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LOCAL CONVEXIFIABILITY OF SOME RIGID DOMAINS

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ABSTRACT. The main obstruction for constructing holomorphic reproducing kernels of Cauchy type on weakly pseudoconvex domains is the Kohn-Nirenberg phenomenon, i.e., nonexistence of supporting functions and local nonconvexifiability. It is well known that "generic" weakly pseudoconvex domains in $\mathbb{C}^2$, domains of type four, do admit supporting functions at every boundary point, but it is still an open question whether such domains are also locally convexifiable. In this paper we prove this under the additional assumption of rigidity of the domain.

1. INTRODUCTION

Let $D \subseteq \mathbb{C}^2$ be a domain with real analytic boundary, and let $p \in \partial D$. For a neighbourhood U of p let $\phi \in C^\omega(U)$ be a real valued function such that

$$\partial D \cap U = \{z \in U \mid \phi(z) = 0\}$$

and grad $\phi \neq 0$ in U. Recall that ∂D is pseudoconvex in U if for all $q \in \partial D \cap U$ the Levi form

$$L_q(\zeta) = \sum_{i,j=1}^{2} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(q)\zeta_i \bar{\zeta}_j \tag{1.1}$$

is nonnegative on the complex tangent space to ∂D at q, i.e., on complex vectors $\zeta = (\zeta_1, \zeta_2)$ satisfying $\sum_{i=1}^2 \frac{\partial \phi}{\partial z_i}(q)\zeta_i = 0$. When ∂D is strictly pseudoconvex (the Levi form is positive definite at each point), fundamental work of Henkin and Ramirez provides a construction of a holomorphic reproducing kernel, a direct analogy of Cauchy's integral kernel from one complex variable. The construction is

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based on holomorphic supporting functions $h_q(z)$, which are defined in a neighbourhood of each $q \in \partial D$ and whose zero set intersects $\bar{D}$ only at q. The reciprocal of a supporting function provides an analogy to the all-important function $\frac{1}{z-q}$ from one variable. Local analysis on strongly pseudoconvex domains is further simplified by Narasimhan's lemma, which gives local holomorphic coordinates in which ∂D is strongly convex, and suporting functions can be taken linear.

One of active areas of current research in several complex variables concentrates around the attempt to extend Henkin's construction to (at least some) weakly pseudoconvex domains (see [DF], [DM], [M] for related geometric results).

The main obstacle for this was discovered by Kohn and Nirenberg in [KN]. Their example shows that a weakly pseudoconvex domain need not be locally convexifiable and need not admit supporting functions. As a consequence, it became an important problem to find a characterization of those domains for which the Kohn-Nirenberg phenomenon does not occur.

On finite type domains local convexifiability is a much stronger property than mere existence of supporting functions in all neighbouring points. Explicit conditions for local convexifiability were derived in [K1], [K2]. They are applicable on domains of type six and higher.

When studying the Kohn-Nirenberg phenomenon, it is natural to consider first domains of type four, which are in some sense "generic" among weakly pseudoconvex domains and closest to strong pseudoconvexity. It is well known that the Kohn-Nirenberg phenomenon, in the sense of nonexistence of holomorphic support functions does not occur on such domains ([FS], [K3]). On the other hand, the answer to the related question, whether all such domains are locally convexifiable, is still not known. The conjectured answer is affirmative. The main result of this paper is the following theorem, which adds further support to this conjecture.

Theorem 1.1. *Let D be a rigid domain in $\mathbb{C}^2$ with real analytic boundary and $p \in \partial D$ be a point of finite type four. Then D is locally convexifiable near p.*

2. Rigid domains of type four

The study of pseudoconvex domains for which the Levi form degenerates at some points leads to the following definition of type of a boundary point, which measures the maximal order of contact of complex curves with ∂D at p. The original definition of finite type is due to J. J. Kohn ([K]). Here we give an equivalent definition, introduced by J. D'Angelo ([D]).

For a smooth function f defined in a neighbourhood of 0 in $\mathbb{C}$ let $\nu(f)$ denote the order of vanishing of f at 0.

Definition 2.1. p is a point of finite type, if there exists an integer k such that

$$\nu(\phi \circ \gamma) \leq k$$

for all holomorphic maps γ from a neighbourhood of $0 \in \mathbb{C}$ into $\mathbb{C}^2$, satisfying $\gamma(0) = p$ and $\gamma'(0) \neq 0$.

The smallest such integer is called the type of p.

We will assume that $p \in \partial D$ is a point of type four. We will describe the boundary of D in its neighbourhood using local coordinates (z, w), where $z = x+iy$, $w = u+iv$. These coordinates are assumed to be centered at p and the direction of the positive v-axis is the direction of the inner normal to ∂D at p. Hence the real tangent space to ∂D at p is given by $v = 0$. By the implicit function theorem we can describe ∂D near p as a graph of a function $v = F(z, \bar{z}, u)$.

Recall that rigidity means that the above coordinates can be chosen in such a way that F does not depend on u. It may be rather complicated to decide whether a given domain is rigid, but once we find coordinates with this property, the changes of variables which we consider below will not change this form. So we assume that ∂D is desribed by

$$v = \Psi(z, \bar{z})$$

for a real valued real analytic function Ψ. Using $\phi(z, w) = v - \Psi(z, \bar{z})$ in (1.1), the complex tangent space at a point $q \in \partial D$, defined by $\sum_{i=1}^{2} \frac{\partial \phi}{\partial z_i}(q)\zeta_i = 0$, is spanned by the vector $(1, -\phi_z)$. Hence, there is only one nonzero term in (1.1), and the Levi form on ∂D is given by $\frac{1}{4}\Delta\Psi$. Consequently, ∂D is pseudoconvex if and only if Ψ is subharmonic.

Next we consider the Taylor expansion of Ψ at p. By our choice of coordinates, it starts with second order terms

$$\Psi(z, \bar{z}) = a_{11}|z|^2 + 2\operatorname{Re}(a_{20}z^2) + 2\operatorname{Re}(a_{30}z^3) + 2\operatorname{Re}(a_{21}z^2\bar{z}) + O(|z|^4)\,.$$

By assumption, $L(p) = \frac{1}{4}\Delta\Psi(p) = a_{11} = 0$, and pseudoconvexity implies also $a_{21} = 0$. The harmonic terms are absorbed into v in a standard way, by replacing w by $w + 2ia_{20}z^2 + 2ia_{30}z^3$. Ψ then starts with fourth order terms. After a normalization by a linear change in the z variable we have

$$\Psi(z, \bar{z}) = |z|^4 + 2a\operatorname{Re}\ z^2|z|^2 + 2b\operatorname{Re}\ z^4 + O(|z|^4)\,. \tag{2.1}$$

Evaluating the Levi form we get a necessary condition for pseudoconvexity, $a \leq \frac{2}{3}$. In [K3] we considered the case $a < \frac{2}{3}$ (without assuming rigidity), and proved that in this case D is locally convexifiable near p. Here we consider the limiting case, $a = \frac{2}{3}$.

The leading fourth order polynomial can be transformed by another linear change in z and a suitable change of the harmonic term into $(\frac{z+\bar{z}}{2})^4 = x^4$.

Hence we assume that

$$\Psi(x, y) = x^4 + \sum_{i+j=5}^{\infty} b_{ij}x^i y^j\,. \tag{2.2}$$

We will consider only transformations which preserve this form, therefore it is enough to consider the 2-dimensional surface $M = \partial D \cap \{u = 0\}$, and identify ∂D with $M \times \mathbb{R}$. Our aim is to show that there exist new coordinates in which the defining function is convex near p.

3. Proof of Theorem 1.1

Pseudoconvexity of ∂D is equivalent to $\Psi_{z\bar{z}} \geq 0$. On the other hand, it is easily verified that in terms of complex variables Ψ is convex if and only if $|\Psi_{zz}| \leq \Psi_{z\bar{z}}$. Note that the Laplacian of the leading term vanishes identically on the y-axis. We split Ψ into two parts, $\Psi = \Psi_1 + \Psi_2$, where

$$\Psi_1(x,y) = \frac{1}{2}x^4 + \sum_{i=0}^{3} x^i \sum_{j=5-i}^{\infty} b_{ij}y^j ,$$

$$\Psi_2(x,y) = \frac{1}{2}x^4 + o(x^4) .$$

The first part includes all terms which are relevant for the sign of the Hessian of Ψ along the y-axis. By a suitable change of variables we can achieve that b_{1j} and b_{3j} vanish and b_{0j} and b_{2j} have the same sign. More precisely, in order to attain convexity of Ψ_1 we will make Ψ_{zz} and Ψ_{xxx} vanish along the y-axis to a sufficiently high order.

Lemma 3.1. *For every $n \geq 4$ there exist local holomorphic coordinates in which Ψ has form (2.2) and satisfies*

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} \quad \Psi_{zz}(0,y) = 0 + o(y^{n-2}) , \\ &\text{(ii)} \quad \Psi_{xxx}(0,y) = 0 + o(y^{n-3}) . \end{aligned}$$

Proof. Let us consider holomorphic transformations of the form

$$\text{(3.2)} \qquad w^* = w + \sum_{i=5}^{n} f_i z^i, \quad z^* = z + \sum_{i=2}^{n-3} g_i z^i.$$

Using induction we will determine the coefficients f_i, $5 \leq f_i \leq n$ and g_i, $2 \leq g_i \leq n-3$, so that after the polynomial transformation conditions (i) and (ii) are satisfied. Let Ψ^* be the defining function in coordinates z^*, w^*. The general change of variables formula (cf. [CM]):

$$\begin{aligned} &\Psi^*(z + g(z, u + i\Psi(z,\bar{z},v)), \overline{z + g(z, u + i\Psi(z,\bar{z},v))}, v + \text{Im } f(z, u + i\Psi(z,\bar{z},v)) \\ \text{(3.3)} \qquad &= \Psi(z,\bar{z},u) + \text{Re } f(z, u + \text{Im } f(z, u + i\Psi(z,\bar{z},v)) \end{aligned}$$

now reduces to

$$\text{(3.4)} \qquad \Psi^*(x^*,y^*) = \Psi^*(x + \text{Re } g(x,y), y + \text{Im } g(x,y)) = \Psi(x,y) + \text{Im } f(x,y) .$$

It follows from (3.4) that the coefficients of Ψ^* of order k depend only on f_i for $i \leq k$ and g_i for $i \leq k-3$. Also, it follows that the Taylor expansion of Ψ^* has also form (2.2), and we can equate the coefficients of $x^i y^j$, $i + j = k$ in (3.4). We get

$$\text{(3.5)} \qquad b^*_{ij} + 4(\text{Re } g_{k-3}z^{k-3})_{i-3,j} + A = b_{ij} + (\text{Im } f_k z^k)_{ij} ,$$

where $(h)_{ij}$ denotes the coefficient of $x^i y^j$ in a polynomial h, and A is a number depending on g_i with $i < k - 3$ and on b^*_{ij} with $i + j < k$. Now we proceed by induction.
1. For $k = 4$ we take the identity transformation.
2. Assume that for some integer $k > 4$ the coefficients f_i, $i < k$ and g_i, $i < k - 3$ are already determined, so that after the polynomial transformation the conditions are satisfied for $n = k - 1$. Using (3.5), with A being some already determined number, we will determine f_k and g_{k-3}.

First we will use (3.5) for (i, j) equal to $(0, n)$ and $(2, n - 2)$. The following formulas hold for n even:

$$b^*_{0n} = b_{0n} + (-1)^{\frac{n}{2}} \operatorname{Im} f_n - A$$
$$b^*_{2,n-2} = b_{2,n-2} + (-1)^{\frac{n}{2}+1} \binom{n}{2} \operatorname{Im} f_n - A.$$

For n odd we have to replace Im f by Re f in both rows, and the sign by $(-1)^{\frac{n-1}{2}}$ in the first row and by $(-1)^{\frac{n+1}{2}}$ in the second row. We want $(b^*_{0n}, b^*_{2,n-2})$ to be a multiple of $(2, n(n-1))$ in order to get Re $\Psi^*_{z^* z^*}(0, y) = 0 + o(|z|^{n-2})$. Considering Im f_n as a variable, the right hand side forms a line in $\mathbb{R}^2$ in direction $(1, -\binom{n}{2})$. The same direction is obtain for n odd. It follows that there exists Im f_n such that $(b^*_{0n}, b^*_{2,n-2}) = c(2, n(n-1))$ for some $c \in \mathbb{R}$.
For the other two terms involved in (i) and (ii) we get for n even

$$b^*_{1,n-1} = b_{1,n-1} + (-1)^{\frac{n}{2}+1} n \operatorname{Re} f_n - A$$
$$b^*_{3,n-3} = 4 \operatorname{Re} g_{n-3} + b_{3,n-3} + (-1)^{\frac{n}{2}} \binom{n}{3} \operatorname{Re} f_n - A.$$

The first equation gives Re f_n, the second one then Re g_{n-3}. Im g_{n-3} may be arbitrary. For n odd the formulas are modified as before. □

This sequence of changes of coordinates will either reveal an integer for which $b_{0n} \neq 0$ in the coordinates given by Lemma 3.1, or we always get $b_{0n} = 0$. In the first case, let s be the smallest such integer. Lemma 3.1 gives coordinates in which

$$\Psi(x, y) = x^4 + o(x^4) + b_{2,s-2} x^2 y^{s-2} + b_{0s} y^s + o(|z|^n). \tag{3.6}$$

Here $b_{2,s-2} = \binom{s}{2} b_{0s} > 0$, from condition (ii) and positivity of $\Delta\Psi(0, y)$. In the second case we have

Lemma 3.2. *If $b_{0n} = 0$ in the coordinates from Lemma 3.1 for all n, then there are local holomorphic coordinates in which*

$$\Psi(x, y) = x^4 + o(x^4). \tag{3.7}$$

Proof. First we notice that M has to contain a curve of weakly pseudoconvex points passing through p. Indeed, if there were no such curve, then by the curve selection lemma applied to $\Delta\Psi$, p is an isolated weakly pseudoconvex point in M. By Lojasiewicz inequality $\Delta\Psi > \epsilon|z|^m$ for some $\epsilon > 0$ and $m \in \mathbb{N}$. Taking $m = n$ in Lemma 3.1 we have $b_{0n} = b_{2,n-2} = 0$, hence $\Delta\Psi(0, y) = o(y^m)$, a contradiction.

Let γ be such a curve and let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be its coordinates in the z-plane and v-axis. First we transform coordinates so that γ lies above the y-axis, i.e., Re $\gamma_1 = 0$. This is achieved by a transformation $z^* = g(z)$, $w^* = w$, where g is the holomorphic extension of a real analytic map from the image of γ_1 to the y-axis. In the new coordinates we omit stars and parametrize γ by y, so that $\gamma(y) = (iy, \gamma_2(y))$.

Now we want to extend the function $\gamma_2(y)$ into a harmonic function $\tilde{h}(x, y)$ which satisfies $\tilde{h}_x(0, y) = \Psi_x(0, y)$, in addition to $\tilde{h}(0, y) = \gamma_2(y)$. This is an initial value problem for the Laplace equation, hence there is a unique solution $\tilde{h}$ in some neighbourhood of 0. Let h be a holomorphic function such that Im $h = \tilde{h}$. After the change of coordinates $z^* = z$, $w^* = w - h(z)$ we get

$$\Psi^* = \Psi - \operatorname{Im}\, h = \Psi - \tilde{h}\,.$$

That gives, after dropping stars, $\Psi(0, y) = 0$ and $\Psi_x(0, y) = 0$. Since $L \geq 0$ and L vanishes at $(0, y)$, we have $L(0, y) = \Psi_{xx}(0, y) = 0$ and $\Psi_{xxx}(0, y) = L_x(0, y) = 0$. In other words, Ψ has form (3.7). □

Now we transform the second part of Ψ into a convex function, not changing the form given by (3.6) and (3.7), and complete the proof of Theorem 1.1.

Lemma 3.3. *There exist local holomorphic coordinates such that Ψ satisfies* (3.6) *or* (3.7)*, and moreover, $b_{41} = 0$ and $b_{42} = 1$. In such coordinates, Ψ is convex in a neighbourhood of p.*

Proof. Consider first the case when Ψ has form (3.7). Let us take a change of coordinates whose inverse is given by

$$z = z^* + i(\frac{1}{4}b_{41})(z^*)^2, \quad w = w^*.$$

After substituting for x and y in

$$v^* = v = x^4 + b_{41}x^4y + o(x^4y, x^5)\,,$$

the 5-th order term cancels while form (3.7) is preserved. In the new coordinates, with stars omited, we have

$$v = x^4 + b_{42}x^4y^2 + o(x^4y^2, x^5)\,.$$

In the same way we use these coordinates to define new ones by

$$z = z^* + i\frac{1}{4}(b_{42} - 1)(z^*)^3, \quad w = w^*\,.$$

That gives $b^*_{42} = 1$.

If Ψ has form (3.6), the same argument applies. We only need to notice that the leading n-th order term $(c_1x^2y^{n-2}+c_2y^n)$ is not affected by the two changes of variables.

In order to prove convexity of Ψ, we first consider (3.6) and split Ψ into two parts,

$$\Psi_1(x,y)=\frac{1}{2}x^4+x^4y^2+o(x^4y^2,x^5)$$
$$\Psi_2(x,y)=\frac{1}{2}x^4+c_1x^2y^{n-2}+c_2y^n+o(|z|^n)\,,$$

where c_1, c_2 are positive constants. It is straightforward to verify that the Hessian of both is positive semidefinite in a neighbourhood of p. For (3.7) the same calculation applies. □

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