{12}$$

as well and obtain the dual σ-model on $\tilde{G}$. On the other hand some Drinfel'd doubles can be decomposed into more than two dual Manin triples and the fact that (9) doesn't depend on the choice of the Manin triple enables to construct more than two equivalent σ-models on $G, \tilde{G}, B, \tilde{B}, \ldots$ from

$$l = \gamma \cdot \tilde{h} = \tilde{\gamma} \cdot h = \beta \cdot \tilde{c} = \tilde{\beta} \cdot c = \ldots$$

This property can be called Poisson–Lie T-plurality and we call the models that are in this way transformable to each other PLT-equivalent.

All the models we are going to present are three dimensional so that the Drinfel'd doubles in these cases have dimension six. The background manifolds of the models are equipped with the Lie group structures that are at least locally given by the Bianchi classification of three-dimensional Lie algebras.

3. Three dimensional models

By inspection of a certain class of models in [3] we have found many matrices E that produce PLT-pluralizable models. Here we shall present explicit examples of corresponding tensors F and their properties.

Let us start with two σ-models given by the following background tensors F:

Example 1: σ-model with

$$F(y) = \begin{pmatrix} 0 & 0 & e^{-y_1} \\ 0 & e^{-2y_1} & 0 \\ e^{-y_1} & 0 & 0 \end{pmatrix} . \tag{13}$$

Example 2: σ-model with

$$F(y) = \kappa \begin{pmatrix} 1 & 0 & y_2 \\ 0 & -1 & -y_1 \\ y_2 & -y_1 & 1 - {y_1}^2 + {y_2}^2 \end{pmatrix} . \tag{14}$$

Both these backgrounds are flat and torsionless. They satisfy the condition (2) for $(G, \tilde{G})$ equal to $(\mathbf{G5}, \mathbf{G1})$ and $(\mathbf{G6_0}, \mathbf{G1})$, respectively, where $\mathbf{G1}, \mathbf{G5}, \mathbf{G6_0}$ are three-dimensional Lie groups given locally by the Bianchi Lie algebras $\mathbf{1}, \mathbf{5}, \mathbf{6_0}$ (see [3], [6], [1]).

These two models are not PLT-equivalent but for each of the tensors a rather large set of mutually PLT-equivalent models can be found with properties not encountered before in the context of Poisson–Lie T-plurality.

Firstly, there are models that allow different dilatons. It is quite surprising to have two seemingly distinct solutions of the vanishing β-function equations (3)–(5) for the same F.

Example: σ-model for the Manin triple $(\mathcal{D}, \mathbf{1}, \mathbf{6_0})$ with

$$F(y) = \left((y_1-1)^2 - {y_2}^2\right)^{-1} \begin{pmatrix} {y_1}^2 & y_1(1-y_2) & 1-y_1-y_2 \\ -y_1(1+y_2) & -1+{y_2}^2 & -1+y_1+y_2 \\ 1-y_1+y_2 & 1-y_1+y_2 & 0\,. \end{pmatrix} \tag{15}$$

This background is flat and torsionless. The σ-model model is PLT-equivalent to (14). The β-equations hold both for

$$\Phi = \ln|(y_1-1)^2 - {y_2}^2| \quad \text{and} \quad \Phi = \text{const.}$$

Secondly, there are PLT-equivalent σ-models on the same Manin triple with different matrices E, i.e. on isomorphic Manin triples with the same subspaces $\mathcal{E}^{\pm}$, some of them being flat with constant dilaton, the others being curved and with nontrivial dilaton.

Example: σ-models for the Manin triple $(\mathcal{D}, \mathbf{1}, \mathbf{6_0})$ with (15) and with

$$F(y) = (1-{y_1}^2+{y_2}^2)^{-1} \begin{pmatrix} 1-{y_1}^2 & y_1 y_2 & y_2 \\ y_1 y_2 & -1-{y_2}^2 & -y_1 \\ -y_2 & y_1 & 1 \end{pmatrix}, \tag{16}$$

$$\Phi(y) = \ln|1+y_1^2-y_2^2|\,.$$

The former model is flat and torsionless while the latter one is curved with scalar curvature

$$R = \frac{-10-4y_1^2+4y_2^2}{(1-y_1^2+y_2^2)^2}$$

and torsion

$$H_{123} = -\frac{2}{(1-y_1^2+y_2^2)^2}\,.$$

Thirdly, by different choices of matrix E one can get PLT-inequivalent models on the same Manin triple with rather different properties. Of course, in general it is highly unprobable that a dilaton satisfying (3)–(5) exists for a generic choice of matrix E. On the other hand there can be various E's for which the dilatons exist even though the σ-models are not PLT-equivalent.

Example: σ-models for $(\mathcal{D}, \mathbf{6_0}, \mathbf{1})$ with (14) and

$$F(y) = \begin{pmatrix} 1 & 1 & \rho \\ 1 & 1 & 1+\rho \\ \rho & 1+\rho & 2y_1+\rho^2 \end{pmatrix}, \quad \rho = y_1+y_2\,, \quad \Phi(y) = 2y_3\,. \tag{17}$$

This model model is PLT-equivalent to (13) but not to (14). It is torsionless but curved with nontrivial Ricci tensor and vanishing scalar curvature.

The curvatures of some models diverge on hypersurfaces where the corresponding dilatons are also divergent. Nevertheless, the metric, torsion potential and dilaton appear to have a reasonable continuation behind the singularity since all of them are well-defined. We don't know at the moment whether such backgrounds have meaningful physical interpretation, e.g. as branes.

Example: See (16).

Presented examples should have illustrated the variety of phenomena that arise by application of Poisson–Lie T-plurality transformations between geometric backgrounds of σ-models.

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