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ISOMETRY GROUPS OF LOBACHEVSKIAN SPACES, SIMILARITY TRANSFORMATION GROUPS OF EUCLIDEAN SPACES AND LORENTZIAN HOLONOMY GROUPS

ANTON S. GALAEV

ABSTRACT. Weakly-irreducible not irreducible subalgebras of $\mathfrak{so}(1, n+1)$ were classified by L. Berard Bergery and A. Ikemakhen. In the present paper a geometrical proof of this result is given. Transitively acting isometry groups of Lobachevskian spaces and transitively acting similarity transformation groups of Euclidean spaces are classified.

INTRODUCTION

In 1952 A. Borel and A. Lichnerowicz showed that *the holonomy group of a Riemannian manifold is a product of irreducible holonomy groups of Riemannian manifolds*, see [9]. The main reason is the following. If a subgroup $G \subset SO(n)$ preserves a proper vector subspace, then G preserves also its orthogonal complement $U^{\perp}$ and we have $R^n = U \oplus U^{\perp}$, i.e. the group G is totally reducible. In 1955 M. Berger classified possible connected irreducible holonomy groups of Riemannian manifolds, see [8].

The Borel and Lichnerowicz theorem does not work in the pseudo-Riemannian case. Suppose a subgroup $G \subset SO(r, s)$ preserves a proper vector subspace $U \subset \mathbb{R}^{r,s}$ such that the restriction of the inner product to U is degenerate, then $U \cap U^{\perp} \neq \{0\}$ and we have no orthogonal decomposition of $\mathbb{R}^{r,s}$ into G-irreducible subspaces. A subgroup $G \subset SO(r, s)$ is called *weakly-irreducible* if it preserves no nondegenerate proper subspace of $\mathbb{R}^{r,s}$. There is Wu's theorem that states that *the holonomy group of a pseudo-Riemannian manifold is a product of weakly-irreducible holonomy groups of pseudo-Riemannian manifolds*, see [21]. If a holonomy group is irreducible, then it is weakly-irreducible. In [8] M. Berger gave a classification of irreducible holonomy groups for pseudo-Riemannian manifolds. In particular, *the only connected irreducible holonomy group of Lorentzian manifolds is* $SO^0(1, n+1)$, see [12] and [11] for direct proofs of this fact.

There is still no classification of weakly-irreducible not irreducible holonomy groups of pseudo-Riemannian manifolds. The first step towards a classification of weakly-irreducible not irreducible holonomy groups of Lorentzian manifolds was made by L. Berard Bergery and A. Ikemakhen, who classified weakly-irreducible not irreducible

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subalgebras of $\mathfrak{so}(1,n+1)$, see [6]. More precisely, they divided weakly-irreducible not irreducible subalgebras of $\mathfrak{so}(1,n+1)$ into 4 types. The proof of this result was purely algebraical.

We introduce a geometrical proof of the result of L. Berard Bergery and A. Ikemakhen. We consider an $n+2$-dimensional Minkowski vector space (V,η) and fix an isotropic vector $p \in V$. We denote by $SO(V)_{\mathbb{R}p}$ the Lie subgroup of $SO(V)$ that preserves the isotropic line $\mathbb{R}p$. We denote by E a vector subspace $E \subset V$ such that $(\mathbb{R}p)^{\perp_\eta} = \mathbb{R}p \oplus E$, and by q an isotropic vector $q \in V$ such that $\eta(q,E) = 0$ and $\eta(p,q) = 1$. The vector space E is an Euclidean space. We consider the vector model of the $n+1$-dimensional Lobachevskian space L^{n+1} and its boundary ∂L^{n+1}, which is diffeomorphic to the n-dimensional unit sphere. We have the natural isomorphisms

$$SO(V) \simeq \text{Isom } L^{n+1} \simeq \text{Conf } \partial L^{n+1} \quad \text{and} \quad SO(V)_{\mathbb{R}p} \simeq \text{Sim } E\,,$$

where Isom L^{n+1} is the group of all isometries of L^{n+1}, Conf ∂L^{n+1} is the group of all conformal transformations of ∂L^{n+1} and Sim E is the group of all similarity transformations of E. We identify the set $\partial L^{n+1}\backslash\{\mathbb{R}p\}$ with the Euclidean space E. Then any subgroup $G \subset SO(V)_{\mathbb{R}p}$ acts on E, moreover we have $G \subset$ Sim E. We prove that *a connected subgroup $G \subset SO(V)_{\mathbb{R}p}$ is weakly-irreducible iff the corresponding subgroup $G \subset$ Sim E under the isomorphism $SO(V)_{\mathbb{R}p} \simeq$ Sim E acts transitively on E.* This gives us a *one-to-one correspondence between connected weakly-irreducibly acting subgroups of $SO(V)_{\mathbb{R}p}$ and connected transitively acting subgroups of* Sim E. Using a description for connected transitive subgroups of Sim E (see [2], [3]), we divide such subgroups into 4 types. We show that the corresponding weakly-irreducible subgroups of $SO(V)_{\mathbb{R}p}$ have the same type introduced by L. Berard Bergery and A. Ikemakhen.

We also classify transitively acting isometry groups of the Lobachevskian space L^{n+1}. We show that these groups are exhausted by $SO^0(V)$ and by the weakly-irreducible not irreducible subgroups of $SO(V)_{\mathbb{R}p}$ of type 1 and type 3.

Remark. In another paper we will use similar ideas for complex Lobachevskian space in order to classify connected weakly-irreducible not irreducible subgroups of $SU(1,n+1) \subset SO(2,2n+2)$.

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1. Results of L. Berard Bergery and A. Ikemakhen

Let (V,η) be a Minkowski space of dimension $n+2$, where η is a metric on V of signature $(1,n+1)$. We fix a basis $p, e_1, \dots, e_n, q$ of V with respect to which the Gram matrix of η has the form $\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$ where E_n is the n-dimensional identity matrix.

Let $E \subset V$ be the vector subspace spanned by $e_1, \dots, e_n$. The vector space E is an Euclidean space with respect to the inner product $\eta|_E$.

Denote by $\mathfrak{so}(V)$ the Lie algebra of all η-skew symmetric endomorphisms of V and by $\mathfrak{so}(V)_{\mathbb{R}p}$ the subalgebra of $\mathfrak{so}(V)$ that preserves the line $\mathbb{R}p$.

The Lie algebra $\mathfrak{so}(V)_{\mathbb{R}p}$ can be identified with the following algebra of matrices:

$$\mathfrak{so}(V)_{\mathbb{R}p} = \left\{ \begin{pmatrix} a & -X^t & 0 \\ 0 & A & X \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R},\, X \in E,\, A \in \mathfrak{so}(E) \right\}.$$

The above matrix can be identified with the triple (a, A, X). Define the following subalgebras of $\mathfrak{so}(V)_{\mathbb{R}p}$, $\mathcal{A} = \{(a,0,0) : a \in \mathbb{R}\}$, $\mathcal{K} = \{(0,A,0) : A \in \mathfrak{so}(E)\}$ and $\mathcal{N} = \{(0,0,X) : X \in E\}$. We see that $\mathcal{A}$ commutes with $\mathcal{K}$, and $\mathcal{N}$ is an ideal. We have the decomposition

$$\mathfrak{so}(V)_{\mathbb{R}p} = (\mathcal{A} \oplus \mathcal{K}) \ltimes \mathcal{N}.$$

A subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ is called *irreducible* if it preserves no proper subspace of V; $\mathfrak{g}$ is called *weakly-irreducible* if it preserves no nondegenerate proper subspace of V.

Obviously, if $\mathfrak{g} \subset \mathfrak{so}(V)$ is irreducible, then it is weakly-irreducible. If $\mathfrak{g} \subset \mathfrak{so}(V)$ preserves a degenerate proper subspace $U \subset V$, then it preserves the isotropic line $U \bigcap U^{\perp}$; any such algebra is conjugated to a subalgebra of $\mathfrak{so}(V)_{\mathbb{R}p}$.

Let $\mathcal{B} \subset \mathfrak{so}(E)$ be a subalgebra. Recall that $\mathcal{B}$ is a compact Lie algebra and we have the decomposition $\mathcal{B} = \mathcal{B}' \oplus \mathfrak{z}(\mathcal{B})$, where $\mathcal{B}'$ is the commutant of $\mathcal{B}$ and $\mathfrak{z}(\mathcal{B})$ is the center of $\mathcal{B}$.

The following result is due to L. Berard Bergery and A. Ikemakhen.

Theorem. *Suppose $\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}p}$ is a weakly-irreducible subalgebra. Then $\mathfrak{g}$ belongs to one of the following types*

- **type 1:** $\mathfrak{g} = (\mathcal{A} \oplus \mathcal{B}) \ltimes \mathcal{N}$, *where $\mathcal{B} \subset \mathfrak{so}(E)$ is a subalgebra;*
- **type 2:** $\mathfrak{g} = \mathcal{B} \ltimes \mathcal{N}$;
- **type 3:** $\mathfrak{g} = (\mathcal{B}' \oplus \{\varphi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes \mathcal{N}$, *where $\varphi : \mathfrak{z}(\mathcal{B}) \to \mathcal{A}$ is a non-zero linear map;*
- **type 4:** $\mathfrak{g} = (\mathcal{B}' \oplus \{\psi(A) + A : A \in \mathfrak{z}(B)\}) \ltimes \mathcal{N}_W$, *where we have a non-trivial orthogonal decomposition $E = U \oplus W$ such that $\mathcal{B} \subset \mathfrak{so}(W)$; $\mathcal{N}_W = \{(0,0,X) : X \in W\}$; $\mathcal{N}_U = \{(0,0,X) : X \in U\}$ and $\psi : \mathfrak{z}(\mathcal{B}) \to \mathcal{N}_U$ is a surjective linear map.*

Denote by $SO(V)$ the Lie group of all automorphisms of V that preserve the form η, and with $\det f = 1$, and by $SO(V)_{\mathbb{R}p}$ the Lie subgroup of $SO(V)$ that preserves the isotropic line $\mathbb{R}p$. Obviously, $\mathfrak{so}(V)$ and $\mathfrak{so}(V)_{\mathbb{R}p}$ are the Lie algebras of $SO(V)$ and $SO(V)_{\mathbb{R}p}$ respectively.

By definition, the *type* of a connected weakly-irreducible Lie subgroup $G \subset SO(V)_{\mathbb{R}p}$ is the type of its Lie algebra $\mathfrak{g} \subset \mathfrak{so}(V)_{\mathbb{R}p}$.

2. Transitive similarity transformation groups of Euclidean spaces

In this section we recall a description for connected transitively acting groups of similarity transformations and isometries of Euclidean spaces, see [2] or [3].

Let (E, η) be an Euclidean space. A map $f : E \to E$ is called a *similarity transformation* of E if there exists a $\lambda > 0$ such that $\|f(x_1) - f(x_2)\| = \lambda \|x_1 - x_2\|$ for all $x_1, x_2 \in E$, where $\|x\|^2 = \eta(x,x)$. If $\lambda = 1$, then f is called an *isometry*. Denote by Sim E and Isom E the groups of all similarity transformations and isometries of E respectively. A subgroup $G \subset \text{Sim } E$ such that $G \not\subset \text{Isom } E$ is called *essential*. A subgroup $G \subset \mathrm{Sim}\, E$ is called *irreducible* if it preserves no proper affine subspace of E.

We need the following theorem from [2] and [3].

Theorem 1.

(1) *Let $G \subset \mathrm{Isom}\, E$ be a connected subgroup that acts transitively on E. Then there exists a decomposition $G = H \leftthreetimes F$, where H is the stabilizer of a point $x \in E$ and F is a normal subgroup of G that acts simply transitively on E.*

(2) *Let $F \subset \mathrm{Isom}\, E$ be a connected subgroup that acts simply transitively on E. Then there exists an orthogonal decomposition $E = U \oplus W$ and a Lie groups homomorphism $\Psi : U \to SO(W)$ such that $F = U^\Psi \leftthreetimes W$, where*

$$U^\Psi = \{\Psi(u) \cdot u : u \in U\} \subset SO(W) \times U$$

is a group of screw isometries.

(3) *Let $G \subset \mathrm{Sim}\, E$ be an essential connected subgroup that acts transitively on E. Then $G = (A_1 \times H) \leftthreetimes F$, where $A_1 \subset \mathrm{Sim}\, E$ is a 1-parameterized essential subgroup that preserves a point x, $H \subset \mathrm{Isom}\, E$ commutes with A_1 and preserves the point x, and F is a normal subgroup of G that acts simply transitively on E.*

(4) *A connected subgroup $G \subset \mathrm{Isom}\, E$ acts irreducibly on E if and only if it acts transitively on E.*

From parts (3) and (4) of the theorem it follows that *a connected subgroup $G \subset \mathrm{Sim}\, E$ acts irreducibly on E if and only if it acts transitively on E.*

3. Isometries of Lobachevskian spaces

Let $p, e_1, \dots, e_n, q$ be a basis of the vector space V as above. Consider the basis $e_0, e_1, \dots, e_n, e_{n+1}$ of V, where $e_0 = \frac{\sqrt{2}}{2}(p - q)$ and $e_{n+1} = \frac{\sqrt{2}}{2}(p + q)$. With respect to this basis the Gram matrix of η has the form $\begin{pmatrix} -1 & 0 \\ 0 & E_{n+1} \end{pmatrix}$, where E_{n+1} is the $n+1$-dimensional identity matrix.

The vector model of the $n+1$-dimensional *Lobachevskian space* is defined in the following way

$$L^{n+1} = \{x \in V : \ \eta(x,x) = -1, \ x_0 > 0\}.$$

Recall that L^{n+1} is an $n+1$-dimensional Riemannian submanifold of V. The tangent space at a point $x \in L^{n+1}$ is identified with the vector subspace $(x)^{\perp_\eta} \subset V$ and the restriction of the form η to this subspace is positively definite.

Any element $f \in SO(V)$ preserves the space L^{n+1}. Moreover, for any $f \in SO(V)$, the restriction $f|_{L^{n+1}}$ is an isometry of L^{n+1} and any isometry of L^{n+1} can be obtained in this way. Hence we have the isomorphism

$$SO(V) \simeq \mathrm{Isom}\, L^{n+1},$$

where $\mathrm{Isom}\, L^{n+1}$ is the group of all isometries of L^{n+1}.

Consider the *light-cone* of V,

$$C = \{x \in V : \ \eta(x,x) = 0\}.$$

The subset of the $n+1$-dimensional projective space $\mathbb{P}V$ that consists of all *isotropic lines* $l \subset C$ is called the *boundary of the Lobachevskian space* L^{n+1} and is denoted by ∂L^{n+1}.

We identify ∂L^{n+1} with the n-dimensional unit sphere S^n in the following way. Consider the vector subspace $E_1 = E \oplus \mathbb{R}e_{n+1}$. Each isotropic line intersects the hyperplane $e_0 + E_1$ at a unique point. The intersection $(e_0 + E_1) \cap C$ is the set

$$\{x \in V : x_0 = 1,\, x_1^2 + \cdots + x_{n+1}^2 = 1\},$$

which is the n-dimensional sphere S^n. This gives us the identification $\partial L^{n+1} \simeq S^n$.

Denote by Conf S^n the group of all conformal transformations of S^n. Any transformation $f \in SO(V)$ takes isotropic lines to isotropic lines. Moreover, under the above identification, we have $f|_{\partial L^{n+1}} \in \text{Conf}\, \partial L^{n+1}$ and any transformation from Conf ∂L^{n+1} can be obtained in this way. Hence we have the isomorphism

$$SO(V) \simeq \text{Conf}\, \partial L^{n+1}.$$

Suppose $f \in SO(V)_{\mathbb{R}p}$. The corresponding element $f \in \text{Conf}\, S^n$ (we denote it by the same letter) preserves the point $p_0 = \mathbb{R}p \cap (e_0 + E_1)$. Clearly, $p_0 = \sqrt{2}p$. Denote by s_0 the stereographic projection $s_0 : S^n \backslash \{p_0\} \to e_0 + E$. Since $f \in \text{Conf}\, S^n$, we see that $s_0 \circ f \circ s_0^{-1} : E \to E$ (here we identify $e_0 + E$ with E) is a similarity transformation of the Euclidean space E. Conversely, any similarity transformation of E can be obtained in this way. Thus we have the isomorphism

$$SO(V)_{\mathbb{R}p} \simeq \text{Sim}\, E.$$

A *plane* in the Lobachevskian space L^{n+1} is the nonempty intersection of L^{n+1} and of a vector subspace $U \subset V$. The intersection $L^{n+1} \cap U$ is not empty if and only if the restriction of the form η to U has signature $(1, \dim U - 1)$. A subgroup $G \subset \text{Isom}\, L^{n+1}$ is called *irreducible* if it preserves no proper plane in L^{n+1}.

The following theorem is due to F. I. Karpelevich, see [3] or [16].

Theorem 2. *Let G be a proper connected closed subgroup of* Isom L^{n+1}. *Then G acts irreducibly on L^{n+1} if and only if it preserves an isotropic line $l \in \partial L^{n+1}$ and acts transitively on the Euclidean space $E_l = \partial L^{n+1} \backslash \{l\}$.*

Since the holonomy group of a Lorentzian manifold can be not closed, we need an analog of this theorem for not closed groups. In [12] was proved the following theorem.

Theorem 3. *Let G be a connected (non nec. closed) subgroup of $SO(V)$ that acts weakly-irreducibly. Then either G acts transitively on L^{n+1} or G acts transitively on the Euclidean space $E_l = \partial L^{n+1} \backslash \{l\}$.*

We will prove the following theorem.

Theorem 4. *Let G be a proper connected subgroup of $SO(V)_{\mathbb{R}p}$. Then G acts weakly-irreducibly on V if and only if it acts transitively on the Euclidean space $E = \partial L^{n+1} \backslash \{\mathbb{R}p\}$.*

Proof. We claim that the subgroup $G \subset SO(V)_{\mathbb{R}p}$ acts weakly-irreducibly on V if and only if the corresponding subgroup $G \subset \text{Sim}\, E$ acts irreducibly on E. If $G \subset SO(V)_{\mathbb{R}p}$ is not weakly-irreducible, then it preserves a not degenerate proper subspace $U \subset V$. Since the orthogonal complement $U^\perp \subset V$ is also preserved and either $U \cap C \neq \{0\}$ or $U^{\perp} \cap C \neq \{0\}$, we can assume that $U \cap C \neq \{0\}$. The subgroup $G \subset \mathrm{Sim}\ E$ preserves the affine subspace $s_0((e_0+E)\cap C \cap U) \subset E$, which is not empty. Conversely, if the subgroup $G \subset \mathrm{Sim}\ E$ preserves a proper affine subspace $W \subset E$, then $G \subset SO(V)_{\mathbb{R}p}$ preserves the vector subspace of V spanned by $s_0^{-1}(W) \subset e_0 + E$, which is not degenerate. Now the proof of the theorem follows from parts (3) and (4) of Theorem 1. □

4. Application to holonomy groups of Lorentzian manifolds

Now we consider connected weakly-irreducible not irreducible subgroups of $SO(V)$. Any such group G preserves an isotropic line and is conjugated to a subgroup of $SO(V)_{\mathbb{R}p}$.

In section 2 we have constructed the isomorphism $SO(V)_{\mathbb{R}p} \simeq \mathrm{Sim}\ E$. This isomorphism and theorem 4 gives us a *one-to-one correspondence between connected weakly-irreducible subgroups $G \subset SO(V)_{\mathbb{R}p}$ and connected transitively acting subgroups $G \subset \mathrm{Sim}\ E$.*

Theorem 5. *Let $G \subset \mathrm{Sim}\ E$ be a transitively acting connected subgroup. Then G belongs to one of the following types*

type 1: *$G = (A \times H) \ltimes E$, where $A = \mathbb{R}^+$ is the unite component for the group of all dilations of E about the origin 0, $H \subset SO(E)$ is a connected Lie subgroup, and E is the group of all translations in E;*

type 2: *$G = H \ltimes E$;*

type 3: *$G = (A^{\Phi} \times H) \ltimes E$, where $\Phi : A \to SO(E)$ is a homomorphism and*

$$A^{\Phi} = \{\Phi(a) \cdot a : a \in A\} \subset SO(E) \times A$$

is a group of screw dilations of E;

type 4: *$G = (H \times U^{\Psi}) \ltimes W$, where $E = U \oplus W$ is an orthogonal decomposition, $\Psi : U \to SO(W)$ is a homomorphism, and*

$$U^{\Psi} = \{\Psi(u) \cdot u : u \in U\} \subset SO(W) \times U$$

is a group of screw isometries of E.

The corresponding subgroups of $SO(V)_{\mathbb{R}p}$ under the isomorphism $SO(V)_{\mathbb{R}p} \simeq \mathrm{Sim}\ E$ are the groups of the same type introduced by L. Berard Bergery and A. Ikemakhen.

Proof. Denote by A, K and N the connected Lie subgroups of $SO(V)_{\mathbb{R}p}$ corresponding to the subalgebras $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{N} \subset \mathfrak{so}(V)_{\mathbb{R}p}$. With respect to the basis $p, e_1, \dots, e_n, q$ these groups have the following forms $A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R},\ a > 0 \right\}$,

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in SO(E) \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^t X \\ 0 & \mathrm{id} & X \\ 0 & 0 & 1 \end{pmatrix} : X \in E \right\}.$$

We have the decomposition $SO^0(V)_{\mathbb{R}p} = (A \times K) \ltimes N$.

The computation shows that under the isomorphism $SO(V)_{\mathbb{R}p} \simeq \text{Sim } E$

the element $\begin{pmatrix} a & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} \in A$ corresponds to the dilation $X \mapsto aX$,

the element $\begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K$ corresponds to $f \in SO(E)$, and

the element $\begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^tX \\ 0 & \text{id} & X \\ 0 & 0 & 1 \end{pmatrix} \in N$ corresponds to the translation $Y \mapsto Y+X$.

Let a subgroup $G \subset \text{Sim } E$ act transitively. Denote by the same letter G the corresponding weakly-irreducible subgroup of $SO(V)_{\mathbb{R}p}$. Since we are interested in the groups up to conjugacy, in the theorem 1 we choose $x = 0$, then $H \subset SO(E)$.

For the subgroup $G \subset SO(V)_{\mathbb{R}p}$ we have two cases:

Case 1. G preserves the vector p;

Case 2. G preserves the isotropic line $\mathbb{R}p$ but does not preserve the vector p.

Consider these cases.

Case 1. We have $G \subset K \leftthreetimes N$. Hence the corresponding subgroup $G \subset \text{Sim } E$ consists of isometries, i.e. $G \subset \text{Isom } E$. From the transitivity of G it follows that $G = H \leftthreetimes F$, where $H \subset SO(E)$ and F is a normal subgroup of G that acts simply transitively on E. Hence there exists an orthogonal decomposition $E = U \oplus W$ and a homomorphism $\Psi : U \to SO(W)$ such that $F = U^{\Psi} \leftthreetimes W$.

There are two subcases

Subcase 1.1. The homomorphism Ψ is trivial. Hence $F = E$ and $G = H \leftthreetimes E$. From the classification of L. Berard Bergery and A. Ikemakhen we have $G \subset SO(V)_{\mathbb{R}p}$ is *a group of type 2.*

Subcase 1.2. The homomorphism Ψ is not trivial. We can assume that the homomorphism $d\Psi : U \to \mathfrak{so}(W)$ is injective. Indeed, if $\ker d\Psi \neq \{0\}$, then we choose the decomposition $E = U_1 \oplus W_1$, where $W_1 = W \oplus \ker d\Psi$ and $U_1 \subset U$ is the orthogonal complement of $\ker d\Psi$ in U, and we consider $\Psi_1 = \Psi|_{U_1}$.

We claim that H commutes with $\Psi(U) \subset SO(W)$, moreover H acts trivially on U and $H \subset SO(W)$. Let $f \in H$, $u \in U$. Since F is a normal subgroup of G, we have $f \circ \Psi(u) \circ u \circ f^{-1} = w \circ \Psi(u_1) \circ u_1$ for some $w \in W$ and $u_1 \in U$. Hence for all $v \in E$ we have $f(u) + f \circ \Psi(u) \circ f^{-1}(v) = w + u_1 + \Psi(u_1)v$. Since this holds for all $v \in E$, we have $f \circ \Psi(u) \circ f^{-1} = \Psi(u_1)$. We will prove that $\Psi(u) = \Psi(u_1)$. Let $l(\Psi(U))$ and $\mathfrak{h} = l(H)$ be the Lie algebras of the Lie groups $\Psi(U)$ and H respectively. We have $(\mathfrak{h}+l(\Psi(U)))' = \mathfrak{h}' + [\mathfrak{h}, \Psi(U)]$. Since $[\mathfrak{h}, \Psi(U)] \subset \Psi(U)$ and the Lie algebra $l(\Psi(U))$ is commutative, we have $(\mathfrak{h}+l(\Psi(U)))'' = \mathfrak{h}'$. If $\Psi(u) \neq \Psi(u_1)$, then $[\mathfrak{h}, \Psi(U)] \neq \{0\}$ and $(\mathfrak{h}+l(\Psi(U)))' \neq (\mathfrak{h}+l(\Psi(U)))''$. Since the subalgebra $\mathfrak{h}+l(\Psi(U)) \subset \mathfrak{so}(E)$ is compact, we have a contradiction. Thus, $\Psi(u) = \Psi(u_1)$ and H commutes with $\Psi(U)$. Consider now the Lie algebra $l(G)$ of the Lie group G. We have $l(G) = (\mathfrak{h} \oplus l(U^{\Psi})) \ltimes W$. Since $U^{\Psi} = \{\Psi(u) \circ u : u \in U\}$, we see that $l(U^{\Psi}) = \{d\Psi(u) + u : u \in U\}$. For $\xi \in \mathfrak{h}$ and $d\Psi(u) + u \in l(U^{\Psi})$ we have $[\xi, d\Psi(u) + u] = \xi u \subset U$. Since $U \cap l(G) = \{\varnothing\}$, we see that $\xi u = 0$. Hence H acts trivially on U. Since $H \subset SO(E)$ and W is orthogonal to U, we see that $H(W) \subset W$ and $H \subset SO(W)$.

We see now that $d\Psi(U) \subset \mathfrak{so}(W)$ is a commutative subalgebra that commutes with $\mathfrak{h}$. Put $\mathcal{B} = \mathfrak{h} \oplus d\Psi(U)$. We have $\mathfrak{z}(\mathcal{B}) = \mathfrak{z}(\mathfrak{h}) \oplus d\Psi(U)$. Put $\psi = d\Psi^{-1} : d\Psi(U) \to U$ and extend ψ to the linear map $\psi : \mathfrak{z}(\mathcal{B}) \to U$ by putting $\psi|_{\mathfrak{z}(\mathfrak{h})} = 0$. Thus we have

$$l(G) = (\mathcal{B}' \oplus \{\psi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes W\,.$$

We see that $l(G)$ is an algebra of type 4 and G is *a group of type 4*.

Case 2. In this case we have $G \subset \mathrm{Sim}\, E$, hence $G = (A_1 \times H) \leftthreetimes F$, where A_1 is a 1-parameterized subgroup of G that preserves the point 0, $H \subset SO(E)$ commutes with A_1, and F is a normal subgroup that acts simply transitively on E.

There are two subcases

Subcase 2.1. We have $A_1 = A$ is the unity component of the group of all dilations of E about the origin $0 \in E$.

We claim that $F = E$. Indeed, suppose that $F = U^\Psi \leftthreetimes W$ and the homomorphism Ψ is not trivial. Let $u \in U$, $w \in W$ and $1 \neq \lambda \in A = \mathbb{R}^+$. Since the subgroup $F \subset G$ is normal, we see that $\lambda \circ \Psi(u) \circ u \circ w \circ \lambda^{-1} \in U^\Psi \leftthreetimes W$. Let $v \in E$. We have $(\lambda \circ \Psi(u) \circ u \circ w \circ \lambda^{-1})v =$
$\Psi(u)(\lambda \circ u \circ w \circ \lambda^{-1})v = \Psi(u)(\lambda \circ u \circ w(\lambda^{-1}v)) = \Psi(u)(\lambda(u + w + \lambda^{-1}v)) = \Psi(u)(\lambda u + \lambda w + v)$.
Hence, $\lambda \circ \Psi(u) \circ u \circ w \circ \lambda^{-1} = \Psi(u) \circ (\lambda u) \circ (\lambda w) \in U^\Psi \leftthreetimes W$. This implies $u = \lambda u$ for all $u \in U$, hence, $\lambda = 1$. This gives us a contradiction. Thus, $F = E$.

Now we see that $G = (A_1 \times H) \leftthreetimes F$ is *a group of type 1*.

Subcase 2.2. In this case $A_1 \neq A$, then $A_1 \subset A \times SO(E)$. By analogy with subcase 2.1, we can prove that $F = E$.

Let $\xi : \mathbb{R} \to A_1$ be a parameterization of the group A_1. Define the homomorphisms $\xi_1 : \mathbb{R} \to A$ and $\xi_2 : \mathbb{R} \to SO(E)$ by condition $\xi(t) = \xi_1(t) \cdot \xi_2(t)$ for all $t \in \mathbb{R}$. Since $A_1 \not\subset SO(E)$, we see that the homomorphism ξ_1 is an isomorphism. Put $\Phi = \xi_2 \circ \xi_1^{-1} : A \to SO(E)$. We have

$$A_1 = \{\Phi(a) \cdot a : a \in A\} \subset SO(n) \times \mathbb{R}\,.$$

We see that $l(G) = (l(A_1) \oplus \mathfrak{h}) \ltimes E$ and

$$l(A_1) = \{d\Phi(a) + a : a \in l(A)\}\,.$$

Note that the subalgebra $l(d\Phi(l(A))) \subset \mathfrak{so}(E)$ is commutative and commutes with $\mathfrak{h}$. Put $\mathcal{B} = \mathfrak{h} \oplus l(d\Phi(l(A)))$. We see that $\mathfrak{z}(\mathcal{B}) = \mathfrak{z}(\mathfrak{h}) \oplus l(d\Phi(l(A)))$. Put $\varphi = (d\Phi)^{-1} : d\Phi(l(A)) \to l(A)$ and extend φ to the linear map $\varphi : \mathfrak{z}(\mathcal{B}) \to l(A)$ by putting $\varphi|_{\mathfrak{z}(\mathfrak{h})} = 0$. Thus,

$$l(G) = (\mathcal{B}' \oplus \{\varphi(A) + A : A \in \mathfrak{z}(\mathcal{B})\}) \ltimes E.$$

We see that G is *a group of type 3*. This completes the proof of the theorem. □

5. Transitive isometry groups of the Lobachevskian space L^{n+1}

Recall that we consider a Minkowski space (V, η) of dimension $n + 2$ and a basis $p, e_1, \ldots, e_n, q$ of V with respect to which the Gram matrix of η has the form

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & E_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$, where E_n is the n-dimensional identity matrix. We consider the vector subspace $E \subset V$ spanned by $e_1, \dots, e_n$ as an Euclidean space with respect to the inner product $\eta|_E$. We denote by $SO(V)_{\mathbb{R}p}$ the subgroup of $SO(V)$ that preserves the line $\mathbb{R}p$. For the Lie group $SO^0(V)_{\mathbb{R}p}$ we have the decomposition $SO^0(V)_{\mathbb{R}p} = (A \times K) \leftthreetimes N$, where with respect to the basis $p, e_1, \dots, e_n, q$ the groups A, K and N have the following matrix forms $A = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R},\ a > 0 \right\}$,

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} : f \in SO(E) \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^tX \\ 0 & \mathrm{id} & X \\ 0 & 0 & 1 \end{pmatrix} : X \in E \right\}.$$

Theorem 6. *Let $G \subset SO(V)$ be a connected subgroup that acts transitively on the Lobachevskian space L^{n+1}. Then either $G = SO^0(V)$ or G preserves an isotropic line $l \subset V$ and there exists a basis $p, e_1, \dots, e_n, q$ of V as above such that $l = \mathbb{R}p$ and G is one of the following groups*

(1) *$(A \times H) \leftthreetimes N$, where $H \subset K$ is a subgroup;*

(2) *$(A^\Phi \times H) \leftthreetimes N$, where $\Phi : A \to K$ is a not trivial homomorphism and*

$$A^\Phi = \{\Phi(a) \cdot a : a \in A\} \subset K \times A.$$

Moreover the groups of the form $A \leftthreetimes N$ and $A^\Phi \leftthreetimes N$ exhaust all connected subgroups of $SO(V)$ that act simply transitively on L^{n+1}.

Note that A is the group of translations in L^{n+1} along the line $h = (\mathbb{R}p \oplus \mathbb{R}q) \cap L^{n+1}$, K is the group of rotations about h, N is the group of parabolic translations along 2-dimension planes in L^{n+1} and A^Φ is a group of screw translations along the line h.

Proof. Suppose a subgroup $G \subset SO(V)$ acts transitively on L^{n+1}. Then it preserves no plane in L^{n+1}, hence G acts weakly-irreducibly on V. If G acts irreducibly on V, then $G = SO^0(V)$, see [12] or [11].

If G acts weakly-irreducibly not irreducibly on V, then G preserves an isotropic line $l \subset V$, we assume that $l = \mathbb{R}p$. Then G is the group of type 1, 2, 3 or 4.

We claim that the subgroup $K \leftthreetimes N \subset SO(V)$ does not act transitively on L^{n+1}. Indeed, any element of $K \leftthreetimes N$ takes the vector $\frac{1}{2}p - q \in L^{n+1}$ to some vector $u - q$, where $u \in \operatorname{span}\{p, e_1, \dots, e_n\}$, hence there is no element of $K \leftthreetimes N$ that takes $\frac{1}{2}p - q \in L^{n+1}$ to $p - \frac{1}{2}q \in L^{n+1}$. Hence the groups of type 2 and 4 does not act transitively on L^{n+1}.

We must prove that groups of type 1 and 3, i.e. groups of the form $A \times H \leftthreetimes N$ and $A^\Phi \times H \leftthreetimes N$ act transitively on L^{n+1}. Let $v = xp + \alpha + yq \in L^{n+1}$ and $w = xp + \beta + yq \in L^{n+1}$, where $\alpha, \beta \in E$. Then we have $2xy + \eta(\alpha, \alpha) = -1$ and $2xy + \eta(\beta, \beta) = -1$. Let $X = \frac{\alpha - \beta}{y}$. The element $\begin{pmatrix} 1 & -X^t & -\frac{1}{2}X^tX \\ 0 & \mathrm{id} & X \\ 0 & 0 & 1 \end{pmatrix} \in N$ takes u to w.

Let $v = x_1 p + \beta + y_1 q \in L^{n+1}$, i.e. $2x_1y_1 + \eta(\beta, \beta) = -1$.

The element $\begin{pmatrix} \frac{x_1}{x} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \frac{x}{x_1} \end{pmatrix} \in A$ takes w to v. The element $\begin{pmatrix} \frac{x_1}{x} & 0 & 0 \\ 0 & \Phi(\frac{x_1}{x}) & 0 \\ 0 & 0 & \frac{x}{x_1} \end{pmatrix} \in A^{\Phi}$ takes w to $xp + \Phi(\frac{x_1}{x})(\beta) + yq \in L^{n+1}$. Thus there exist elements in $(A \times H) \leftthreetimes N$ and $(A^{\Phi} \times H) \leftthreetimes N$ that take u to v, i.e. the groups $(A \times H) \leftthreetimes N$ and $(A^{\Phi} \times H) \leftthreetimes N$ act transitively on L^{n+1}.

Note that the elements of the subgroup $H \subset G$ preserve the point $p - \frac{1}{2}q \in L^{n+1}$. Since $\dim L^{n+1} = \dim(A \leftthreetimes N) = \dim(A^{\Phi} \leftthreetimes N)$ and L^{n+1} is simply connected, we see that the groups of the form $A \leftthreetimes N$ and $A^{\Phi} \leftthreetimes N$ are the only connected subgroups of $SO(V)$ that act simply transitively on L^{n+1}. □

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