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MULTISYMPLECTIC STRUCTURES OF DEGREE THREE OF PRODUCT TYPE ON 6-DIMENSIONAL MANIFOLDS

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ABSTRACT. Let ω be a multisymplectic 3-form on 6-dimensional manifold of constant type at each point, which induces product structure on manifold. There are geometric structures related with the form ω. Properties of the structures as integrability conditions, related metrics, connections etc. are studied. Some natural examples are presented.

1. ALGEBRAIC PROPERTIES OF REAL 3-FORMS IN DIMENSION SIX

Consider a 6-dimensional real vector space V. Multisymplectic 3-form on V is a 3-form ω such that the associated homomorphism

$$\kappa : V \to \Lambda^2 V^*, \quad \kappa v = \iota_v \omega = \omega(v, \cdot, \cdot)$$

is injective.

Denote $\Lambda^3_{ms} V^*$ the subset of $\Lambda^3 V^*$ consisting of all multisymplectic forms. $\Lambda^3_{ms} V^*$ is an open subset. The natural action of $GL(V)$ on $\Lambda^3 V^*$ preserves $\Lambda^3_{ms} V^*$. Under this action $\Lambda^3_{ms} V^*$ decomposes into three orbits (types). Two of them are open orbits, the third one is a submanifold of codimension 1.

As representatives of these orbits we can take the following 3-forms. (We choose a basis $e_1, \dots, e_6$ of V, and we denote $\alpha_1, \dots, \alpha_6$ the corresponding dual basis.)

1. $\omega_+ = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,
2. $\omega_- = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,
3. $\omega_0 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

The open set containing the form ω_+ (ω_-) we shall denote U_+ (U_-), and the codimension 1 submanifold containing ω_0 we shall denote U_0. There is also another characterization of these orbits. Namely, for any 3-form ω we define

$$\Delta^2(\omega) = \{v \in V; (\iota_v \omega) \wedge (\iota_v \omega) = 0\}.$$

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In other words, the subset $\Delta^2(\omega) \subset V$ consists of all vectors $v \in V$ such that the 2-form $\iota_v \omega$ is decomposable.

A computation shows that for our representatives we get:

$$\Delta^2(\omega_+) = [e_1, e_2, e_3] \cup [e_4, e_5, e_6],$$
$$\Delta^2(\omega_-) = \{0\},$$
$$\Delta^2(\omega_0) = [e_1, e_2, e_3].$$

In general case for a form $\omega \in \Lambda^3_{ms} V^*$ we have

1. $\omega \in U_+$ if and only if $\Delta^2(\omega)$ consists of the union of two transversal 3-dimensional subspaces.
2. $\omega \in U_-$ if and only if $\Delta^2(\omega) = \{0\}$.
3. $\omega \in U_0$ if and only if $\Delta^2(\omega)$ is a 3-dimensional subspace.

Choose a nonzero 6-form ϑ on V. There exists a unique endomorphism $Q : V \to V$ such that

$$(\iota_v \omega) \wedge \omega = \iota_{Q(v)} \vartheta .$$

We shall now study the properties of the endomorphism Q.

Case(ω_+). Let us assume that $\omega \in U_+$. Then $\Delta^2(\omega) = V_3' \cup V_3''$, where V_3' and V_3'' are transversal 3-dimensional subspaces and we have

$$V = V_3' \oplus V_3''.$$

The linear map Q preserves the subspaces V_3' and V_3'' and satisfies

$$Q^2 = \lambda^2 \cdot \mathrm{Id}, \quad \lambda \neq 0$$

and defines a product structure $P_\omega = \lambda^{-1} Q$ on V.

Case (ω_-). Let us assume that $\omega \in U_-$. Then $\Delta^2(\omega) = \{0\}$, and the linear map Q satisfies

$$Q^2 = -\lambda^2 \cdot \mathrm{Id}, \quad \lambda \neq 0$$

and defines a complex structure $J_\omega = \lambda^{-1} Q$ on V.

Case (ω_0). Let us assume that $\omega \in U_0$. For simplicity we denote $V_0 = \Delta^2(\omega_3)$. The linear map Q satisfies

$$Q^2 = 0, Q(V) = V_0, Q(V_0) = 0$$

and defines a tangent structure $T_\omega = Q$ on V. $(T_\omega^2 = 0, \operatorname{Ker} T_\omega = \operatorname{Im} T_\omega)$

The structures P_ω, J_ω are unique up to sign, the structure T_ω up to a nonzero multiple.

We shall introduce the group $\mathcal{G}(\omega_i)$ consisting of all automorphisms $\varphi \in \mathrm{Gl}(V)$ preserving the form ω_i, i.e. such that for every vectors $v, v', v'' \in V$ there is

$$\omega_i(\varphi v, \varphi v', \varphi v'') = \omega_i(v, v', v'') .$$

It is clear that $\mathcal{G}(\omega_i)$ is a closed subgroup of $\mathrm{Gl}\,(V) \simeq \mathrm{Gl}\,(6,\mathbf{R})$. Consequently, $\mathcal{G}(\omega_i)$ is a Lie group. We have

$$\begin{aligned}\mathcal{G}(\omega_+) &\simeq \mathrm{Sl}\,(3,\mathbf{R}) \times \mathrm{Sl}\,(3,\mathbf{R})\\ \mathcal{G}(\omega_-) &\simeq \mathrm{Sl}\,(3,\mathbf{C})\\ \mathcal{G}(\omega_0) &\simeq \mathrm{Gl}\,(3,\mathbf{R}) \times_{sd} \mathbf{R}^8\end{aligned}$$

Remark 1.1. There is possible comparison with Hitchin's results ([5]).

2. Multisymplectic 3-forms on manifolds.

Definition 2.1. A multisymplectic 3-form Ω on a six-dimensional manifold M is a section of $\Lambda^3 T^*M$ such that its restriction to the tangent space T_xM is multisymplectic for any $x \in M$, and is of type ω_ϵ at $x \in M$, $\epsilon = \pm, 0$, if the restriction to T_xM is of type ω_ϵ.

The couple (M, Ω), M a six-dimensional manifold and Ω a multisymplectic 3-form of type ϵ, is called almost multisymplectic structure of type ϵ on M. Almost multisymplectic structure of type ϵ is called integrable (or multisymplectic structure)on M if at any point $x \in M$ it is possible to choose a cordinate neigborhood W with coordinates $(x_1, ..., x_6)$ such that

1. For $\epsilon = +$

$$\Omega = dx_1 \wedge dx_2 \wedge dx_3 + dx_4 \wedge dx_5 \wedge dx_6$$

2. For $\epsilon = -$

$$\Omega = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_4 \wedge dx_6 - dx_3 \wedge dx_5 \wedge dx_6$$

3. For $\epsilon = 0$

$$\Omega = dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_5 \wedge dx_6$$

on W.

There are examples that a multisymplectic form on M can change its type, as follows:

Example ([7]). Consider on $\mathbf{R}^6$ the form

$$\begin{aligned}\sigma = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_4 \wedge dx_6\\ + \sin(x_3 + x_4) dx_3 \wedge dx_5 \wedge dx_6 + \sin(x_3 + x_4) dx_4 \wedge dx_5 \wedge dx_6\,,\end{aligned}$$

a 3-form on $\mathbf{R}^6$. σ is of type 3 on the submanifold given by the equation $x_3 + x_4 = k\pi$, $k \in \mathbb{N}$. If $x_3 + x_4 \in (k\pi, (k+1)\pi)$, k even, then σ is of type 1 and if $x_3 + x_4 \in (k\pi, (k+1)\pi)$, k odd, then σ is of type 2. σ is closed and invariant under the action of the group $(2\pi\mathbb{Z})^6$ and we can factor σ to get a form changing the type on $\mathbb{R}^6/(2\pi\mathbb{Z})^6$, which is the 6-dimensional torus, that is σ is closed on a compact manifold.

2.1. **3-forms of type ω_+.(Product type).** Let us restrict ourselves to the product type only, other types were studied by N. Hitchin [5], M. Panák and J. Vanžura [7], and J. Vanžura [8].

Let us recall first basic facts from the theory of almost paracomplex structures on a manifold M. An almost paracomplex structure is a special case of almost product structure (there are only two distributions in the decomposition, but of the same dimension), the theory of almost product structures with with the corresponding restriction condition from [9] will be used in the following text.

Definition 2.2. Almost paracomplex structure on a manifold M of dimension $2n$ is a smooth tensor field P of the type (1,1) on M such that $P^2 = Id$ and there are two distributions (D_+, D_-) on M of dimension n such that

$$D_\pm(x) = \{X \in T_xM \mid P(x)(X) = \pm X\}.$$

Thus the tangent bundle is a direct sum

$$TM = D_+ \oplus D_-$$

and we have projectors

$$\pi_\pm = \frac{1}{2}(Id \pm P)$$

on $D_\pm$ with $\pi_\pm^2 = \pi_\pm$.

Almost paracomplex structure is called integrable if both distributions are integrable, it means that for any two vector fields

$$X, Y \in \mathcal{X}(M),\ X, Y \in \Gamma(D_\pm) \Rightarrow [X, Y] \in \Gamma(D_\pm)$$

or at any point of M there are local coordinates $(x_1, \dots, x_{2n})$ on M such that $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ is basis of sections of D_+ and $\{\frac{\partial}{\partial x_{n+1}}, \dots, \frac{\partial}{\partial x_{2n}}\}$ basis of sections of D_-.

Let A, B are tensor fields of type $(1,1)$ on a manifold M. The Nijenhuis bracket of the fields A, B is a tensor field $[A, B]$ of type $(1,2)$ on M

$$\begin{aligned}[A, B](X, Y) = {} & [AX, BY] + [BX, AY] + AB[X, Y] + BA[X, Y] \\ & - A[BX, Y] - B[AX, Y] - A[X, BY] - B[X, AY].\end{aligned}$$

It is antisymmetric, namely we have $[A, B](Y, X) = -[A, B](X, Y)$ and $[A, I] = 0$ for $I = \mathrm{Id}$.

Theorem 2.3 ([9]). *Almost paracomplex structure P on M is integrable if and only if the Nijenhuis bracket $[P, P] = 0$.*

For our multisymplectic structures of product type on six-dimensional manifolds we have the following theorem:

Theorem 2.4. *Let $\Omega \in \Omega^3(M)$ be an almost multisymplectic 3-structure on six-dimensional orientable manifold M of type ω_+. Thus M has an almost paracomplex structure P with distributions (D_+, D_-) and there are fixed volume forms $\beta_\pm$ on $D_\pm$ determined by Ω and vice versa.*

Moreover the structure Ω is integrable iff the almost paracomplex structure is integrable (the Nijenhuis bracket $[P, P] = 0$) and the form Ω is closed ($d\Omega = 0$).

Proof. If the structure is integrable the implication is trivial. The second implication we get as follows: From the condition $[P, P] = 0$ it follows that D_i are integrable, namely there exist local coordinates $(x_1, x_2, \ldots, x_6)$ such that $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\}$ is basis of sections of D_+ and $\{\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}\}$ is basis of sections of D_-. In these coordinates we have

$$\Omega = f(x_1, \ldots, x_6) dx_1 \wedge dx_2 \wedge dx_3 + g(x_1, \ldots, x_6) dx_4 \wedge dx_5 \wedge dx_6$$

and the condition $d\Omega = 0$ is equivalent to the conditions

$$\frac{\partial f}{\partial x_4} = \frac{\partial f}{\partial x_5} = \frac{\partial f}{\partial x_6} = 0$$

and

$$\frac{\partial g}{\partial x_1} = \frac{\partial g}{\partial x_2} = \frac{\partial g}{\partial x_3} = 0$$

and it is possible to change coordinates to new coordinates $(y_1, \ldots, y_6)$ in such a way that

$$\Omega = dy_1 \wedge dy_2 \wedge dy_3 + dy_4 \wedge dy_5 \wedge dy_6 . \qquad \square$$

Remark 2.5. It is possible to generalize the notion of integrability in two directions: First supposing that P is integrable but Ω is not closed and second that P is not integrable but Ω is closed. The study of properties and examples of these semi-integrable structures will be postponed to some of the forthcoming papers

Definition 2.6. Let (M, Ω) be a multisymplectic 3-structure of product type and P the corresponding product structure. The Nijenhuis bracket tensor $H = \frac{1}{8}[P, P]$ is called the torsion tensor of the structure.

2.2. **Examples.**

Example 1. Let $M = M_1 \times M_2$ be the product of two oriented 3-dimensional manifolds (M_i, α_i) endowed with volume forms α_i. Denote

$$\pi_i : M_1 \times M_2 \to M_i$$

the projections. Thus

$$\Omega = \pi_1^* \alpha_1 + \pi_2^* \alpha_2$$

and Ω is an integrable multisymplectic 3-structure on M

Example 2. Let M be an oriented 3-dimensional manifold endowed with volume form α, suppose θ be a linear connection on M. On the tangent bundle TM we have two distributions, say H is the horizontal and V the vertical distribution. For any $v \in T_x M$ we have

$$T_v(T_x M) = H_v + V_v, \; V_v \equiv T_x(M) .$$

The connection θ defines a multisymplectic 3-structure of type ω_+ on M if we set

$$\omega = \pi^* \alpha + t_v \alpha_x$$

with $t_v \alpha_x$ being the canonical shift of α_x in the fibre $T_x(M)$.

Example 3. Let M be a Riemannian oriented 3-dimensional manifold. Suppose θ be an invariant connection on M. On the special orthogonal frame bundle

$$B = B_{SO(3)} \to M$$

we have two distributions

$$T_r(B) = H_r + V_r,\ H_r \equiv T_x(M).$$

The connection θ defines the 3-structure of type ω_+ on M composed from the volume form on M lifted to TB and invariant volume form on the fibre.

Example 4. Further examples we get as induced structures on special hypersurfaces of seven-dimensional manifolds endowed with multisymplectic 3-forms. Following the classification of types from [1] the structure $\Omega = F^*\tilde{\Omega}$ of product type we get on some hypersurfaces $F : N^6 \subset (M^7, \tilde{\Omega})$ in case that $\tilde{\Omega}$ is of type 1,2, or 5.

2.3. **Existence problem.** Let us restrict ourselves to the case of orientable manifolds. Let (M, Ω) be the multisymplectic 3-structure and (D', D'') the corresponding distributions on M.

The restrictions $\Omega|D'$ and $\Omega|D''$ are everywhere nonzero 3-forms on D' and D'', respectively. This means that both the distributions D' and D'' are orientable. We get easily:

Theorem 2.7. *Let M be an orientable manifold. Then there exists on M a multisymplectic form Ω of type ω_+ if and only if $TM = D_+ \oplus D_-$, where D_+ and D_- are orientable 3-dimensional distributions.*

Proof. If the multisymplectic form Ω_+ exists then we have the almost product structure P on M and two distributions D_+ and D_- such that restrictions of Ω on $D_\pm$ are nonzero three forms giving the orientations of $D_\pm$. On the contrary suppose that D_+ and D_- are orientable and Θ_+, resp. Θ_- the corresponding forms of orientation, defined on M. Thus $\Omega = \Theta_+ + \Theta_-$ is the desired form. □

3. Further properties

3.1. **Related metrics and connections.** Let (M^6, Ω) be an oriented connected manifold endowed with a 3-form Ω of the product type. Let one of the almost product structures P can be chosen. Then we have an ordered couple of distributions (D_+, D_-) and decomposition

$$\Omega = \Theta_+ + \Theta_-$$

of Ω. Moreover we can define conjugate form

$$\bar{\Omega} = \Theta_+ - \Theta_-$$

and the volume form

$$\Xi = \frac{1}{2}(\Omega \wedge \bar{\Omega}).$$

Definition 3.1. The riemannian metric g on (M, Ω) is called adapted metric if the distributions D_1 and D_2 are orthogonal with respect to g, the forms Θ_i are volume forms on D_i with respect to $g|D_i$.

Theorem 3.2. *On any multisymplectic 3-structure (M, Ω) of the product type there exists an adapted riemannian metric.*

Definition 3.3. Affine connection ∇ on (M, Ω) of product type is called adapted connection if $\nabla\Omega = 0$.

Theorem 3.4. *Let ∇ be an adapted connection on (M, Ω) of the product type. Then for the related almost product structure P with distributions (D_+, D_-) we have:*

1. *The distributions D_+, D_- are parallel with respect to ∇ and $\nabla P = 0$.*
2. *There exists an adapted torsionless connection on (M, Ω) if and only if the multisymplectic structure is integrable.*

Proof. The distributions D_+, D_- are defined by the condition

$$Y \in T_xM,\ Y \in D_+(x) \cup D_-(x) \Leftrightarrow i_Y\Omega \wedge i_Y\Omega = 0\,.$$

Remark, that at any $x \in M$ we have $D_+(x) \cup D_-(x) = \Delta^2(\Omega(x))$, it follows from the discussions of types in section 1.

Suppose $\nabla\Omega = 0$, and take e.g. vector field $Y \in \Gamma(D_+)$. We would like to prove

$$\nabla_X Y \in \Gamma(D_+)$$

for any vector field $X \in \Gamma(TM)$.

By a direct computation it is possible to prove

$$\nabla_X(i_Y\Omega) = i_{\nabla_X Y}\Omega\,.$$

Because $Y \in \Gamma(D_+) \subset \Gamma(\Delta)$ we have

$$i_Y\Omega \wedge i_Y\Omega = 0$$

and for any X

$$(i_{\nabla_X Y}\Omega) \wedge i_Y\Omega = 0\,.$$

Applying $i_{\nabla_X Y}$ to the equation we get

$$(i_{\nabla_X Y}\Omega) \wedge (i_{\nabla_X Y} i_Y\Omega) = 0$$

and there is one form α such that

$$(i_{\nabla_X Y}\Omega) = \alpha \wedge (i_{\nabla_X Y} i_Y\Omega)\,.$$

From this condition it follows immediately

$$(i_{\nabla_X Y}\Omega) \wedge (i_{\nabla_X Y}\Omega) = 0$$

and we have $\nabla_X Y \in \Gamma(D_+)$. The same result we get for $\Gamma(D_-)$.

Let us prove that $\nabla P = 0$. It is equivalent to

$$\nabla_X(PY) = P(\nabla_X Y)$$

We can decompose $Y = Y_+ + Y_-$ with $Y_\pm \in \Gamma(D_\pm)$, and thus

$$\nabla_X(PY) = \nabla_X(Y_+ - Y_-) = \nabla_X Y_+ - \nabla_X Y_- = P(\nabla_X Y)$$

because the distributions are invariant.

If the multisymplectic structure is integrable, then Ω is parallel with respect to flat connection.

On the contrary, suppose that there is connection ∇ without torsion such that $\nabla\Omega = 0$, then the form Ω is closed. Moreover we have $\nabla P = 0$ and the distributions are parallel with respect to connection with zero torsion, which implies ([9]) that $H = 0$ and the structure is integrable. □

Let me mention a few facts from ([9]) used above.

Asociated with any almost product (almost paracomplex) structure P are certain affine connections on M called special connections ([9]).

Definition 3.5. Affine connection ∇ on a manifold with an almost product structure (M, P) is called special if both distributions D_1, D_2 are parallel with respect to the connection ∇ and the connection ∇ has torsion $T = \frac{1}{2}H$.

Theorem 3.6 ([9]). *Any almost product structure on M admits a special connection ∇.*

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