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ON A DISTINGUISHED CLASS OF INFINITE DIMENSIONAL REPRESENTATIONS OF $\mathfrak{sp}(2n, \mathbb{C})$

SVATOPLUK KRÝSL

ABSTRACT. In this paper we have shown how a tensor product of an infinite dimensional representation within a certain distinguished class of infinite dimensional irreducible representations of $\mathfrak{sp}(2n, \mathbb{C})$ with the defining representation decomposes. Further we have proved a theorem on complete reducibility of a k-fold tensor product of the defining representation (tensored) with a member of the distinguished class.

1. INTRODUCTION

The main aim of the paper is to study a distinguished class of irreducible infinite dimensional representations of the symplectic algebra C_n (so called bounded modules) and tensor products of elements in this class with the defining representation. The motivation for such a study is coming from a study of invariant differential operators on manifolds with a given parabolic structure. Invariant operators in question are acting on sections associated to finite dimensional representations of suitable parabolic subgroups of a semisimple Lie group G. In the case $G = \mathrm{Spin}\,(n)$ a particular role is played by operators acting on sections of bundles on Spin manifolds associated to spinor-tensor representations.

On a class of manifolds with a projective contact structure ([3]), the corresponding vector bundles are associated to representations of symplectic group. But there are no analogues of spinor representations among finite dimensional modules.

It was suggested by Kostant (see [6]) that certain infinite dimensional representations form a suitable analogue of spinor representations of the orthogonal Lie algebra D_n in this case. They were used, for example, in a definition of a symplectic version of the Dirac operator by K. Habermann (see [4]). These two infinite dimensional representations are modules of the metaplectic group (a double cover of the symplectic group), they are called the Segal-Shale-Weil representations. By analogy with the orthogonal case, it is interesting to understand the structure of tensor product of

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these spinor representations with finite dimensional modules. It leads to the family of bounded representations introduced in [1].

In the second section, we shall review basic facts about the symplectic algebra and we shall introduce its spinor representations. Then we shall present some result on decomposition of tensor products of finite and infinite dimensional representation proved by B. Kostant ([7]) in the third section. In the fourth section, tensor products of spinor modules and finite dimensional representations are described following [2], which leads to a definition of a distinguished class of bounded representations. The fifth section contains new results on the decomposition of bounded modules with the defining representations and its powers.

2. Spinor representations of $\mathfrak{sp}(2n, \mathbb{C})$

Let us recall first some basic facts on the symplectic algebra $C_n = \mathfrak{sp}(2n, \mathbb{C})$. This algebra consists of $2n \times 2n$ matrices over complex numbers of the form

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)$$

where $A_1 = -A_4^T$, $A_2 = A_2^T$ and $A_3 = A_3^T$. The Cartan algebra $\mathfrak{h}$ of C_n consists of all diagonal $2n \times 2n$ matrices. If ϵ_i denotes the projection onto the (i,i) element of the matrix, then the set of all roots Φ equals

$$\Phi = \{\pm(\epsilon_i \pm \epsilon_j); 1 \leq i < j \leq n\} \cup \{\pm 2\epsilon_i; i = 1, \dots, n\}.$$

The set of all simple roots Δ equals

$$\Delta = \{\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n\}.$$

The Chevalley basis of C_n is given by

$$\begin{aligned}
X_{\epsilon_i - \epsilon_j} &= E_{i,j} - E_{n+j,n+i}, && 1 \leq i < j \leq n, \\
X_{2\epsilon_i} &= E_{i,n+i}, && i = 1, \dots, n \\
X_{\epsilon_i + \epsilon_j} &= E_{i,n+j} - E_{j,n+i}, && 1 \leq i < j \leq n, \\
Y_\mu &= X_\mu^T, && \mu \in \Phi, \\
H_i &= E_{i,i} - E_{j,j} + E_{n+j,n+j} - E_{n+i,n+i}, && i = 1, \dots, n-1, \\
H_n &= E_{n,n} - E_{2n,2n},
\end{aligned}$$

where $E_{i,j}$ is a matrix having 1 at the place (i,j).

The algebra C_n has a very useful realisation consisting of differential operators on $\mathbb{C}[x^1, \dots, x^n]$. It is shown in [5] that the Lie algebra generated by $\{x^i\partial_{i+1}, x^{i+1}\partial_i; i = 1, \dots, n-1\} \cup \{\partial_1^2, (x^1)^2\}$ (where ∂_i is the partial differentiation in x^i, $i = 1, \dots, n$) is isomorphic to the algebra C_n via the isomorphism $\psi : C_n \to \mathrm{End}(\mathbb{C}[x^1, \dots, x^n])$, defined by

$$\begin{aligned}
\psi(X_{\epsilon_i-\epsilon_{i+1}}) &= x_{n-i}\partial_{n-i+1}\,, && i=1,\dots,n-1\,,\\
\psi(X_{-(\epsilon_i-\epsilon_{i+1})}) &= x_{n-i+1}\partial_{n-i}\,, && i=1,\dots,n-1\\
\psi(X_{2\epsilon_n}) &= -\frac{1}{2}\partial_1^2\,, &&\\
\psi(X_{-2\epsilon_n}) &= \frac{1}{2}(x^1)^2\,. &&
\end{aligned}$$

The requirement that the basis $\{\epsilon_i\}_{i=1}^n$ is an orthonormal basis defines the inner product $(\,,\,)$ on $\mathfrak{h}$. Using the standard notation

$$\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}\,,$$

the fundamental weights $\{\omega_i\}_{i=1}^n$ are defined as the basis dual to the basis $\{\check{\alpha}_i\}_{i=1}^n$, i.e. $(\omega_i, \check{\alpha}_j) = \delta_{ij}$.

There is a very close analogy between representations of $C_n = \mathfrak{sp}(2n,\mathbb{C})$ and $D_n = \mathfrak{so}(2n,\mathbb{C})$. Finite dimensional representations of C_n have their counterpart in tensor representations of D_n (i.e., representations of D_n with highest weights consisting from integers). On the other hand, there is no finite dimensional representation of C_n similar to spinor representations of D_n.

It was suggested by Kostant ([6]) that a proper analogy of spinor representations of orthogonal groups are certain infinite-dimensional representations of symplectic groups called the Segal-Shale-Weil representations. They both share the property that they are representations of the double cover of the corresponding groups. The analogy can be nicely seen using the following realisation of these representations.

Consider first the orthogonal algebras $\mathfrak{so}(2n,\mathbb{C})$ and choose a maximal isotropic subspace V of $\mathbb{C}^{2n}$, it has dimension n. Spinor representations of D_n can be realized on the Grassmann algebra $\mathbb{S} = \bigwedge^*(V) = \oplus_{i=1}^n \bigwedge^i(V)$. It decomposes into two parts $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_-$, where $\mathbb{S}_+ = \oplus_{j\in 2\mathbb{Z}} \bigwedge^j(V)$ and $\mathbb{S}_- = \oplus_{j\in 2\mathbb{Z}+1} \bigwedge^j(V)$, the so called half-spinor representations.

In the case of symplectic algebra, there is a similar construction. Consider the defining representation $\mathbb{C}^{2n}$ of $\mathfrak{so}(2n,\mathbb{C})$ with the corresponding symplectic form and choose again a maximal isotropic subspace $V \simeq \mathbb{C}^n$. The infinite dimensional space $\mathbb{C}[x^1,\dots,x^n] = \oplus_{i=1}^\infty \odot^i(\mathbb{C}^n)$ is a representation of C_n as described above (using the isomorphism ψ). It also decomposes as $\oplus_{i=1}^\infty \odot^i(\mathbb{C}^n) = \mathbb{S}_+ \oplus \mathbb{S}_-$. As in the orthogonal case, the first representation is the direct sum of even dimensional symmetric powers and the second one of the odd dimensional ones. This is a nice example of a supersymmetry, where the space of polynomials in n commuting variables (the symplectic case) has as an analogy the space of polynomials in n anticommuting variables (the orthogonal case). This analogy explains why spinor representations for C_n are infinite dimensional.

Finite dimensional representations of D_n can be all realized as spinor-tensors, i.e., as submodules of tensor products of one of two spinor representations with a tensor representation. Consequently, an analogue of these finite-dimensional representations of D_n is a class of infinite dimensional representations of C_n consisting of submodules of tensor products of one of two infinite dimensional spinor representations of C_n with a finite dimensional representation of C_n. This is a class of representations we are going to study in the paper.

3. Tensor products of finite and infinite dimensional representations

In this section we shall review some basic facts on tensor products of finite and infinite dimensional representations, details can be found in [7]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra and $\mathfrak{U}(\mathfrak{g})$ its universal enveloping algebra. Let us choose a space Δ_+ of positive roots and the corresponding decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm$ are nilpotent subalgebras.

Denote by Z the center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ and by Z' the set of all characters $\chi : Z \to \mathbb{C}$. Consider a a representation $\pi : \mathfrak{g} \to \mathrm{End}(\mathbb{V})$, where $\mathbb{V}$ is a finite or infinite dimensional complex vector space. Assume that π admits a central character $\chi : \mathfrak{h} \in Z'$, i.e., $\pi(X)v = \chi(X)v$ for all $v \in \mathbb{V}$ and $X \in Z$. This is the case, e.g., if π is irreducible.

There is a map

$$\mathfrak{h}' \to Z'$$

given by $\lambda \mapsto \chi_\lambda$, where $\chi_\lambda(u) = f_u(\lambda)$, $u \in Z$. The element f_u is the unique element of the universal enveloping algebra $\mathfrak{U}(\mathfrak{h})$ of the algebra $\mathfrak{h}$, for which

$$u - f_u \in \mathfrak{U}\mathfrak{n}_+ ,$$

where $\mathfrak{n}_+$ is the nilpotent subalgebra of $\mathfrak{g}$ and $\mathfrak{U}\mathfrak{n}_+$ is the left ideal generated by $\mathfrak{n}_+$.

Let W denote the Weyl group of the algebra $\mathfrak{g}$ and let $\tilde{\sigma}$ denote the affine action of a Weyl group element $\sigma \in W$ on the space of weights, i.e.,

$$\tilde{\sigma}(\lambda) = \sigma(\lambda + \rho) - \rho ,$$

where

$$\rho = \frac{1}{2} \sum_{\phi \in \Delta+} \phi \in \mathfrak{h}' .$$

It is well known that the map $\mathfrak{h}' \to Z'$ sending $\lambda \to \chi_\lambda$ is an epimorphism and $\chi_\lambda = \chi_\nu$ if and only if λ and ν are conjugate with respect to the action $\tilde{\sigma}$.

Let us consider a representation $\pi_\lambda : \mathfrak{g} \to \mathrm{End}(\mathbb{V}_\lambda)$ of the algebra $\mathfrak{g}$ on a finite dimensional complex vector space $\mathbb{V}_\lambda$ with the highest weight $\lambda \in \mathfrak{h}'$.

The main result needed from [7] is the following theorem.

Theorem 1. *Let $\{\mu_1, \dots, \mu_k\}$ denotes the set of all weights of the representation π_λ and*

$$Y_i = \{y \in \mathbb{V} \otimes \mathbb{V}_\lambda; uy = \chi_{\nu+\mu_i}(u)y, u \in Z\} , \quad i = 1, \dots , k .$$

Assume that the characters $\chi_{\nu+\mu_i}$ are all distinct. Then

$$\mathbb{V} \otimes \mathbb{V}_\lambda = \bigoplus_{i=1}^{k} Y_i .$$

Moreover, if Y_i is not zero, then Y_i is the maximal submodule of $\mathbb{V} \otimes \mathbb{V}_\lambda$ admitting $\chi_{\nu+\mu_i}$.

4. Completely pointed modules

In this paragraph we review some basic facts on bounded and completely pointed modules from [2]. More details can be found there. The set of bounded modules is a set of infinite dimensional representations of C_n, which is an analogue of the set of finite-dimensional representations of D_n with half-integer highest weights.

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. Let us consider an $\mathfrak{h}$-*diagonalisable* $\mathfrak{g}$-module $\mathbb{V}$, i.e., $\mathbb{V} = \bigoplus_{\nu \in wt(\mathbb{V})} \mathbb{V}_\nu$, where $wt(\mathbb{V}) \subseteq \mathfrak{h}'$ is the space of all weights of the module $\mathbb{V}$. We say that it is a module with *bounded multiplicities* if and only if there is a $k \in \mathbb{N}$ such that $\dim \mathbb{V}_\nu \leq k$ for all $\nu \in wt(\mathbb{V})$. The minimal k is called the *order* of the module. The module is called *completely pointed* provided the order of this module is 1. The bounded modules have some nice properties. For example, it is known that a simple complex Lie algebra has an infinite dimensional irreducible module with bounded multiplicities if and only if it is a either a special linear algebra or a symplectic algebra.

In the paper, we shall consider irreducible highest weight modules. For any weight $\nu \in \mathfrak{h}'$, we shall denote by $L(\nu)$ the unique irreducible module with highest weight ν. Every such module can be realized as a quotient of the Verma module with the highest weight ν. The spinor representations (or the Segal-Shale-Weil representations) belong to this class. It is easy to compute that the weight of a constant polynomial is $\nu_+ = -\frac{1}{2}\omega_n$ and the weight of the monomial x^1 is $\nu_- = \omega_{n-1} - \frac{3}{2}\omega_n$. Hence $\mathbb{S}_+ \simeq L(\nu_+)$ and $\mathbb{S}_- \simeq L(\nu_-)$. Both these representations are completely pointed (different monomials have different weights). It can be shown that the opposite claim is also true. If a highest weight module $L(\nu)$ is a completely pointed C_n-module, then $\nu = \nu_+$ or $\nu = \nu_-$ (see [1]).

The following key facts describe the structure of the tensor product of a spinor representation with a finite dimensional module (for details see [1, 2]).

Theorem 2. *Let $\nu = \sum_{i=1}^n \nu_i\omega_i$ be a dominant integral weight of C_n and let $L(\nu)$ be the corresponding irreducible finite dimensional highest weight module. Let*

$$T_\nu^+ = \Big\{\nu - \sum_{i=1}^n d_i\epsilon_i;\ d_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^n d_i \in 2\mathbb{Z},\ 0 \leq d_i \leq \nu_i,$$
$$i = 1, \dots, n-1,\ 0 \leq d_n \leq 2\nu_n + 1\Big\}$$

$$T_\nu^- = \Big\{\nu - \sum_{i=1}^n d_i\epsilon_i;\ d_j + \delta_{n,j} \in \mathbb{Z}_{\leq 0},\ \sum_{i=1}^n d_i \in 2\mathbb{Z},\ 0 \leq d_i \leq \nu_i,$$
$$i = 1, \dots, n-1,\ 0 \leq d_n \leq 2\nu_n + 1\Big\}.$$

Then

$$L(\nu_\pm) \otimes L(\nu) = \bigoplus_{\kappa \in T_\nu^\pm} L(\nu_\pm + \kappa).$$

Let us denote by $\mathbb{A}$ the following set of weights.

$$\mathbb{A} = \Big\{\sum_{i=1}^n \lambda_i\omega_i;\ \lambda_i \geq 0,\ \lambda_i \in \mathbb{Z},\ i = 1, \dots, n-1;\ \lambda_n \in \mathbb{Z} + \frac{1}{2},\ \lambda_{n-1} + 2\lambda_n + 3 > 0\Big\}$$

and

$$\mathcal{A} = \{L(\nu);\ \nu \in \mathbb{A}\}\,.$$

Theorem 3. *The following conditions are equivalent*

1. $L(\nu) \in \mathcal{A}$
2. $L(\nu)$ *is a direct summand in the decomposition of* $L(\lambda) \otimes L(-\frac{1}{2}\omega_n)$ *for some dominant integral* λ
3. L *has bounded multiplicities.*

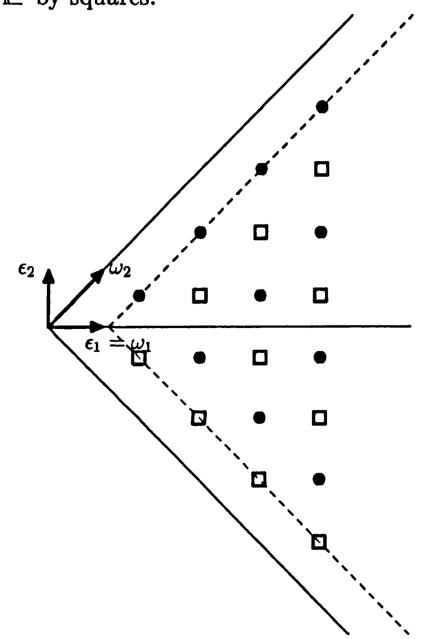
We shall write the set $\mathbb{A}$ as a union of two subsets $\mathbb{A} = \mathbb{A}_+ \oplus \mathbb{A}_-$, where

$$\mathbb{A}_+ = \left\{\lambda \in \mathbb{A};\ \lambda = \nu_+ + \sum_{i=1}^{n} m_i\epsilon_i,\ \sum_{i=1}^{n} m_i \in 2\mathbb{Z}\right\},$$

$$\mathbb{A}_- = \left\{\lambda \in \mathbb{A};\ \lambda = \nu_- + \sum_{i=1}^{n} m_i\epsilon_i,\ \sum_{i=1}^{n} m_i \in 2\mathbb{Z}\right\}.$$

Weights from the set $\mathbb{A} + \rho$ (i.e. we consider weights from $\mathbb{A}$ shifted by ρ) are all included in two Weyl chambers only – the union of the dominant Weyl chamber and its image under the reflection with respect to ϵ_n.

All that can be nicely illustrated in the case of C_2. At the next picture, we can see the corresponding two Weyl chambers in the Cartan-Stiefel diagram of C_2 below. Elements of $\mathbb{A}$ are shifted by ρ, elements of $\mathbb{A}_+$ are denoted by dots and elements of $\mathbb{A}_-$ by squares.



5. Tensor products with the defining representation

In this section, we are going to study tensor products of any bounded module with the defining representation $L(\omega_1)$. We show that these products are completely reducible and that there are no multiplicities in the decomposition. By induction, we get complete reducibility also for a product with powers of $L(\omega_1)$. These are exactly facts needed for future applications in a study of invariant differential operators on projective contact manifolds (see [3]).

Theorem 4. *Let $\lambda \in \mathbb{A}$ and let $\Pi(\omega_1) = \{\pm\epsilon_i, i = 1, \dots, n\}$ denote the set of all weights of the defining representation $L(\omega_1)$.*

Then $L(\lambda) \otimes L(\omega_1)$ is completely reducible and

$$L(\lambda) \otimes L(\omega_1) = \bigoplus_{\kappa \in \mathbb{A}_\lambda} L(\kappa)$$

where $\mathbb{A}_\lambda \subseteq \{\kappa = \lambda + \mu; \kappa \in \mathbb{A}, \mu \in \Pi(\omega_1)\}$.

Proof. Suppose that $L(\lambda)$ is a direct summand in the decomposition of some

$$L(-\frac{1}{2}\omega_n) \otimes L(\nu)$$

for some integral dominant ν (the other case can be treated in a same way). Thus $L(\lambda) \otimes L(\omega_1) \subseteq (L(-\frac{1}{2}\omega_n) \otimes L(\nu)) \otimes L(\omega_1) = L(-\frac{1}{2}\omega_n) \otimes (L(\nu) \otimes L(\omega_1))$ and therefore we know that direct summands in $L(\lambda) \otimes L(\omega_1)$ are in $\mathcal{A}$. We shall prove that the characters $\chi_{\lambda+\nu}$ and $\chi_{\lambda+\mu}$ are distinct for any $\mu, \nu \in \Pi(\omega_1)$. This is equivalent to the fact that $\lambda + \nu$ and $\lambda + \mu$ are not conjugated. This can be seen as follows: Let $\lambda + \nu \in W_1$ and $\lambda + \mu \in W_2$ where W_1, W_2 are two Weyl neighbour chambers. This chambers are described in the $\{\epsilon_i\}_{i=1}^n$ basis as follows

$$W_1 = \left\{ \sum_{i=1}^n \beta_i \epsilon_i;\ \beta_1 > \dots > \beta_n > 0 \right\},$$

$$W_2 = \left\{ \sum_{i=1}^n \beta_i \epsilon_i;\ \beta_1 > \dots > \beta_{n-1} > -\beta_n > 0 \right\}.$$

This two Weyl chambers are mapped to each other by the reflection in the plane orthogonal to ϵ_n. From the structure of the set $\Pi(\omega_1)$ it is evident that $\lambda \in \mathbb{A}_\pm$ implies that $\lambda + \tau \in \mathbb{A}_\mp$ for each $\tau \in \Pi(\omega_1)$. Thus the difference $(\lambda+\nu) - (\lambda+\mu) = \sum_{i=1}^n m_i \epsilon_i$, where $\sum_{i=1}^n m_i \in 2\mathbb{Z}$. Two elements $\lambda + \nu \in W_1$ and $\lambda + \mu \in W_2$ are conjugated (by the reflection in the plane orthogonal to ϵ_n) when their difference is $(\lambda+\nu) - (\lambda+\mu) = (2k+1)\epsilon_n$ – a contradiction.

At the picture of the Cartan-Stiefel diagram for C_2, we can see that if $\lambda + \nu$ and $\lambda + \mu$ are conjugated by the reflection in the plane orthogonal to ϵ_2 then one of them is represented by a dot and the second one by a square. But $\lambda + \nu$ and $\lambda + \mu$ for $\mu, \nu \in \Pi(\omega_1)$ ere both represented either by dots or by squares.

Now, we use the theorem on decomposition of direct product of a finite and infinite representation to conclude that $\kappa = \lambda + \nu$, for some $\nu \in \Pi(\omega_1)$. □

Consider the representation $\otimes^k L(\omega_1) \otimes \mathbb{V}$ for some $\mathbb{V} \in \mathcal{A}$. We know that $L(\omega_1) \otimes \mathbb{V}$ is completely reducible and its direct summands are in $\mathcal{A}$. Let us label these summands by integers, denoting its chosen position in the direct sum. Tensoring $L(\omega_1) \otimes \mathbb{V} = \oplus_{b_1=1}^{n_1} \mathbb{V}_{b_1}$ by $L(\omega_1)$ we obtain an direct sum again since each $\mathbb{V}_{b_1}$, $b_1 = 1, \ldots, n_1$ is in $\mathcal{A}$ and therefore decomposes when tensored by $L(\omega_1)$ due to the previous Theorem 4. We denote the b_2 therm of the direct sum decomposition of $L(\omega_1) \otimes \mathbb{V}_{b_1}$ by $\mathbb{V}_{(b_1,b_2)}$. Continuing in this process (or by induction) we obtain $L(\omega_1)^{\otimes^k} \otimes \mathbb{V} = \oplus_{b_1,\ldots,b_k} \mathbb{V}_{(b_1,\ldots,b_k)}$. Thus we have proved

Corollary. *The representation of $L(\omega_1)^{\otimes^k} \otimes \mathbb{V}$ is completely reducible and decomposes as*

$$L(\omega_1)^{\otimes^k} \otimes \mathbb{V} = \oplus_{b_1,\ldots,b_k} \mathbb{V}_{(b_1,\ldots,b_k)}\,.$$

Example. In this example we shall denote a module $L(\nu)$ with highest weight ν simply by (ν) written in the basis of fundamental weights and we shall describe the set $\mathbb{A}_\lambda$ for $\lambda = (10\ldots01 - \frac{3}{2})$.

We know that

$$(0\ldots01 - \frac{3}{2}) \otimes (10\ldots0) = (0\ldots0 - \frac{1}{2}) \oplus (10\ldots01 - \frac{3}{2})\,.$$

We also know that

$$(10\ldots0) \otimes (10\ldots0) = (20\ldots0) \oplus (010\ldots0) \oplus (0)\,.$$

We can decompose the following tensor products using the prescription of the above theorem to obtain that

$$\begin{aligned}
(0\ldots01 - \frac{3}{2}) \otimes (20\ldots0) &= (20\ldots01 - \frac{3}{2}) \oplus (10\ldots01 - \frac{3}{2}) \oplus (0\ldots01 - \frac{3}{2})\\
(0\ldots01 - \frac{3}{2}) \otimes (010\ldots0) &= (010\ldots01 - \frac{3}{2}) \oplus (10\ldots0 - \frac{1}{2})\\
(0\ldots01 - \frac{3}{2}) \otimes (0) &= (0\ldots01 - \frac{3}{2})
\end{aligned}$$

We also know that

$$(0\ldots0 - \frac{1}{2}) \otimes (10\ldots0) = (0\ldots01 - \frac{3}{2}) \oplus (1\ldots0 - \frac{1}{2})\,.$$

From this we can deduce that:

$$\begin{aligned}
(10\ldots01 - \frac{3}{2}) \otimes (10\ldots0) = (20\ldots01 - \frac{3}{2}) \oplus (0\ldots01 - \frac{3}{2}) \oplus\\
\oplus (010\ldots01 - \frac{3}{2}) \oplus (10\ldots0 - \frac{1}{2})\,.
\end{aligned}$$

Hence we can see here that in this case, the set $\mathbb{A}_\lambda$ is given by

$$\mathbb{A}_\lambda = \{\kappa = \lambda + \mu;\ \kappa \in \mathbb{A},\ \mu \in \Pi(\omega_1)\}\,.$$

Many other examples lead to the same result, hence we conjecture that the same fact will be true in general.

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