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ALGORITHMIC COMPUTATIONS OF LIE ALGEBRAS COHOMOLOGIES

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Abstract. The aim of this work is to advertise an algorithmic treatment of the computation of the cohomologies of semisimple Lie algebras. The base is the Kostant's result (see below) which describes the representation of the proper reductive subalgebra on the cohomologies space. We will show how to (algorithmically) compute the highest weights of irreducible components of this representation using the Dynkin diagrams. The software package *LiE* offers the data structures and corresponding procedures for the computing with semisimple Lie algebras. Thus, using *LiE* it has been easy to implement the (theoretical) algorithm.

0. Introduction

Each standard parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ of the complex semisimple Lie algebra $\mathfrak{g}$ induces decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. Given a representation $\pi : \mathfrak{p}_+ \longrightarrow \mathfrak{gl}(V)$, we define by the usual way the differential $d : \mathrm{Hom}(\bigwedge^n \mathfrak{p}_+; V) \longrightarrow \mathrm{Hom}(\bigwedge^{n+1} \mathfrak{p}_+; V)$. The corresponding cohomologies will be denoted $H^n(\mathfrak{p}_+, V)$. We will be interested only in cases where $\pi = \nu|\mathfrak{p}_+$ for some representation $\nu : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

Following Kostant (see [Ko]), we define the appropriate representation $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H^n(\mathfrak{p}_+, V))$ of the reductive subalgebra $\mathfrak{g}_0$ on the cohomologies space and we derive the algorithm computing the highest weights of the representation β using the notation of Dynkin diagrams. Then we describe Lie algebra cohomologies $H^n(\mathfrak{g}_-, \mathfrak{g})$ where $\pi = \mathrm{ad}\,|\mathfrak{g}_-$, as a dual to $H^n(\mathfrak{p}_+, \mathfrak{g})$. Further we derive how to use these computations effectively for the complex semisimple Lie algebras with more simple components; it will be a special case of the Künnet formula. The web implementation of the resulting algorithm is available on the address `www.math.muni.cz/~silhan/lac`. (These pages compute moreover cohomologies of real semisimple Lie algebras. These cohomologies will be described elsewhere.)

1. Basic notations and definitions

1.1. The Weyl group and the weights. Consider the complex semisimple Lie algebra $\mathfrak{g}$ with the Cartan subalgebra $\mathfrak{h}$, the sets of simple roots, positive roots and roots $\Pi \subseteq \Delta_+ \subseteq \Delta$ and the Weyl group W. The group W is generated by *simple*

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The paper is in final form and no version of it will be submitted elsewhere.

reflections i.e. the reflections corresponding to simple roots. The number of positive roots $\alpha \in \Delta_+$ which are transformed to $w(\alpha) \in \Delta_- = -\Delta$ is called the *length* of w, we write $|w|$. Equivalently (see [FH], [Sa]), the length of w is the minimal number of simple reflections in any expression for w in terms of simple reflections.

The weights of $\mathfrak{g}$ can be described by labeling the nodes of Dynkin diagram by the integer coefficients referring to the linear combination of fundamental weights. The weight is dominant for $\mathfrak{g}$ if and only if all the coefficients are non-negative (such Dynkin diagram describes the irreducible representation of $\mathfrak{g}$). Using this notation it is easy to compute the action of the simple reflection $w = S_{\alpha_i}$ corresponding to $\alpha_i \in \Pi$ on the weight λ:

Let a be the coefficient over the i-th node in the expression of λ. In order to get the coefficients over the nodes corresponding to $S_{\alpha_i}(\lambda)$, add a to the adjacent coefficients, with the multiplicity if there is a multiple edge directed towards the adjacent node, and replace a by $-a$. (This algorithmic background of computing with the Dynkin diagrams was established in [BE]).

For example, $S_{\alpha_2}(\overset{a}{\bullet}\!-\!\!-\!\overset{b}{\bullet}\!-\!\!-\!\overset{c}{\bullet}) = \overset{a+b}{\bullet}\!-\!\!-\!\overset{-b}{\bullet}\!-\!\!-\!\overset{c+b}{\bullet}$, $S_{\alpha_1}(\overset{a}{\bullet}\!\Rightarrow\!\overset{b}{\bullet}) = \overset{-a}{\bullet}\!\Rightarrow\!\overset{2a+b}{\bullet}$ and $S_{\alpha_2}(\overset{a}{\bullet}\!\Rightarrow\!\overset{b}{\bullet}) = \overset{a+b}{\bullet}\!\Rightarrow\!\overset{-b}{\bullet}$ (the simple root α_i corresponds to i-th node of the Dynkin diagram from the left).

The *affine action* of the Weyl group is defined by

$$w.\lambda = w(\lambda + \rho) - \rho$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta} \alpha$ is the lowest weight. It means (in the terms of the Dynkin diagram) to add one over each node, then act with w and finally subtract one over each node.

1.2. **Parabolic subalgebras.** The standard parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ is defined by some set of simple roots $\Sigma \subseteq \Pi$ and is generated by the Cartan subalgebra, root spaces corresponding to positive roots and root spaces corresponding to negative roots except the roots which can be expressed as the negative sum of some roots from Σ. The corresponding Dynkin diagram is obtained from the Dynkin diagram for $\mathfrak{g}$ by crossing out nodes corresponding to simple roots from Σ. Each parabolic subalgebra is conjugated to some standard parabolic subalgebra so we are interested only in the standard cases. It induces the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. (The reductive part $\mathfrak{g}_0$ includes the semisimple part of $\mathfrak{p}$ and the rest of the Cartan subalgebra and $\mathfrak{p}_+$ is the remaining nilpotent part of $\mathfrak{p}$.)

Irreducible representations of $\mathfrak{p}$ are irreducible representations of $\mathfrak{g}_0$ with the trivial action of $\mathfrak{p}_+$. Thus, the weights of $\mathfrak{p}$ can be described by the labeled Dynkin diagram, where coefficients over non-crossed nodes are integers. This weight is dominant for $\mathfrak{p}$ if and only if the coefficients over non-crossed nodes are non-negative (such Dynkin diagrams describe the irreducible representations of the reductive part $\mathfrak{g}_0$ of $\mathfrak{p}$).

For each set $\Sigma \subseteq \Pi$ and the corresponding parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ we define $W^{\mathfrak{p}} \subseteq W$ as the subset of all elements, which map the weights dominant for $\mathfrak{g}$ into the weights dominant for $\mathfrak{p}$. Equivalently, $W^{\mathfrak{p}}$ is the set of all the elements w for which the set $\Phi_w = w(\Delta_-) \cap \Delta_+$ contains only roots of $\mathfrak{p}_+$ i.e. the positive roots of $\mathfrak{g}$ which do not lie in the semisimple part of $\mathfrak{g}_0$ (see [Ko]).

2. Lie algebra cohomologies

2.1. Cohomologies of Lie algebras. For a representation $\pi : \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ of an arbitrary Lie algebra $\mathfrak{a}$ we define the differential $d : \mathrm{Hom}(\bigwedge^n \mathfrak{a}; V) \longrightarrow \mathrm{Hom}(\bigwedge^{n+1} \mathfrak{a}; V)$ by the formula

$$\begin{aligned}(dp)(X_0 \wedge \cdots \wedge X_n) = & \sum_{i<j} (-1)^{i+j} p([X_i, X_j] \wedge X_0 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_n) \\ & + \sum_i (-1)^i \pi(X_i) p(X_0 \wedge \cdots \hat{X}_i \cdots \wedge X_n).\end{aligned}$$

The differential d induces cohomologies $H^n(\mathfrak{a}; V)$, called the cohomologies of $\mathfrak{a}$ with coefficients in V because $d^2 = 0$ (we set $\mathrm{Hom}(\bigwedge^n \mathfrak{a}; V) = 0$ for $n < 0$ and $n > \dim \mathfrak{a}$).

We are interested only in the case, where $\mathfrak{a} = \mathfrak{p}_+$ and $\pi = \nu|\mathfrak{p}_+$ for some representation $\nu : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. It follows from the structure of the parabolic subalgebra that we have the natural action of the elements from $\mathfrak{p}$ on $\mathrm{Hom}(\bigwedge^n \mathfrak{p}_+; V)$ (it is the adjoint action on $\bigwedge^n \mathfrak{p}_+$ and the action given by π on V). This induces the representation of $\mathfrak{p}$ on $\mathrm{Hom}(\bigwedge^n \mathfrak{p}_+; V)$ which factorizes (see [Ko]) to the representation $\beta : \mathfrak{p} \longrightarrow \mathfrak{gl}(H^n(\mathfrak{p}_+, V))$ on cohomologies. This representation is completely reducible and thus we are interested only in the restriction $\beta : \mathfrak{g}_0 \longrightarrow \mathfrak{gl}(H^n(\mathfrak{p}_+, V))$.

2.2. Theorem. [Ko] ***Kostant's result.*** *For the finite dimensional representation $\nu : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ with the highest weight λ and the restriction $\pi = \nu|\mathfrak{p}$, the irreducible components of β are in bijective correspondence with the set $W^{\mathfrak{p}}$ and the multiplicity of each component is one. The highest weight of the irreducible component of the representation β corresponding to $w \in W^{\mathfrak{p}}$ is $w.\lambda = w(\lambda + \rho) - \rho$ and it occurs in degree $|w|$. The generator of this component (the vector of the highest weight) is $\bigwedge_{\alpha \in \Phi_w} \mathfrak{g}_\alpha \longrightarrow s_{w\lambda}$ where $s_{w\lambda} \in V$ is a weight vector of the weight $w\lambda$.*

2.3. *Example.* **Cohomologies with** $V = \overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}$ **and** $\mathfrak{p} = \times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet$. The zero cohomologies are clearly $\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}$. For the first cohomologies we need the element $w \in W^{\mathfrak{p}}$ of the length one i.e. one simple reflection. It is clearly the reflection corresponding to the single crossed node and so we get $\overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}$. For the second cohomologies we need two simple reflections. It is easy to see that we must begin with an adjacent node to the cross and then use the crossed node. As the result we will get $\overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}$.

2.4. Lie algebra cohomologies $H(\mathfrak{g}_-, \mathfrak{g})$. An important case for applications in geometry are *Lie algebra cohomologies* $H(\mathfrak{g}_-, \mathfrak{g})$, where we use the representation $\mathrm{ad}\,|\mathfrak{g}_- : \mathfrak{g}_- \longrightarrow \mathfrak{gl}(\mathfrak{g})$. We have $H(\mathfrak{g}_-, \mathfrak{g}) \simeq H(\mathfrak{p}_+, \mathfrak{g})^*$ because $\mathfrak{g}_- \simeq \mathfrak{p}_+^*$ and the adjoint representation is self-dual. Thus we must describe the dual representation $\hat{\beta}$ for the representation β. The same is true for all self-dual modules V. In general, $H(\mathfrak{g}_-, V) \simeq H(\mathfrak{p}_+, V^*)^*$.

2.5. Dual representations for reductive algebras. It is well known how to compute the dual representations for simple Lie algebras (for A_l, D_{2l+1} and E_6 we use

k	$k \in \{1,\dots,l\}$	$k \in \{1,\dots,l\}$
λ	$\overset{\lambda_1}{\bullet}\cdots\overset{\lambda_{k-1}}{\bullet}-\overset{\lambda_k}{\times}-\overset{\lambda_{k+1}}{\bullet}\cdots\overset{\lambda_l}{\bullet}$	$\overset{\lambda_1}{\bullet}\cdots\overset{\lambda_{k-1}}{\bullet}-\overset{\lambda_k}{\times}-\overset{\lambda_{k+1}}{\bullet}\cdots\overset{\lambda_{l-1}}{\bullet}\Leftarrow\overset{\lambda_l}{\bullet}$
$\hat{\lambda}$	$\overset{\lambda_{k-1}}{\bullet}\cdots\overset{\lambda_1}{\bullet}-\overset{\hat{\lambda}_k}{\times}-\overset{\lambda_l}{\bullet}\cdots\overset{\lambda_{k+1}}{\bullet}$	$\overset{\lambda_{k-1}}{\bullet}\cdots\overset{\lambda_1}{\bullet}-\overset{\hat{\lambda}_k}{\times}-\overset{\lambda_{k+1}}{\bullet}\cdots\overset{\lambda_{l-1}}{\bullet}\Leftarrow\overset{\lambda_l}{\bullet}$
$\hat{\lambda}_k$	$\hat{\lambda}_k = -\sum_{i=1}^{l}\lambda_i$	$\hat{\lambda}_k = -(\sum_{i=1}^{k}\lambda_i + 2\sum_{i=k+1}^{l}\lambda_i)$

TABLE 1. The duals of reductive subalgebras for types A and C.

k	$k \in \{1,\dots,l-1\}$	$k = l$
λ	$\overset{\lambda_1}{\bullet}\cdots\overset{\lambda_{k-1}}{\bullet}-\overset{\lambda_k}{\times}-\overset{\lambda_{k+1}}{\bullet}\cdots\overset{\lambda_{l-1}}{\bullet}\Rightarrow\overset{\lambda_l}{\bullet}$	$\overset{\lambda_1}{\bullet}-\overset{\lambda_2}{\bullet}\cdots\overset{\lambda_{l-1}}{\bullet}\Rightarrow\overset{\lambda_l}{\times}$
$\hat{\lambda}$	$\overset{\lambda_{k-1}}{\bullet}\cdots\overset{\lambda_1}{\bullet}-\overset{\hat{\lambda}_k}{\times}-\overset{\lambda_{k+1}}{\bullet}\cdots\overset{\lambda_{l-1}}{\bullet}\Rightarrow\overset{\lambda_l}{\bullet}$	$\overset{\lambda_{l-1}}{\bullet}-\overset{\lambda_{l-2}}{\bullet}\cdots\overset{\lambda_1}{\bullet}\Rightarrow\overset{\hat{\lambda}_l}{\times}$
$\hat{\lambda}_k$	$\hat{\lambda}_k = -(\sum_{i=1}^{k}\lambda_i + 2\sum_{i=k+1}^{l-1}\lambda_i + \lambda_l)$	$\hat{\lambda}_l = -(2\sum_{i=1}^{l-1}\lambda_i + \lambda_l)$

TABLE 2. The duals of reductive subalgebras for the type B.

a non-trivial symmetry of the Dynkin diagram and the other cases are self-dual). Consider the irreducible representation γ of the reductive subalgebra $\mathfrak{g}_0$ with the highest weight λ and denote $\hat{\lambda}$ the highest weight of the dual. If we eliminate crossed nodes of the Dynkin diagram we will get a semisimple Lie algebra; consider some of it's simple components $\mathfrak{g}_s$. It is easy to see that $\hat{\lambda}|\mathfrak{g}_s = \widehat{\lambda|\mathfrak{g}_s}$ (it follows from the fact that the highest weight and the lowest weight are on the same orbit of actions of the Weyl group).

It remains to compute the coefficients of $\hat{\lambda}$ over the crossed nodes. Suppose that the i-th node is crossed and λ_i is it's coefficient. It is easy to show that we can restrict only to the Dynkin diagram with the i-th node and the maximal simple component(s) adjacent to this node. Thus, the coefficient $\hat{\lambda}_k$ over a crossed node depends only of the coefficients over the adjacent simple components. In other words, it suffices to consider the reductive subalgebra with one dimensional center. In such case, let us consider an arbitrary generator Z of the center. Then $\lambda(Z) = -\hat{\lambda}(Z)$. This yields the hint how to compute the unknown coefficient of $\hat{\lambda}$ (there is only one unknown coefficient): find an element Z of the center in each simple Lie algebra with one crossed node (case by case) and solve the (linear) equation $\lambda(Z) = -\hat{\lambda}(Z)$ (there is only one unknown parameter). The results for classical simple Lie algebras are in the tables.

2.6. *Example.* **Lie algebra cohomologies $H(\mathfrak{g}_-;\mathfrak{g})$ for $\times-\bullet-\bullet$.** Since the adjoint representation has the highest weight $\overset{1}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet}$, we can use the previous example and we only need to compute the duals. With formulas from Table 1 we will

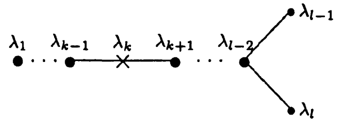
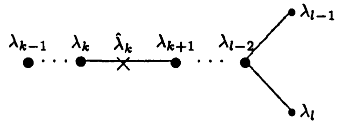
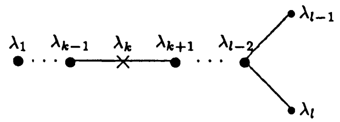
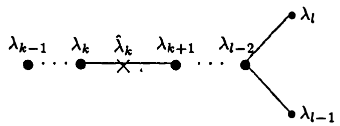
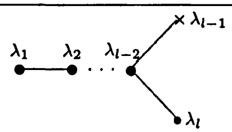
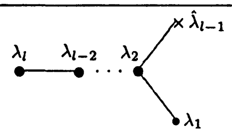
k	$k \in \{1, \dots, l-2\}$, $l-k$ even
λ	
$\hat{\lambda}$	
$\hat{\lambda}_k$	$\hat{\lambda}_k = -(\sum_{i=1}^{k} \lambda_i + 2\sum_{i=k+1}^{l-2} \lambda_i + \lambda_{l-1} + \lambda_l)$
k	$k \in \{1, \dots, l-2\}$, $l-k$ odd
λ	
$\hat{\lambda}$	
$\hat{\lambda}_k$	$\hat{\lambda}_k = -(\sum_{i=1}^{k} \lambda_i + 2\sum_{i=k+1}^{l-2} \lambda_i + \lambda_{l-1} + \lambda_l)$
k	$k = l-1$
λ	
$\hat{\lambda}$	
$\hat{\lambda}_k$	$\hat{\lambda}_{l-1} = -(\lambda_1 + 2\sum_{i=2}^{l-2} \lambda_i + \lambda_{l-1} + \lambda_l)$

TABLE 3. The duals of reductive subalgebras for the type D.

get

$$H^0(\overset{1}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet}) = (\overset{1}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet})^* = \overset{-2}{\times}-\overset{1}{\bullet}-\overset{0}{\bullet}$$
$$H^1(\overset{1}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet}) = (\overset{-3}{\times}-\overset{2}{\bullet}-\overset{1}{\bullet})^* = \overset{0}{\times}-\overset{1}{\bullet}-\overset{2}{\bullet}$$
$$H^2(\overset{1}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet}) = (\overset{-4}{\times}-\overset{1}{\bullet}-\overset{2}{\bullet})^* = \overset{1}{\times}-\overset{2}{\bullet}-\overset{1}{\bullet}.$$

$H^2(\mathfrak{p}^1_+ \oplus \mathfrak{p}^2_+; \mathfrak{sl}(3,\mathbb{C}))$	$H^2(\mathfrak{p}^1_+ \oplus \mathfrak{p}^2_+; \mathfrak{sl}(4,\mathbb{C}))$
$\overset{1}{\times}-\overset{1}{\bullet}\boxtimes\overset{-3}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet}$	$\overset{0}{\times}-\overset{0}{\bullet}\boxtimes\overset{-4}{\times}-\overset{1}{\bullet}-\overset{2}{\bullet}$
$\overset{-3}{\times}-\overset{3}{\bullet}\boxtimes\overset{-2}{\times}-\overset{1}{\bullet}-\overset{0}{\bullet}$	$\overset{-2}{\times}-\overset{1}{\bullet}\boxtimes\overset{-3}{\times}-\overset{2}{\bullet}-\overset{1}{\bullet}$
$\overset{-5}{\times}-\overset{1}{\bullet}\boxtimes\overset{0}{\times}-\overset{0}{\bullet}-\overset{0}{\bullet}$	$\overset{-3}{\times}-\overset{0}{\bullet}\boxtimes\overset{1}{\times}-\overset{0}{\bullet}-\overset{1}{\bullet}$

TABLE 4. The cohomologies $H(\mathfrak{p}_+;\mathfrak{g})$ for $\mathfrak{g}$ semisimple.

2.7. The semisimple Lie algebras composite from more simple components. If we begin with a semisimple Lie algebra $\mathfrak{g}$, we can compute the cohomologies by the procedure described in the examples above. But there is a more effective approach: we can "put together" the required cohomologies from the cohomologies of the simple parts.

Suppose (for simplicity) that $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$ is the complex semisimple Lie algebra with the simple components $\mathfrak{g}^1$ and $\mathfrak{g}^2$ and that $\mathfrak{p} = \mathfrak{p}^1 \oplus \mathfrak{p}^2$ where $\mathfrak{p}^1 \subseteq \mathfrak{g}^1$ and $\mathfrak{p}^2 \subseteq \mathfrak{g}^2$, is the parabolic subalgebra. Denote $W_1, W_2 \subseteq W$ the Weyl groups of $\mathfrak{g}^1$ and $\mathfrak{g}^2$. Each element $w \in W$ can be expressed as composition $w = w_1w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$ commute. Clearly $|w| = |w_1| + |w_2|$ and the condition $w \in W^{\mathfrak{p}}$ is satisfied if and only if $w_1 \in W^{\mathfrak{p}^1}$ and $w_2 \in W^{\mathfrak{p}^2}$. Consider again an irreducible representation $\nu : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ with the highest weight λ. Denoting $\lambda_1 = \lambda|\mathfrak{g}^1$ and $\lambda_2 = \lambda|\mathfrak{g}^2$, we can write $\lambda = (\lambda_1, \lambda_2)$ and clearly $w.\lambda = (w_1.\lambda_1, w_2.\lambda_2)$. Moreover, the representation ν is equivalent to (external) tensor product of representations $\nu_i : \mathfrak{g}_i \longrightarrow \mathfrak{gl}(V_i)$ with highest weights λ_i, $i \in \{1,2\}$ i.e. $\nu \sim \nu_1 \boxtimes \nu_2 : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_1 \boxtimes V_2)$. Summarizing, we have shown that the highest weight $w.\lambda$ of each component in the cohomologies $H(\mathfrak{p}_+;V)$ can be written as a couple $(w_1.\lambda_1, w_2.\lambda_2)$ where $w_i.\lambda_i$ is the highest weight of some component from the cohomologies $H(\mathfrak{p}^i_+;V_i)$, $i \in \{1,2\}$. Thus, the cohomologies of $H(\mathfrak{p}_+;V)$ in degree n can be described as couples of the cohomologies $H(\mathfrak{p}^1_+;V_1)$ in degree n_1 and the cohomologies $H(\mathfrak{p}^2_+;V_2)$ in degree n_2 such the $n_1 + n_2 = n$. This is the special case of the Künnet formula for Lie algebras:

Consider simple Lie algebras and parabolic subalgebras $\mathfrak{p}^i \subseteq \mathfrak{g}^i$ and their irreducible representations $\nu_i : \mathfrak{g}^i \longrightarrow \mathfrak{gl}(V_i)$, $i \in \{1,2\}$. Further consider algebra $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$ with parabolic subalgebra $\mathfrak{p} = \mathfrak{p}^1 \oplus \mathfrak{p}^2$ and its (irreducible) representation $\nu = \nu_1 \boxtimes \nu_2 : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_1 \boxtimes V_2)$; let us denote $V = V_1 \boxtimes V_2$. Then holds

$$H^n(\mathfrak{p}_+;V) = H^n(\mathfrak{p}^1_+ \oplus \mathfrak{p}^2_+; V_1 \boxtimes V_2) = \bigoplus_{i+j=n} (H^i(\mathfrak{p}^1_+;V_1) \boxtimes H^j(\mathfrak{p}^2_+;V_2)).$$

If we compute cohomologies $H(\mathfrak{g}_-;V)$, it suffices to compute the duals of cohomologies $H(\mathfrak{p}_+;V)$. As an example, cohomologies of $\times-\bullet \oplus \times-\bullet-\bullet$ and the adjoint representation are displayed in Table 4. (This representation is not irreducible. In such case, we obtain cohomologies as the sum of cohomologies corresponding to irreducible components of the requested representation.)

3. The algorithm

3.1. The algorithm in the terms of the roots and the weights. The input for our algorithm will be the simple Lie algebra (represented by it's type), the parabolic subalgebra (represented by the set $\Pi \subseteq \Delta_+$), the highest weight of the representation ν (represented by the coefficients over the nodes of the corresponding Dynkin diagram) and the degree j of the required cohomologies.

The computation of the affine action of $w \in W$ is easy if we have w expressed as a composition of simple reflections. The problem is so retrenched into two steps: to find all the elements of $W^{\mathfrak{p}}$ of the length j in the form of the composition of the simple reflections and then to compute the dual weights. The second step is easy (you can use formulas from the tables). The first step can be done in the following way:
1. Generate all sequences of j nodes of the Dynkin diagram.
2. Compute a length of each sequence which is understood as an element of the Weyl group (as a number of the positive roots transformed into the negative ones) and remove the elements with the length smaller then j.
3. Remove the duplicities (by using some canonical form).
4. Compute the set $w(\Delta_-) \cap \Delta_+$ and check that there are only the roots of the semisimple part of $\mathfrak{g}_0$ in this set. The elements satisfying this condition are the required elements of $W^{\mathfrak{p}}$ with the length $|j|$.

3.2. Implementation. The most tedious part of the implementation is to make some representation of the data of simple Lie algebras (i.e. the root systems, the Weyl groups etc.). The software package *LiE* [1] offers such data and procedures for manipulating with them. Thus the implementation in *LiE* corresponds to the "theoretical" algorithm above. (*LiE* contains all necessary steps not specified above, e.g. using of some canonical form of elements of the Weyl group). The web implementation (based on *LiE* too) is on the address www.math.muni.cz/~silhan/lac.

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[1] www.can.nl/SystemOverview/Special/GroupTheory/LiE/index.html