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CONFORMAL KILLING FORMS WITH NORMALISATION CONDITION

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ABSTRACT. We introduce in this paper normalised equations for conformal Killing forms and study some sorts of their solutions. The normalised equations are conformally covariant and arise naturally from the canonical Cartan connection of conformal geometry. The existence of solutions induces a reduction of the holonomy of the canonical connection. Typical examples with (weakly) irreducible holonomy representation are the so-called Fefferman spaces.

1. INTRODUCTION

A classical object of interest in differential geometry are conformal maps and symmetries. Examples for conformal symmetries arise from the flow of Killing and conformal Killing vector fields on a semi-Riemannian manifold. The notion of conformal vector field has a natural generalisation to differential forms and spinor fields, namely the so-called conformal Killing forms and spinors (cf. [10], [18], [16], [17], [15], [3], [5]).

We aim to introduce in this paper a special class of conformal Killing forms, which we call the normal conformal Killing forms (shortly: nc-Killing forms). These objects are solutions of certain normalised equations for differential forms, which are conformally covariant. Their existence reflects a special part of the conformal symmetry for a metric (or conformal structure) on a semi-Riemannian manifold. The normalised equations can be derived from a so-called conformal tractor calculus, which was originally invented in the 1930's by Tracy Thomas (cf. [19]) and then rediscovered in the work [1].

We will proceed with our discussion as follows. In section 2 to 4 we develop briefly the basic theory for the construction of the canonical normal connection of conformal geometry and present finally the normalised equations for conformal Killing forms. In section 5 we derive integrability conditions in terms of curvature expressions for the existence of solutions. In section 6 we study solutions for nc-Killing forms on Einstein spaces (cf. Theorem 1). Finally, we will present a class of examples for (generically) non-Einstein spaces with nc-Killing forms, the so-called Fefferman spaces.

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2. The representations

Let $\mathbb{R}^{r,s}$ denote the (pseudo)-Euclidean space of signature (r,s) with dimension $n=r+s$. The Lie algebra of conformal Killing vector fields on $\mathbb{R}^{r,s}$ is isomorphic to $\mathfrak{so}(r+1,s+1)$. We describe here the usual action of $\mathfrak{so}(r+1,s+1)$ on the spaces $\Lambda^p_{r+1,s+1}$ of p-forms over $\mathbb{R}^{r+1,s+1}$ in terms of 2-forms with respect to the irreducible parts of the subrepresentation belonging to the subalgebra $\mathfrak{so}(r,s)$. The latter one is the Lie algebra of the special orthogonal group $\mathrm{SO}(r,s)$, which is isomorphic to the set of Killing vector fields on $\mathbb{R}^{r,s}$ having a zero at the origin, i.e., these are generators of orthogonal rotations. In the following, we denote by $\flat$ and $\sharp$ the mappings between $\mathbb{R}^{r,s}$ and its dual $\mathbb{R}^{r,s*}$, which are naturally induced via the metric product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{r,s}$. Moreover, we denote by $e=(e_1,\dots,e_n)$ the standard orthonormal basis in $\mathbb{R}^{r,s}$ such that $\varepsilon_i := \langle e_i,e_i\rangle = -1$ for $i<r+1$.

The space of 2-forms on $\mathbb{R}^{r,s}$ is naturally isomorphic to $\mathfrak{so}(r,s)$ via the mapping

$$\iota : \Lambda^2_{r,s} \to \mathfrak{so}(r,s) \subset \mathfrak{gl}(n)\,.$$
$$\omega \mapsto (\ x \mapsto (x \lrcorner\ \omega)^\sharp\ )$$

The natural action of $\Lambda^2_{r,s}$ on $\alpha\in\Lambda^p_{r,s}$ is then given by

$$\begin{aligned} e_i^\flat \wedge e_j^\flat \circ \alpha &= -e_i^\flat \wedge (e_j \lrcorner\ \alpha) + e_j^\flat \wedge (e_i \lrcorner\ \alpha) \\ &= e_j \lrcorner\ (e_i^\flat \wedge \alpha) - e_i \lrcorner\ (e_j^\flat \wedge \alpha)\,. \end{aligned}$$

The Lie algebra $\mathfrak{so}(r+1,s+1)$ of the group of conformal transformations on the conformal compactification space $S^{r,s}$ of $\mathbb{R}^{r,s}$ (Möbius space of signature (r,s)) is $|1|$-graded:

$$\mathfrak{so}(r+1,s+1) = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+\,,$$

where $\mathfrak{g}_- \cong \mathbb{R}^{r,s}$, $\mathfrak{g}_0 \cong \mathfrak{co}(r,s)$ and $\mathfrak{g}_+ \cong \mathbb{R}^{r,s*}$ (see their brackets below). To set up explicit identifications for these three subspaces, let $(e_t,e_s,e_1,\dots,e_n)$ be an orthonormal frame of $\mathbb{R}^{r+1,s+1}$, where e_t is timelike, e_s spacelike and the e_i's are the basis of $\mathbb{R}^{r,s}$. We denote $e_- = \frac{1}{\sqrt{2}}(e_s-e_t)$ and $e_+ = \frac{1}{\sqrt{2}}(e_s+e_t)$. Then we identify

$$\iota : \mathbb{R}^{r,s} \to \mathfrak{g}_-\,, \qquad x \mapsto e_-^\flat \wedge x^\flat \qquad\qquad \iota : \mathbb{R}^{r,s*} \to \mathfrak{g}_+\,, \qquad y^\flat \mapsto e_+^\flat \wedge y^\flat$$

$$\iota : \mathbb{R}\oplus\mathfrak{so}(r,s) \to \mathfrak{g}_0\,,$$
$$(l,\omega) \mapsto l\cdot e_-^\flat \wedge e_+^\flat + \omega\,.$$

Besides the usual bracket on $\mathfrak{g}_0 \cong \mathfrak{co}(r,s)$, the non-vanishing Lie brackets are

$$[\omega,x] = (x \lrcorner\ \omega)^\sharp, \qquad [\omega,y^\flat] = y \lrcorner\ \omega \quad \text{and} \quad [x,y^\flat] = \langle x,y\rangle\cdot e_-^\flat\wedge e_+^\flat + x^\flat \wedge y^\flat$$

where $x\in\mathfrak{g}_-$, $y^\flat\in\mathfrak{g}_+$ and $\omega\in\mathfrak{g}_0$. The brackets $[\mathfrak{g}_-,\mathfrak{g}_-]$ and $[\mathfrak{g}_+,\mathfrak{g}_+]$ all vanish.

An arbitrary $(p+1)$-form $\alpha\in\Lambda^{p+1}_{r+1,s+1}$ on $\mathbb{R}^{r+1,s+1}$ decomposes into

$$\alpha = e_-^\flat\wedge\alpha_- + \alpha_0 + e_-^\flat\wedge e_+^\flat\wedge\alpha_\mp + e_+^\flat\wedge\alpha_+$$

with uniquely determined forms $\alpha_-, \alpha_+ \in \Lambda^p_{r,s}$, $\alpha_0 \in \Lambda^{p+1}_{r,s}$ and $\alpha_\mp \in \Lambda^{p-1}_{r,s}$. The summands are with respect to the decomposition of $\Lambda^{p+1}_{r+1,s+1}$ into the irreducible submodules

$$\Lambda^p_{r,s} \oplus \Lambda^{p+1}_{r,s} \oplus \Lambda^{p-1}_{r,s} \oplus \Lambda^p_{r,s}$$

of the restricted action to $\mathfrak{so}(r,s)$. The action of $\mathfrak{so}(r+1,s+1)$ on $\Lambda^{p+1}_{r+1,s+1}$ with respect to this decomposition is given by

$$\begin{aligned}
e^\flat_- \wedge e^\flat_i \circ \alpha_- &= 0 \\
e^\flat_- \wedge e^\flat_i \circ \alpha_0 &= -e^\flat_- \wedge (e_i \lrcorner\, \alpha_0) \\
e^\flat_- \wedge e^\flat_i \circ \alpha_\mp &= e^\flat_- \wedge e^\flat_i \wedge \alpha_\mp \\
e^\flat_- \wedge e^\flat_i \circ \alpha_+ &= e^\flat_i \wedge \alpha_+ + e^\flat_- \wedge e^\flat_+ \wedge (e_i \lrcorner\, \alpha_+)
\end{aligned}$$

for $e^\flat_- \wedge e^\flat_i \in \mathfrak{g}_-$. For $e^\flat_+ \wedge e^\flat_i \in \mathfrak{g}_+$, we have

$$\begin{aligned}
e^\flat_+ \wedge e^\flat_i \circ \alpha_- &= e^\flat_i \wedge \alpha_- - e^\flat_- \wedge e^\flat_+ \wedge (e_i \lrcorner\, \alpha_-) \\
e^\flat_+ \wedge e^\flat_i \circ \alpha_0 &= -e^\flat_+ \wedge (e_i \lrcorner\, \alpha_0) \\
e^\flat_+ \wedge e^\flat_i \circ \alpha_\mp &= -e^\flat_+ \wedge e^\flat_i \wedge \alpha_\mp \\
e^\flat_+ \wedge e^\flat_i \circ \alpha_+ &= 0
\end{aligned}$$

and it is

$$e^\flat_- \wedge e^\flat_+ \circ \alpha = -e^\flat_- \wedge \alpha_- + e^\flat_+ \wedge \alpha_+ .$$

The action of $\mathfrak{co}(r,s)$ on the components of α is the usual one.

3. The canonical connection

Let $(M^{n,r}, g)$ be an oriented (pseudo)-Riemannian manifold of dimension $n \geq 3$, where g is a metric of signature $(r, n-r)$. The metric g induces a conformal structure $c := [g]$ on $M^{n,r}$, which is by definition the equivalence class of metrics, which differ from g only by multiplication with a positive function in $C^\infty(M)$. Such a conformal structure on M is equivalently defined by a reduction of the general linear frame bundle $\mathrm{Gl}(M)$ to a principal fibre bundle $G_0(M)$ with structure group $\mathrm{CO}(r,s) = \mathbb{R}^+ \times \mathrm{SO}(r,s)$. The canonical form with values in $\mathbb{R}^{r,s} \cong \mathfrak{g}_-$, reduced to $G_0(M)$, is denoted by θ_-. Moreover, the metric g gives rise to the Levi-Civita connection form ω_{LC} on $G_0(M)$. Every choice of a connection form on $G_0(M)$ represents an additional information to the given conformal structure. However, we will recall now the construction of a conformally invariant connection on a frame bundle of second order.

The conformal structure $c = [g]$ on M is equivalently defined by a P-reduction $P(M)$ of the second order frame bundle $\mathrm{Gl}^{(2)}(M)$, where the structure group P is the parabolic subgroup of the Möbius group $\mathrm{SO}(r+1, s+1)$ with Lie algebra

$$\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \, .$$

The principal fibre bundle $P(M)$ inherits an invariant canonical form $\theta = \theta_- + \theta_0$ from $\mathrm{Gl}^{(2)}(M)$. Thereby, it is

$$d\theta_- = [\theta_-, \theta_0],$$

i.e., the canonical form has no torsion (cf. [11], [8]).

Now let ω be an arbitrary Cartan connection on $P(M)$ with values in $\mathfrak{so}(r+1, s+1) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Its curvature 2-form is defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

and the corresponding curvature function with values in $\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}$ is given in a point $u \in P(M)$ by

$$\nu(u)(x,y) := \Omega(\omega^{-1}(x), \omega^{-1}(y))(u), \qquad x, y \in \mathfrak{g}_-.$$

Since the map $ad : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_-)$ is injective, the $\mathfrak{g}_0$-part ν_0 of the curvature function can be seen as function on $P(M)$ with values in $\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_+^*$.

It is a fact of conformal geometry that there exists a unique Cartan connection

$$\omega_{NC} = \omega_{-1} \oplus \omega_0 \oplus \omega_1$$

on $P(M)$ with the following two properties (cf. [11], [8]):

(1) It is

$$\omega_{-1} = \theta_{-1} \qquad \text{and} \qquad \omega_0 = \theta_0,$$

i.e., the torsion of ω_{NC} vanishes and

(2)

$$tr(\nu_0)(x,y) := \sum_{i=1}^{n} \nu_0(e_i, x)(y)(e_i^\flat) = 0, \quad x, y \in \mathfrak{g}_-,$$

i.e., the trace of the $\mathfrak{g}_0$-part of the curvature function is trivial.

The so determined Cartan connection ω_{NC} on the bundle $P(M)$ is called the canonical normal Cartan connection of conformal geometry and is the basic object for our considerations.

Next we want to describe the normal conformal Cartan connection ω_{NC} in terms of the metric g in the conformal class c. First, we notice that if $\pi : P(M) \to G_0(M)$ denotes the natural projection then θ_- projects to the canonical form on $G_0(M) \subset \mathrm{Gl}^{(1)}(M)$. Furthermore, the G_0-equivariant lifts σ of $G_0(M)$ to $P(M)$ correspond bijectively to the Weyl connections ω^σ (i.e. connections without torsion) on $G_0(M)$ by

$$\omega^\sigma = \sigma^* \theta_0.$$

In particular, if σ^g is the equivariant lift induced by the Levi-Civita connection ω_{LC}^g then the $\mathfrak{g}_0$-part of ω_{NC} is related to ω_{LC}^g by $\omega_{LC}^g = \sigma^* \theta_0$. It remains to determine the $\mathfrak{g}_1$-part of ω_{NC} with respect to g. This part must be calculated from the trace-free condition on the curvature function ν_0 and the result is

$$\omega_1 = -\Gamma \circ \theta_{-1},$$

where the function $\Gamma : P(M) \to \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1$ is the pullback of the so-called *rho*-tensor on (M, g), which is given by the expression

$$K_g = \frac{1}{n-2}\Big(\frac{\mathrm{scal}_g}{2(n-1)} - \mathrm{Ric}_g\Big).$$

Thereby, Ric_g denotes the Ricci-tenor and scal_g is the scalar curvature of g. In short, we see that ω_{NC} is given with respect to $g \in c$ by θ_-, ω_{LC}^g and K_g.

We aim to assign a usual principal bundle connection to ω_{NC}. For this, let

$$\mathfrak{M}(M) = P(M) \times_P \mathrm{SO}(r+1, s+1)$$

be the extended bundle with structure group $\mathrm{SO}(r+1, s+1)$. With respect to a metric g and the inclusion of $\mathfrak{so}(r,s)$ in $\mathfrak{so}(r+1, s+1)$ as described above, we can express this bundle also as

$$\mathfrak{M}(M) = SO(M, g) \times_{\mathrm{SO}(r,s)} \mathrm{SO}(r+1, s+1),$$

where $SO(M, g)$ is the orthonormal frame bundle to g on M. Then a local frame $s = (s_1, \ldots, s_n)$ on M, which is a local section in $SO(M, g)$, has a natural extension to a section $s_c = (s_-, s_+, s_1, \ldots, s_n)$ in $\mathfrak{M}(M)$. Thereby, the $(s_c)_i$'s can be understood as 1-form tractors in $T_{\mathfrak{M}}(M) \cong \Lambda^1_{\mathfrak{M}}(M)$ (see below).

The Cartan connection ω_{NC} can now be extended to a usual principal bundle connection on $\mathfrak{M}(M)$ by right translation on the fibres. We denote this connection on $\mathfrak{M}(M)$ also by $\omega_{NC} = \omega_{-1} \oplus \omega_0 \oplus \omega_1$. We have already calculated the components of the connection ω_{NC} with respect to the metric g. Let s be a local frame on (M, g). Then we have the following expression for the local connection form on (M, g):

$$\omega_{NC} \circ ds_c(X) = e_-^\flat \wedge \theta_-(X)^\flat + \omega_{LC} \circ ds_c(X) - e_+^\flat \wedge \theta_-(K_g(X))^\flat, \quad X \in TM,$$

where $(e_-, e_+, e_1, \ldots, e_n)$ is the standard basis in $\mathbb{R}^{r+1,s+1}$ and θ_- is evaluated at s. We denote by $Hol(\omega_{NC})$ the holonomy group of the principal bundle connection ω_{NC}, which sits in the Möbius group $\mathrm{SO}(r+1, s+1)$.

With the approach of a principal bundle connection form we can introduce covariant derivatives to ω_{NC} on vector bundles with structure group $\mathrm{SO}(r+1, s+1)$ associated to $\mathfrak{M}(M)$ in the usual manner. In particular, ω_{NC} induces derivatives ∇^{NC} on the p-form tractor bundles defined as

$$\Lambda^p_{\mathfrak{M}}(M) := \mathfrak{M}(M) \times_\iota \Lambda^p_{r+1,s+1}.$$

With respect to the metric g these bundles split into a sum of usual p-form bundles on M:

$$\Lambda^{p+1}_{\mathfrak{M}}(M) = \Lambda^p(M) \oplus \Lambda^{p+1}(M) \oplus \Lambda^{p-1}(M) \oplus \Lambda^p(M).$$

The covariant derivative ∇^{NC} acts on sections in these bundles with respect to the above splitting by the matrix expression

$$\nabla^{NC}_X \alpha = \begin{pmatrix} \nabla^{LC}_X & -X \lrcorner & X^\flat \wedge & 0 \\ -K(X)^\flat \wedge & \nabla^{LC}_X & 0 & X^\flat \wedge \\ K(X) \lrcorner & 0 & \nabla^{LC}_X & X \lrcorner \\ 0 & K(X) \lrcorner & K(X)^\flat \wedge & \nabla^{LC}_X \end{pmatrix} \begin{pmatrix} \alpha_- \\ \alpha_0 \\ \alpha_\mp \\ \alpha_+ \end{pmatrix}.$$

Thereby, ∇^{LC} denotes the Levi-Civita connection. This expression can be easily calculated from the local form of ω_{NC} and the formulae for the action of $\mathfrak{so}(r+1,s+1)$ on $\Lambda^{p+1}_{r+1,s+1}$ (cf. section 2).

Vector bundles $W_{\mathfrak{M}}$ over a conformal space (M,c) of the form $\mathfrak{M}(M)\times_\ell W$, where (W,ℓ) is a $SO(r+1,s+1)$-represenation, are called conformal tractor bundles (cf. [19], [1]). A general discussion of covariant derivatives and their action on tractor bundles in the context of parabolic geometries can be found in [7].

4. The normalised equations

Let $(M^{n,r},g)$ be an oriented (pseudo)-Riemannian manifold and let $\Lambda^{p+1}_{\mathfrak{M}}(M)$ be the associated bundle of $(p+1)$-form tractors to the principal fibre bundle $\mathfrak{M}(M)$ with normal conformal covariant derivative ∇^{NC}. We call a section $\alpha\in\Omega^{p+1}_{\mathfrak{M}}(M)$ a (normal) parallel tractor iff $\nabla^{NC}\alpha=0$. The tractor α corresponds via the metric g to a set of differential forms on $M^{n,r}$:

$$\alpha\;\longleftrightarrow\;(\alpha_-,\alpha_0,\alpha_\mp,\alpha_+)\,,$$

where $\alpha_-,\alpha_+\in\Omega^p(M)$, $\alpha_0\in\Omega^{p+1}(M)$ and $\alpha_\mp\in\Omega^{p-1}(M)$. The condition $\nabla^{NC}\alpha=0$ is then equivalent to the set of conformally covariant equations given by

$$\nabla^{LC}_X\alpha_- - X\lrcorner\,\alpha_0 + X^\flat\wedge\alpha_\mp = 0 \tag{1}$$

$$-K(X)^\flat\wedge\alpha_- + \nabla^{LC}_X\alpha_0 + X^\flat\wedge\alpha_+ = 0 \tag{2}$$

$$K(X)\lrcorner\,\alpha_- + \nabla^{LC}_X\alpha_\mp + X\lrcorner\,\alpha_+ = 0 \tag{3}$$

$$K(X)\lrcorner\,\alpha_0 + K(X)^\flat\wedge\alpha_\mp + \nabla^{LC}_X\alpha_+ = 0\,. \tag{4}$$

We calculate from α_- of a given solution α the remaining differential forms in order to get equations for α_- only. It is

$$d=\sum_{i=1}^n\varepsilon_i s_i^\flat\wedge\nabla^{LC}_{s_i}\qquad\text{and}\qquad d^*=-\sum_{i=1}^n\varepsilon_i s_i\lrcorner\,\nabla^{LC}_{s_i}$$

the exterior differential resp. the codifferential with respect to a local orthonormal frame s. The equations (1) – (3) imply for a parallel tractor α of degree $p+1$ that

$$d\alpha_-=(p+1)\alpha_0,\qquad d^*\alpha_-=(n-p+1)\alpha_\mp$$

$$\frac{1}{p+1}d^*d\alpha_-=(n-p)\alpha_+-\sum_i^n\varepsilon_i s_i\lrcorner\,(K(s_i)^\flat\wedge\alpha_-)$$

$$\frac{1}{n-p+1}dd^*\alpha_-=-p\alpha_+-\sum_i^n\varepsilon_i s_i^\flat\wedge(K(s_i)\lrcorner\,\alpha_-)\,.$$

For $n\neq 2p$ the sum of the latter two equations results to

$$\alpha_+=\frac{1}{n-2p}\cdot\Big(-\frac{\mathrm{scal}}{2(n-1)}\alpha_-+\frac{1}{p+1}d^*d\alpha_-+\frac{1}{n-p+1}dd^*\alpha_-\Big)\,,$$

which is

$$\alpha_+ = \frac{1}{n-2p}(\nabla^*\nabla - \frac{\mathrm{scal}}{2(n-1)})\alpha_- ,$$

where $\nabla^*\nabla$ denotes the Bochner-Laplacian. In the middle dimensional case $2p = n$ we have

$$\alpha_+ = \frac{1}{n}\cdot\big[\frac{1}{p+1}(d^*d - dd^*)\alpha_- + \sum_{i=1}^{n}\varepsilon_i\cdot\big(s_i \lrcorner (K(s_i)^\flat \wedge \alpha_-) - s_i^\flat \wedge (K(s_i) \lrcorner \alpha_-)\big)\big] .$$

We observe that $\alpha_- \equiv 0$ if and only if the tractor α is trivial.

With the so derived expressions for the components of a parallel tractor α we now formulate the normalised conformal Killing equations for a p-form α_- on a (pseudo)-Riemannian manifold $(M^{n,r}, g)$. They are

$$0 = \nabla^{LC}_X\alpha_- - \frac{1}{p+1}X \lrcorner\, d\alpha_- + \frac{1}{n-p+1}X^\flat \wedge d^*\alpha_- \tag{5}$$

$$0 = -K(X)^\flat \wedge \alpha_- + \frac{1}{p+1}\nabla^{LC}_X d\alpha_- + X^\flat \wedge \Box_p\alpha_- \tag{6}$$

$$0 = K(X) \lrcorner\, \alpha_- + \frac{1}{n-p+1}\nabla^{LC}_X d^*\alpha_- + X \lrcorner\, \Box_p\alpha_- \tag{7}$$

$$0 = \frac{1}{p+1}K(X) \lrcorner\, d\alpha_- + \frac{1}{n-p+1}K(X)^\flat \wedge d^*\alpha_- + \nabla^{LC}_X\Box_p\alpha_- , \tag{8}$$

whereby we set

$$\Box_p := \frac{1}{n-2p}\cdot\big(-\frac{\mathrm{scal}}{2(n-1)}\mathrm{id} + \nabla^*\nabla\big) \qquad \text{for } n \neq 2p$$

and

$$\Box_{n/2} := \frac{1}{n}\cdot\big[\frac{1}{p+1}(d^*d - dd^*) + \sum_{i=1}^{n}\varepsilon_i\cdot\big(s_i \lrcorner\, (K(s_i)^\flat \wedge \cdot) - s_i^\flat \wedge (K(s_i) \lrcorner\, \cdot)\big)\big] .$$

In the following, we say that a p-form $\alpha_- \in \Omega^p(M)$, which satisfies the normalised equations (5) - (8), is a normal conformal Killing p-form (or shortly: nc-Killing p-form). The conformal covariance of the equations implies that if α_- is a nc-Killing p-form to g on M then the rescaled p-form

$$\tilde{\alpha}_- := e^{-(p+1)\phi}\cdot\alpha_-$$

is nc-Killing with respect to the conformally changed metric $\tilde{g} = e^{-2\phi}\cdot g$.

However, the equations (5) - (8) are not only conformally covariant, but a further natural symmetry appears. Let $*$ denote the Hodge-star operator on $\Lambda^*(M)$ defined by

$$\alpha_- \wedge *\alpha_- = g(\alpha_-, \alpha_-)dM ,$$

where dM denotes the volume form of $(M^{n,r}, g)$. It is

$$**|_{\Lambda^p} = (-1)^{p(n-p)+r} \qquad \text{and} \qquad d^* = (-1)^{n(p-1)+r+1} * d * .$$

There is also a 'Hodge' operator $*_{\mathfrak{M}}$ on $\Lambda^*_{\mathfrak{M}}(M)$ defined in the same manner:

$$\alpha \wedge *_{\mathfrak{M}}\alpha = c_{\mathfrak{M}}(\alpha,\alpha) dM_{\mathfrak{M}}\,,$$

where $dM_{\mathfrak{M}} := -e_-^\flat \wedge e_+^\flat \wedge dM$ and $c_{\mathfrak{M}}$ is the obvious $\mathrm{SO}(r+1, s+1)$-invariant scalar product on $\Lambda^*_{\mathfrak{M}}(M)$. The operator $*_{\mathfrak{M}}$ is parallel:

$$\nabla^{NC} *_{\mathfrak{M}} = *_{\mathfrak{M}} \nabla^{NC}\,.$$

Therefore, if α is a parallel $(p+1)$-tractor then $*_{\mathfrak{M}}\alpha$ is a parallel $(n-p+1)$-tractor. The parallel tractor $*_{\mathfrak{M}}\alpha$ corresponds to the set

$$\big((-1)^p * \alpha_-\,,\ *\alpha_\mp\,,\ -*\alpha_0\,,\ (-1)^{p+1} * \alpha_+\big)$$

of differential forms. This shows that if α_- is a nc-Killing p-form then $*\alpha_-$ is a nc-Killing $(n-p)$-form. Indeed, with

$$*(X \lrcorner\ \beta^p) = (-1)^{p+1} X^\flat \wedge *\beta \qquad \text{and} \qquad *\,(X^\flat \wedge \beta^p) = (-1)^p X \lrcorner\ *\beta\,,$$

and since $*\Box_p = -\Box_{n-p}*$ is anti-commuting, the normalised equations (5) – (8) are seen to be $*$-invariant as well.

Finally, we remark that for a 1-form α_-, equation (5) just means that the dual to α_- is a conformal vector field. In general, solutions of (5) are known as conformal Killing p-forms (cf. [10], [18], [5]). Equation (5) is Hodge $*$-invariant itself. The additional equations (6) – (8) then impose further conditions on a conformal Killing p-form to be 'normal'.

5. Curvature conditions

We derive here integrability conditions for the existence of nc-Killing p-forms on a (pseudo)-Riemannian manifold $(M^{n,r}, g)$ in terms of curvature expressions. Let

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

be the Riemannian curvature tensor, where $X, Y, Z \in TM$ are tangent vectors. By contraction, we obtain

$$\mathrm{Ric}(X) = \sum_{i=1}^n \varepsilon_i \cdot R(X, s_i)s_i\,, \qquad \mathrm{scal} = \mathrm{tr}(\mathrm{Ric})\,,$$

the Ricci tensor and the scalar curvature, where s denotes a local frame as usual. The *rho*-tensor is $K = \frac{1}{n-2}\big(\frac{\mathrm{scal}}{2(n-1)} - \mathrm{Ric}\big)$. The trace-free part of the Riemannian curvature tensor is the Weyl tensor W, which can be expressed by

$$W = R - g \star K\,,$$

where $\star$ denotes the Kulkarni-Nomizu product. Moreover, we have the Cotton-York tensor C, which is the anti-symmetrisation of the covariant derivative of the *rho*-tensor:

$$C(X,Y) := (\nabla_X K)(Y) - (\nabla_Y K)(X)\,.$$

Furthermore, we find the Bach tensor

$$B(X,Y) = \sum_{i=1}^n \varepsilon_i \cdot \nabla_{s_i} C(X,Y,s_i) - \sum_{i=1}^n \varepsilon_i \cdot W(K(s_i), X, Y, s_i)\,,$$

where $C(X,Y,Z) := C_X(Y,Z) = g(C(Y,Z),X)$. The Weyl tensor considered as a symmetric map on the space of 2-forms is conformally invariant. The Bach tensor is symmetric and divergence-free. Moreover, we have the Bianchi identities

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0$$

for all $X, Y, Z \in TM$, which also imply

$$\sum_{i=1}^{n} \varepsilon_i \cdot \nabla_{s_i} W(X,Y,Z,s_i) = (3-n) \cdot C(Z,X,Y) \quad \text{and} \quad \sum_{i=1}^{n} \varepsilon_i \cdot C(s_i,s_i,X) = 0\,.$$

A straightforward calculation proves that the curvature tensor to ∇^{NC} takes the matrix form

$$R^{\nabla} = \begin{pmatrix} W & 0 & 0 & 0 \\ -C(X,Y)^{\flat}\wedge & W & 0 & 0 \\ C(X,Y) \lrcorner & 0 & W & 0 \\ 0 & C(X,Y) \lrcorner & C(X,Y)^{\flat}\wedge & W \end{pmatrix}.$$

As integrability condition for the existence of a normal parallel tractor α we obtain

$$\begin{aligned} W(X,Y) \circ \alpha_- &= 0 \\ W(X,Y) \circ \alpha_0 &= \ C(X,Y)^{\flat} \wedge \alpha_- \\ W(X,Y) \circ \alpha_{\mp} &= -C(X,Y) \lrcorner \ \alpha_- \\ W(X,Y) \circ \alpha_+ &= -C(X,Y) \lrcorner \ \alpha_0 - C(X,Y)^{\flat} \wedge \alpha_{\mp}\,. \end{aligned}$$

By taking the divergence on both sides of these equations we get

$$\begin{aligned} (n-4) \cdot C_T \circ \alpha_- &= 0 \\ (n-4) \cdot C_T \circ \alpha_0 &= -B(T)^{\flat} \wedge \alpha_- \\ (n-4) \cdot C_T \circ \alpha_{\mp} &= \ B(T) \lrcorner \ \alpha_- \\ (n-4) \cdot C_T \circ \alpha_+ &= \ B(T) \lrcorner \ \alpha_0 + B(T)^{\flat} \wedge \alpha_{\mp}\,. \end{aligned}$$

The curvature conditions expressed for the nc-Killing p-form α_- take the form:

(9) $$W(X,Y) \circ \alpha_- = 0$$

(10) $$W(X,Y) \circ d\alpha_- = (p+1) \cdot C(X,Y)^{\flat} \wedge \alpha_-$$

(11) $$W(X,Y) \circ d^*\alpha_- = -(n-p+1)C(X,Y) \lrcorner \ \alpha_-$$

(12) $$W(X,Y) \circ \Box_p \alpha_- = -\frac{1}{p+1} C(X,Y) \lrcorner \ d\alpha_- - \frac{1}{n-p+1} C(X,Y)^{\flat} \wedge d^*\alpha_-\,.$$

The sets of integrability conditions are conformally covariant and invariant under the Hodge $*$-operator. The curvature conditions in this section can also be deduced from the work [7], where a general discussion of the action of the Cartan curvature on (parabolic) form tractors is exercised.

6. Normal conformal Killing p-forms on Einstein manifolds

We consider in this section solutions of the normalised equations (5) - (8) on Einstein manifolds. We start with a criterion when a space $(M^{n,r}, g)$ is conformally equivalent to an Einstein space, i.e., there is a metric $\tilde{g}$ in the conformal class $c = [g]$, which satisfies

$$\mathrm{Ric}_{\tilde{g}} = \frac{\mathrm{scal}_{\tilde{g}}}{n} \cdot \tilde{g} .$$

For this, let us assume that $f_- = \alpha_-$ is a nc-Killing function without zero. We have mentioned before that the rescaled function $\tilde{\alpha}_- = \frac{1}{f_-}\alpha_- = 1$ is nc-Killing with respect to the metric $\tilde{g} = \frac{1}{f_-^2} \cdot g$. From the normalised equations (5) - (8), it follows immediately

$$K_{\tilde{g}} = -\frac{\mathrm{scal}_{\tilde{g}}}{2n(n-1)} \cdot \tilde{g} ,$$

which means that $\tilde{g}$ is Einstein. On the other hand, every constant function on an Einstein space is nc-Killing. The criterion then says that a metric is conformally Einstein (i.e., there is an Einstein metric in the conformal class $[g]$) if and only if there exists at least one nc-Killing function without zero. By the way, two nc-Killing functions without zero (whose quotient is not constant) induce a conformal transformation (which is not a homothety) between two Einstein metrics in the conformal class $[g]$. The gradient of the quotient is conformal and induces a warped-product structure on the Einstein metrics in question (cf. [6], [12]). In general, the normalised equations (5) - (8) for a function $f_- = \alpha_-$ reduce to the single second order equation

$$\text{trace free part of } \ (\nabla^2 f_- - f_- \cdot K) \ = \ 0 .$$

Obviously, in case that a solution f_- of this equation has a zero the rescaling in the way as above is not possible. Indeed, examples of nc-Killing functions on non-conformally Einstein spaces are well known (cf. [13]). However, since in general the set of zeros of nc-Killing forms is singular on the underlying conformal space, we can at least say that the existence of a nc-Killing function f_- implies that, up to singularities, an Einstein metric exists in the conformal class $[g]$. This is exactly the case when the holonomy group $Hol(\omega_{NC})$ of the conformal connection ω_{NC} fixes at least one vector in $\mathbb{R}^{r+1,s+1}$. For the case that the fixed vector is lightlike there exists a rescaled metric, which is Ricci-flat. The case of a fixed timelike vector is for 'Einstein scalings' with scal > 0, the spacelike case when scal < 0.

Now we assume that $(M^{n,r}, g)$ is a (pseudo)-Riemannian Einstein manifold. The 1-form tractor $o := s_-^\flat - \frac{\mathrm{scal}}{2(n-1)n} s_+^\flat$ satisfies the equations (1) - (4) and it corresponds to the set of forms $(1, 0, 0, -\frac{\mathrm{scal}}{2(n-1)n})$, i.e., the nc-Killing function o_- is constant one. Furthermore, let

$$\alpha = s_-^\flat \wedge \alpha_- + \alpha_0 + s_-^\flat \wedge s_+^\flat \wedge \alpha_\mp + s_+^\flat \wedge \alpha_+$$

be any additional parallel tractor of degree $(p+1)$ with $d\alpha_- \not\equiv 0$ on M. It follows immediately that $o \wedge \alpha$ is a parallel $(p+2)$-form tractor. This tractor corresponds to the set

$$\left(\alpha_0\ ,\ 0\ ,\ \alpha_+ + \frac{\mathrm{scal}}{2(n-1)n}\alpha_-\ ,\ \frac{-\,\mathrm{scal}}{2(n-1)n}\alpha_0\right)$$

of differential forms, which shows that $d\alpha_-$ is a (closed) nc-Killing $(p+1)$-form. The $(n-p-1)$-form $*d\alpha_-$ is then nc-Killing and coclosed.

In general, the set of normalised equations (5) - (8) reduces for a coclosed p-form β_- on an Einstein space to

$$\nabla^{LC}_X \beta_- = \frac{1}{p+1}\cdot X \lrcorner\ d\beta_- \tag{13}$$

$$\nabla^{LC}_X d\beta_- = -\frac{(p+1)\cdot \mathrm{scal}}{n\cdot(n-1)}\cdot X^\flat \wedge \beta_-\,, \tag{14}$$

which implies $\Delta_p\beta_- = \frac{(p+1)(n-p)\cdot\mathrm{scal}}{n\cdot(n-1)}\beta_-$ for the Laplacian $\Delta_p = dd^* + d^*d$. A differential form that satisfies the equations (13) and (14) is called a special Killing p-form to the Killing constant $-\frac{(p+1)\cdot\mathrm{scal}}{n\cdot(n-1)}$ (cf. [5]). There is an effective way to describe the geometry of spaces with special Killing forms. The cone metric with scaling $b \neq 0$ is defined on the space $\mathbb{R}_+ \times M$ as

$$\hat{g}_b := b\,dt^2 + t^2 g\,,$$

where t denotes the parameter in $\mathbb{R}_+$. This metric has either signature $(r, s+1)$ or $(r+1, s)$.

Proposition 1 (cf. [5]). *Let $(M^{n,r}, g)$ be a (pseudo)-Riemannian manifold and $\hat{M}_b$ its cone with metric $\hat{g}_b$ to the constant $b \neq 0$. Then the special Killing p-forms on $M^{n,r}$ with Killing constant $-\frac{(p+1)}{b}$ correspond bijectively to the parallel $(p+1)$-forms on the cone $\hat{M}_b$. The correspondence is explicitly given by*

$$\beta_- \in \Omega^p(M) \ \mapsto\ t^p dt \wedge \beta_- + \frac{\mathrm{sign}(b)\cdot t^{p+1}}{p+1} d\beta_- \in \Omega^{p+1}(\hat{M}_b)\,.$$

In the Riemannian case a geometric characterisation of complete spaces (M^n, g) with positive scalar curvature admitting special Killing forms was established by using the above correspondence with the cone and the holonomy classification of the cone (cf. [4], [5]). Thereby, we remember to the fact that if the holonomy group of a Riemannian cone $\hat{M}_b$, $b > 0$, over a complete Riemannian space M is reducible then the cone is automatically flat. We apply the classification to the nc-Killing forms.

Theorem 1 (cf. [4], [5]). *Let (M^n, g) be a simply connected and complete Riemannian Einstein space of positive scalar curvature admitting a nc-Killing p-form. Then M^n is either*

1. *the round (conformally flat) sphere S^n,*
2. *an Einstein-Sasaki manifold of odd dimension $n \geq 5$ with a special Killing 1-form α_-,*
3. *an Einstein-3-Sasaki space of dimension $n = 4m+3 \geq 7$ with three independent special Killing 1-forms α^1_-, α^2_- and α^3_-,*
4. *a nearly Kähler manifold of dimension 6, where the Kähler form ω_- is a special Killing 2-form or*

(5) *a nearly parallel G_2-manifold in dimension 7 with its fundamental form γ_- as special Killing 3-form.*

Proof. We only have to show that a special Killing form appears on M under the assumptions. We have seen above that if α_- is nc-Killing and $d\alpha_- \neq 0$ then $*d\alpha_-$ is nc-Killing and coclosed, i.e., $*d\alpha_-$ is special Killing. In case that $d\alpha_- = 0$, it holds $d^*\alpha_- \neq 0$, since scal > 0. Hence, $*\alpha_-$ is special Killing. □

As we mentioned already, in the conformal Einstein case the holonomy $Hol(\omega_{NC})$ fixes a vector in $\mathbb{R}^{r+1,s+1}$. In particular, the standard representation of $Hol(\omega_{NC})$ on $\mathbb{R}^{r+1,s+1}$ acts not irreducible. An example of a class of metrics (conformal structures) which are generically non-conformally Einstein, but admit nc-Killing 1-forms, are the Fefferman metrics in Lorentzian geometry (cf. e.g. [9]).

Example (generically non-Einstein; cf. [2]). Let (N^3, H, J, θ) be a strictly pseudoconvex manifold of dimension 3. In detail, the data mean that H is a smooth distribution in TN of codimension 1 and J is an integrable complex structure on H (i.e. $J^2 = -\operatorname{id}|_H$) and the pair (H, J) is an (abstract) CR-structure on N^3. Moreover, θ is a contact form on N such that $\theta|_H = 0$ and the (symmetric) Levi-form $L_\theta := H \times H \to \mathbb{R}$ defined by

$$L_\theta(X, Y) := d\theta(X, JY)$$

is required to be positive definit. Under these assumptions the tensor field

$$g_\theta := L_\theta + \theta \circ \theta$$

defines a Riemannian metric on N and this metric admits a special metric covariant derivative ∇^W, which is called the Tanaka-Webster connection and is determined by the torsion conditions

$$\operatorname{Tor}^W(X, Y) = L_\theta(JX, Y) \cdot T \qquad \text{and} \qquad \operatorname{Tor}^W(T, X) = -\frac{1}{2}([T, X] + J[T, JX]),$$

where $X, Y \in \Gamma(H)$ and T is the Reeb vector field to θ determined by

$$\theta(T) = 1 \qquad \text{and} \qquad T \lrcorner\, d\theta = 0 .$$

Now let (F, π, N) denote the S^1-principal fibre bundle associated to the canonical complex line bundle $\Lambda^{2,0}(N)$ of the underlying CR-structure (H, J). The principal bundle (F, π, N) has a total space of dimension 4 and admits a connection form A^W with values in the Lie algebra $i\mathbb{R}$ of S^1, which induces the Tanaka-Webster covariant derivative on $\Lambda^{2,0}(N)$. The Fefferman metric h_θ is now defined on the total space F^4 by the expression

$$h_\theta := \pi^* L_\theta - i\frac{4}{3}\pi^*\theta \circ (A^W - \frac{i}{4}R^W \cdot \theta),$$

where R^W is the Tanaka-Webster scalar curvature, which is a complete contraction of the curvature tensor belonging to ∇^W. We set $A_\theta := A^W - \frac{i}{4}R^W \cdot \pi^*\theta$. Then the 1-forms $\pi^*\theta$ and A_θ are lightlike with respect to h_θ and the signature of the Fefferman metric h_θ is Lorentzian $(3, 1)$. The important point of the construction is that the conformal class $[h_\theta]$ of the Fefferman metric is independent from the choice of the pseudo-hermitian form θ, i.e., $[h_\theta]$ is an invariant of the CR-structure (H, J) on N^3.

The Fefferman metric admits by construction a canonical lightlike Killing vector field V_- along the fibres of F over N^3, namely the fundamental vector field

$$V_-(b) := \frac{d}{dt}\Big|_{t=0}(b \cdot e^{3it}), \quad b \in F,$$

generated by the fibre action. The dual of V_- with respect to h_θ is the Killing 1-form

$$\alpha_- := 2\pi^*\theta,$$

which is twice the lift of the pseudo-hermitian form θ to F^4. The Killing field V_- satisfies

$$V_- \lrcorner W = 0, \qquad V_- \lrcorner C = 0 \qquad \text{and} \qquad K(V_-, V_-) = \text{const.} < 0$$

(cf. conditions (9) to (12)). In fact, a straightforward calculation shows that the dual α_- to V_- is a nc-Killing 1-form, i.e., it solves the normalised equations (5) to (8). Thereby, it is

$$\alpha_\pm = \frac{1}{4}d^*\alpha_- = 0 \qquad \text{and} \qquad \alpha_0 = \frac{1}{2}d\alpha_- = \pi^* d\theta.$$

For the codifferential of $\pi^* d\theta$ we find

$$d^*\pi^* d\theta = -\frac{2i}{3}A_\theta + \frac{\mathrm{scal}_{h_\theta}}{3}\pi^*\theta,$$

which results to $\alpha_+ = -\frac{i}{3}A_\theta$. Then the normal parallel 2-form tractor α, corresponding to the nc-Killing form α_-, is expressed by

$$\alpha = s_-^\flat \wedge 2\pi^*\theta + s_+^\flat \wedge \frac{1}{3i}A_\theta + \pi^* d\theta.$$

It turns out that this expression equals the standard Kähler form ω_o in the 2-form tractors $\Omega^2_{\mathfrak{M}}(F)$ over $(F, [h_\theta])$. In particular, this implies that the holonomy group $Hol(\omega_{NC})$ is a subgroup of $\mathrm{U}(1,2)$ in $\mathrm{SO}(2,4)$. In fact, one can prove that (at least locally) any Fefferman metric admits a solution of the conformal Killing spinor equation (cf. [14], [2]). This finally shows that $Hol(\omega_{NC})$ is reduced at least to $\mathrm{SU}(1,2)$.

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